INTRODUCTORY
PHYSICS

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N.N. GHOSH

INTRODUCTORY PHYSICS

[Part I]

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** In S. I. Unit
** With Standard Notations

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FOREWORD

I have seen the book 'Introductory Physics' authored by Sri N. N. Ghosh.

For sometime, there has been a pressing need for a book, written in English, which will meet the requirements of the new Intermediate Examinations (+2) syllabus prescribed.

This book will fill that void. I have great hopes that this book will prove to be useful for the students, for whom it is meant.

Dr. M. P. Gupta
Ex.University Professor and Head of the Deptt. of Physics
Ranchi University, Ranchi.

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A nice flower takes time to come up. First a good seed is planted in sunny, yet shaded place. Slowly it comes up, one leaf after the other pierces the hard ground. In due time a bud changes into a flower. It basks in the sun for a little time, then forms a fruit for all see and taste.

For 23 years generations of bright students have heard Prof. N. N. Ghosh teach them Physics. To satisfy their intellectual eagerness, Prof. N. N. Ghosh patiently gathered one by one thousand of problems, which he published in book form—a book whose second edition will soon be exhausted. He then wrote a book on Practicals, which I find in everybody's hands. He now presents us with this book on the Theory of Physics. His well-known thoroughness and clarity of thought have fully matured into this fruit of his labour.

May this book nurture the seed that lays dormant in the soul of the next generations of bright young scholars of Physics.

C. De Brouwer sj
Ex. Head of the Dept. of Physics
St. Xavier's College, Ranchi

PREFACE

This book is prepared with the sole aim to provide a text book for the Intermediate Education and a base book for students preparing for various competitive examinations of the state level as well as all India level. The systeme Internationale d'unites (SI units) have been used throughout the text as is desired by the Bihar Intermediate Education Council and other councils of India. Modern trends of presenting chapters have been adopted throughout the text. New concepts like 'force and environment', 'free-body diagram', 'real and pseudo forces' etc. have been introduced in appropriate chapters. All the three sign conventions in light have been dealt with. The magnetic terms and definitions have been approached from the magnetic effect of moving charged particle, that is, electric current and not from the old concept of magnetic poles. Numerical examples and physical illustrations have been provided to make the principles involved more lucid. In each chapter questions are given in five groups—A—objective questions, B—short questions, C—traditional essay type questions, D—numericals and E—very short ponderable questions.

Though I am indebted to all my colleagues for discussion and inspiration from time to time, I am specially indebted to Rev. Fr. C. D. Brouwer, Head of the department of Physics, St. Xavier's College, Ranchi for going patiently through almost the entire manuscript and writing a nicely worded foreword. I am also indebted to Prof. D. Dey, our brilliant ex-student and now esteemed colleague in the department for going partially through the manuscript of magnetism and electrostatics. I am deeply grateful to Dr. M. P. Gupta, University Professor and Head of the Deptt. of Physics, Ranchi University, Ranchi for going through the manuscript and writing few words for the foreword of this book inspite of his heavy engagements before going abroad as visiting professor. Dr. A. K. Rakshit, Head of the Deptt. of Physics, B. N. College, Patna deserves my special thanks for making suggestions and corrections, as a friendly gesture, while the book was in progress in the press. Lastly, I wish to express my gratitude to Prof. Asit Dasgupta, Deptt. of Mathematics, Science College, Patna for helping me in bringing precision and accuracy in the various rules and expressions in the Calculus chapter.

Constructive criticisms and suggestions for improvement will always be thankfully acknowledged.

June. 1983
St. Xavier's College, Ranchi **N. N. Ghosh**

PREFACE TO THE FOURTH EDITION

In this edition utmost care has been taken to remove the minor misprints of the previous edition. To serve the needs of a wider section of students it has become imperative to add a few new chapters (starmarked) and elaborate some of the existing chapters. Students of different universities, councils and boards will carefully select their topics according to their syllabii but for competitive examinations they must read all the chapters. I hope this edition will prove all the more useful for the students.

I look forward earnestly to constructive criticisms and suggestions for further improvement of the book.

October, 1988
St. Xavier's College, Ranchi **N. N. Ghosh**

SOME PHYSICAL CONSTANTS

Speed of light	c	3×10^{8} ms^{-1}
Gravitational constant	G	$6{\cdot}67\times10^{-11}$ Nm^{2}kg^{-2}
Universal gas constant	R	$8{\cdot}31$ Jmol^{-1} K^{-1}
Permeability constant	μ_0	$4\pi\times10^{-7}$Hm^{-1}
Permittivity constant	ϵ_0	$8{\cdot}85\times10^{-12}$ Fm^{-1}
Avogadro's number	N	$6{\cdot}02\times10^{23}$ mol^{-1}
Boltzmann constant	k	$1{\cdot}38\times10^{-23}$ JK^{-1}
Planck's constant	h	$6{\cdot}63\times10^{-34}$ Js
Elementary charge	e	$1{\cdot}60\times10^{-19}$ C
Electron mass (rest)	m	$9{\cdot}11\times10^{-31}$ kg
Electron magnetic moment (Bohr magneton)		$9{\cdot}27\times10^{-24}$ JT^{-1}
Acceleration due to gravity	g	$9{\cdot}80$ms^{-1}
Standard atmospheric pressure		$1{\cdot}01\times10^{5}$Nm^{-2}
Density of water at 4°C		1000 kgm^{-3}
Specific heat capacity of water		4200 Jkg^{-1}K^{-1}
Specific latent heat of ice		$3{\cdot}36\times10^{5}$ Jkg^{-1}
Specific latent heat of steam at 100°C and normal pressure		$2{\cdot}25\times10^{6}$ Jkg^{-1}
A faraday	F	$9{\cdot}65\times10^{4}$ C mol^{-1}
Rydberg constant		$1{\cdot}10\times10^{7}$ m^{-1}
Stefan constant	σ	$5{\cdot}67\times10^{-8}$ Wm^{-2}K^{-4}
Bohr's radius	a_0	$5{\cdot}29\times10^{-11}$ m

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IMPORTANT TRIGONOMETRIC RELATIONS

1. (i) $\sin(A+B)=\sin A\cos B+\cos A\sin B$

(ii) $\cos(A+B)=\cos A\cos B-\sin A\sin B$

(iii) $\sin C+\sin D=2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

(iv) $\sin C-\sin D=2\sin\frac{C-D}{2}\cos\frac{C+D}{2}$

(v) $\cos C+\cos D=2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

(vi) $\cos C-\cos D=-2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

(vii) $\sin 2A=2\sin A\cos A$

(viii) $\cos 2A=\cos^2 A-\sin^2 A=2\cos^2 A-1=1-2\sin^2 A.$

2. (i) sine property of a triangle : In a triangle ABC

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

(ii) cosine property : $c^2=a^2+b^2-2ab\cos C$; $a^2=b^2+c^2-2bc\cos A$; $b^2=c^2+a^2-2ca\cos B$

(iii) Area of a triangle : $\Delta=\frac{1}{2}ab\sin C$ or $\frac{1}{2}bc\sin A$ or $\frac{1}{2}ca\sin B$.

3. Expansion formula :

$$(1+x)^n=1+nx+\frac{n(n-1)}{1.2}x^2+\frac{n(n-1)(n-2)}{1.2.3}x^3+\ldots\ldots\ldots$$

When x is very small, $(1+x)^n=1+nx$.

Standard Notations

Physical quantity	Accepted notations in SI	Old notations
Thermal conductivity	λ	K
Magnetic pole	q_m	m
Magnetic moment	m	M
Magnetisation	M	I

CONTENTS

GENERAL PHYSICS

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CHAPTER 1

ELEMENTARY IDEAS ABOUT CALCULUS

1.1. Definitions

Quantity. Anything that can be measured is termed as 'quantity'. e.g., mass, distance, time, temperature, area, volume, velocity, charge,, energy..... They are denoted by alphabets a, b, c, d....x, y, z.

Variable Quantity. A quantity is termed *variable* when it has not always a fixed or the same value, i e., when it may have a series of values, e.g., the velocity of a body falling under gravity. They are denoted by $x, y, z, \ldots$

Constant Quantity. A quantity which has always a fixed value is termed as a constant quantity. It is generally denoted by $a, b, c, \ldots$

An Independent Variable. A variable to which any arbitrary value may be assigned is termed as an independent variable, e.g., in a simple pendulum the length of the pendulum may be set to any value one likes, and hence the length of the pendulum is an independent variable.

Function. When a variable quantity depends on another variable for its value, the former is called a function of the latter, e.g., in a simple pendulum, at a given place, the time period is a function of the length of the pendulum.

The volume of a given mass of a gas at a fixed temperature is a function of the pressure of the gas.

The frequency of vibration of a stretched string is a function of its tension when its length and linear density remain constant.

Note. A function is a dependent variable. The quantity, of which it is a function, is an independent variable.

Notation of a Function. A function is denoted by symbols like $f(x)$, $F(x)$, $\varphi(x)$, $\psi(x)$ .....and is read as *function of x* or *function x*. Thus if y is a function of x, we may write $y=f(x)$. Note that y and $f(x)$ represent one and the same thing. This symbolic relation merely denotes that y is a function of x.

1.2. Evaluation of Functions

Example

If $y=f(x)=x^2-3x+5$, *find* $f(0)$, $f(1)$, $f(-1)$, $f(3)$.

Sol. To find the value of the function at a particular value of the independent variable, put the given value in the expression of the function and simplify. The result follows :

Here, $f(x)=x^2-3x+5.$

$\therefore$ $f(0)=0^2-3.0+5=5;$

$f(1)=1^2-3.1+5=3;$

$f(-1)=(-1)^2-3.(-1)+5=9;$

$f(3)=3^2-3.3+5=5.$

EXERCISES

1. If $f(t)=3t+4t^2$, find $f(3)$, $f(1)$, $f(-1)$ and $f(0)$.
[Ans. 45, 7, 1, 0.]

2. If $f(\theta)=\sin\theta+\cos\theta$, find $f(0)$, $f(\pi/2)$, $f(\pi)$ and $f(-\pi)$.
[Ans. 1, 1, −1, −1.]

3. If $\psi(x)=\dfrac{x^2+3}{x-3}$, find $\psi(1)$, $\psi(-1)$, $\psi(2)$, $\psi(5)$ and $\psi(10)$.

[Ans. −2, −1, −7, 14, 14·7.]

1.3. Limits of Functions : Meaning of the Symbol $x \to a$

When the independent variable is gradually taken to a definite value say, a, the dependent variable, i.e., the function will tend to another definite value, say, l. This value is defined as the limiting value of the function as the independent variable approaches the given value.

The 'arrow' in the above symbol stands for gradual approach of x to a and the symbol is read as 'x tending to a'. If $y=f(x)$ approaches a value l, as x approaches a, we say that the limiting value of $f(x)$ is l when x approaches a and this is symbolically written as

$$\lim_{x \to a} f(x)=l.$$

This symbol is read as "the limit of the function x is l when x tends to a".

ILLUSTRATIONS

1. Inscribe a polygon of n sides in a circle of radius a. The area A of the polygon will depend on the number of sides n. Hence $A = f(n)$.

Now as we increase the number of sides, the sides will be shorter and shorter in size, the area of the polygon will increase and ultimately when n is made infinitely large, the area of the polygon will become equal to the area of the circle. Thus we may say that the limiting value of the area of a polygon of n sides inscribed in a circle is the area of the circle itself, as n tends to infinity.

2. The emf of a cell is a constant, but its terminal voltage V depends on the external resistance R. We say that V is a function of R. As R increases, the value of V also increases and ultimately when R becomes infinitely large, the value of V attains the value of emf. Hence we may say that the limiting value of the terminal voltage is the emf of the cell as the external resistance tends to infinity.

3. Consider the function $y = f(x) = x^2 - 1$. As $x \to 2$, what is the limiting value of $f(x)$?

When x approaches 2 from the right

x	2·2	2·1	2·01	2·001	2·0001
$f(x)$	3·84000	3·41000	3·04010	3·00400	3·00040

When x approaches 2 from the left

x	1·9	1·99	1·999	1·9999	1·99999
$f(x)$	2·61000	2·96010	2·99520	2·99960	2·99996

From the inspection of the above two tables it is clearly seen that as x tends to 2, $f(x)$ tends to 3.

$$\therefore \qquad \lim_{x \to 2} x^2 - 1 = 3.$$

The process of finding the limiting value of a function in the above way is actually the fundamental way of finding the limiting value of a function. But this is a very lengthy process. Every time it is not possible to find the limiting value by this process. Hence we need a short-cut method. Putting $x = 2$ in the expression we find

that the limiting value automatically follows. Hence the short-cut method of finding the limiting value of a function is to make direct substitution of the limiting value of the independent variable in the expression of the function.

4. Find the limiting value of $y=f(x)=\dfrac{2x+3}{x+6}$ from first principles and check it by a short-cut method when x tends to 0.

x	·1	·01	·001	·0001	·00001
$f(x)$	·52460	·50249	·50025	·50002	·50000

Thus from the table we see that as x is made smaller and smaller, $f(x)$ tends to ·5

$$\therefore \qquad \lim_{x\to 0}\frac{2x+3}{x+6}=\cdot 5.$$

Short-cut Method. Put $x=0$, then $\dfrac{2x+3}{x+6}=\dfrac{3}{6}=\tfrac{1}{2}=\cdot 5.$

$$\therefore \qquad \lim_{x\to 0}\frac{2x+3}{x+6}=\cdot 5.$$

5. Show from first principles that

$$\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)=1 \quad \text{and} \quad \lim_{\theta\to 0}\cos\theta=1.$$

$\theta°$	10°0′	8°0′	6°0′	4°0′	3°0′	2°0′	1°0′	0°48′	0°24′	0°12′	0°6′
θ in radian	·17453	·13963	·10473	·06981	·05236	·03491	·01745	·01396	·00698	·00349	·00175
$\sin\theta$	·17365	·13917	·10453	·06976	·05234	·03490	·01745	·01396	·00698	·00349	·00175
$\cos\theta$	·98481	·99027	·99452	·99756	·99963	·99939	·99985	·99990	·99999	·99999	·99999

From the above table we see clearly that as θ (in radian) tends to zero, $\sin\theta$ comes closer and closer to θ itself and $\frac{\sin\theta}{\theta}$ comes closer and closer to 1. It can be seen from the table by taking a row of values of $\left(\frac{\sin\theta}{\theta}\right)$ that it tends to 1. We further see that $\cos\theta$ approaches 1 as θ tends to zero.

Note. It is seen that $\frac{\sin\theta}{\theta}$ is exactly equal to 1 in the last few columns because we have taken values of $\sin\theta$ and θ up to five places of decimal only.

$$\therefore \qquad \lim_{\theta \to 0}\left(\frac{\sin\theta}{\theta}\right)=1 \text{ and } \lim_{\theta \to 0} \cos\theta = 1.$$

These two results are very important in Calculus. Note that so long θ (in degree) is within $4°$, $\sin\theta=\theta$ holds good. This result will be referred to on many occasions in Physics.

Important Theorems to be committed to memory.

(i) The limit of a constant is always equal to that constant.

(ii) $\lim\limits_{x \to a}[f(x) \pm F(x)] = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} F(x)$ provided both limits are not infinity in case of minus sign.

(iii) $\lim\limits_{x \to a} f(x).\,F(x) = \lim\limits_{x \to a} f(x).\, \lim\limits_{x \to a} F(x)$ [Mind that this result does not hold when one limit is zero and the other is infinity.] i.e. the limit of a product is equal to the product of the limits of the factors.

(iv) $\lim\limits_{x \to a} \dfrac{f(x)}{F(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} F(x)}$ [Mind that both are not zero and both are not infinity.]

i.e. the limit of a quotient is equal to the limit of the numerator divided by the limit of the denominator.

1.4. Determinate and Indeterminate Forms

The value of a function is equal to its limiting value in most of our cases to which short-cut method is applicable. Sometimes the

value of a function cannot be determined but its limiting value can be determined.

The value of a function in some cases comes out in meaningless form like $\frac{0}{0}, \frac{\infty}{\infty}$. We then say that the function is of indeterminate form. Let us explain it by an example. Suppose $y=f(x)=\frac{x^2-4}{x^2-x-2}$ and we are asked to find the limiting value of $f(x)$ when x tends to 2. Putting $x=2$ in the expression of the function we have,

$$f(x)=\frac{x^2-4}{x^2-x-2}=\frac{0}{0}.$$ This is a meaningless form.

So the function is undefined at $x=2$. Therefore the short-cut method fails. In such cases we have to do something to avoid such meaningless form. We put $x=2+h$ so that when $x\to 2$, $h\to 0$.

$$\therefore \quad \lim_{x\to 2}\frac{x^2-4}{x^2-x-2}=\lim_{h\to 0}\frac{(2+h)^2-4}{(2+h)^2-(2+h)-2}$$

$$=\lim_{h\to 0}\frac{4+h}{3+h}=\frac{4}{3} \text{ (putting } h=0).$$

Let us consider another example,

$$y=f(x)=\frac{x^2+3}{2x^2+5}$$

and find the limiting value of $f(x)$ when x tends to ∞.

Putting $x=\infty$ in the expression for $f(x)$ we have,

$$f(x)=\frac{x^2+3}{2x^2+5}=\frac{\infty}{\infty}.$$

This is also a meaningless (undefined) form.

Put $x=\frac{1}{h}$ so that when $x\to\infty$, $h\to 0$.

$$\therefore \quad \lim_{x\to\infty}\frac{x^2+3}{2x^2+5}=\lim_{h\to 0}\frac{\left(\frac{1}{h}\right)^2+3}{2\left(\frac{1}{h}\right)^2+5}=\lim_{h\to 0}\frac{1+3h^2}{2+5h^2}=\frac{1}{2}$$

(putting $h=0$).

Important LImits to be committed to memory.

(i) $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}$.

(ii) $\lim\limits_{\theta \to 0} \left(\dfrac{\sin\theta}{\theta}\right) = 1$ and $\lim\limits_{\theta \to 0} \cos\theta = 1$.

(iii) $\lim\limits_{\theta \to 0} \dfrac{\tan\theta}{\theta} = 1$ and $\lim\limits_{\theta \to 0} \theta \cot\theta = 1$.

EXERCISES

1. $\lim\limits_{x \to 0} \dfrac{x+5}{2x+5}$. **2.** $\lim\limits_{x \to 1} \dfrac{x+1}{x-1}$. **3.** $\lim\limits_{x \to 0} \dfrac{x}{5x-4}$.

4. $\lim\limits_{\theta \to 0} \dfrac{1-\cos\theta}{\sin\theta}$. **5.** $\lim\limits_{h \to 0} \cos\left(\theta + \dfrac{h}{2}\right)$.

6. $\lim\limits_{x \to 3} \dfrac{x^4 - 3^4}{x-3}$. **7.** $\lim\limits_{x \to 1} \dfrac{x^2 - x + 2}{x+7}$.

[**Ans.** **1.** 1; **2.** ∞; **3.** 0; **4.** 0; **5.** $\cos\theta$; **6.** 108; **7.** $\frac{1}{4}$.]

1.5. Differentiation and Differential Co-efficient

If a variable is changed by infinitesimally small amount then that change is called the *differential* of the variable. This is denoted by dx and read as *dee x*. This is never *d into x*. This is merely a symbol standing for a very very small increment in x. This should be treated as a single quantity just as $\sin\theta$ is not the product of sin and θ. This is a symbolic representation of the ratio of the perpendicular to the hypotenuse in a right angled triangle with respect to θ.

For a finite but small increment in x we use the symbol Δx. Just like dx this is also not Δ into x. It stands for a small but finite increment in x. Like dx this should also be treated as a single quantity. This should be read as *delta x*.

Definition of Differential Co-efficient. If y be a function of x and Δy be a small increment of y corresponding to a small increment Δx in x, then the limiting value of the ratio $\dfrac{\Delta y}{\Delta x}$ when Δx tends to zero

is called the differential co-efficient of y with respect to x. It is denoted by the symbol $\frac{dy}{dx}$. Thus

$$\frac{dy}{dx}=\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

The process of finding the differential or the differential co-efficient is called *differentiation.*

Examples

1. *If $y=x^n$, find dy/dx.*

Sol. Let us give a small increment Δx in x and let y be thereby increased by Δy.

Then
$$y+\Delta y=(x+\Delta x)^n$$
$$\therefore \quad \Delta y=(x+\Delta x)^n-y=(x+\Delta x)^n-x^n$$

or
$$\Delta y=\left(x^n+nx^{n-1}\Delta x+\frac{n(n-1)}{2!}x^{n-2}(\Delta x)^2+\ldots\right)-x^n$$
$$=nx^{n-1}\Delta x+\frac{n(n-1)}{2!}x^{n-2}(\Delta x)^2+\ldots\ldots$$

$$\therefore \quad \frac{\Delta y}{\Delta x}=\frac{nx^{n-1}\Delta x+\frac{n(n-1)}{2!}x^{n-2}(\Delta x)^2+\ldots\ldots}{\Delta x}$$

$$=nx^{n-1}+\text{terms containing } \Delta x.$$

According to the definition

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}(nx^{n-1}+\text{terms containing }\Delta x)$$
$$=nx^{n-1} \text{ (putting } \Delta x=0).$$

Thus when
$$y=x^n,$$
$$\frac{dy}{dx}=nx^{n-1}.$$

2. *If $y=\sin ax$, find $\frac{dy}{dx}$* and *dy.*

Sol. Let us increase x by Δx.

Then
$$y+\Delta y=\sin a(x+\Delta x)$$

or $\qquad \triangle y=\sin(ax+a\triangle x)-y=\sin(ax+a\triangle x)-\sin ax.$

$$\therefore \qquad \frac{\triangle y}{\triangle x}=\frac{\sin(ax+a\triangle x)-\sin ax}{\triangle x}.$$

By definition

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\triangle y}{\triangle x}=\lim_{\Delta x\to 0}\frac{\sin(ax+a\triangle x)-\sin ax}{\triangle x}$$

or
$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{2\sin\frac{a\triangle x}{2}\cdot\cos\left(ax+\frac{a\triangle x}{2}\right)}{\triangle x}$$

$$=2.\lim_{\Delta x\to 0}\frac{\sin\frac{a\triangle x}{2}}{\triangle x}\lim_{\Delta x\to 0}\cos\left(ax+\frac{a\triangle x}{2}\right)$$

$$=2.\ a/2.\cos ax$$
$$=a\cos ax.$$

Thus when $y=\sin ax$, $\frac{dy}{dx}=a\cos ax$ and $dy=\left(\frac{dy}{dx}\right)dx=a\cos axdx.$

Important Results to be committed to memory.

(i) If $y=x^n$, $\quad \frac{dy}{dx}=nx^{n-1}$, $\quad$ or $\quad dy=nx^{n-1}dx$.

(ii) If $y=\sin ax$, $\quad \frac{dy}{dx}=a\cos ax$, $\quad$ or $\quad dy=a\cos ax\,dx$.

(iii) If $y=\cos ax$, $\quad \frac{dy}{dx}=-a\sin ax$, $\quad$ or $\quad dy=-a\sin ax\,dx$.

(iv) If $y=\tan ax$, $\quad \frac{dy}{dx}=a\sec^2 ax$, $\quad$ or $\quad dy=a\sec^2 ax\,dx$.

(v) If $y=\cot ax$, $\quad \frac{dy}{dx}=-a\,\mathrm{cosec}^2 ax$, or $\quad dy=-a\,\mathrm{cosec}^2\,ax\,dx$.

(vi) If $y=\mathrm{cosec}\,ax$, $\frac{dy}{dx}=-a\,\mathrm{cosec}\,ax\cot ax$,

or $\quad dy=-a\,\mathrm{cosec}\,ax\cot ax\,dx.$

(vii) If $y=\sec ax$, $\frac{dy}{dx}=a \sec ax \tan ax$,

or $dy=a \sec ax \tan ax\, dx$.

(viii) If $y=e^{ax}$, $\frac{dy}{dx}=ae^{ax}$, or $dy=ae^{ax}dx$.

(ix) If $y=\log x$, $\frac{dy}{dx}=\frac{1}{x}$, or $dy=\frac{dx}{x}$.

(x) If $y=\sin^{-1}\left(\frac{x}{a}\right)$, $\frac{dy}{dx}=\frac{1}{\sqrt{a^2-x^2}}$, or $dy=\frac{dx}{\sqrt{a^2-x^2}}$.

(xi) If $y=\cos^{-1}\left(\frac{x}{a}\right)$, $\frac{dy}{dx}=-\frac{1}{\sqrt{a^2-x^2}}$, or $dy=-\frac{dx}{\sqrt{a^2-x^2}}$.

RULES FOR DIFFERENTIATION

(i) $\frac{dc}{dx}=0$ where c is a constant

i.e., differential co-efficient of a constant is zero.

(ii) $\frac{d}{dx}(cu)=c\frac{du}{dx}$.

(iii) $\frac{d}{dx}(u \pm v)=\frac{du}{dx}\pm\frac{dv}{dx}$

i.e., differential co-efficient of a sum = sum of the differential co-efficients of individual terms.

(iv) $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$

i.e. differential co-efficient of a product = first factor × differential co-efficient of the second factor + the second factor × differential co-efficient of the first factor.

(v) $\frac{du}{dx}=\frac{du}{dz}\cdot\frac{dz}{dx}$.

EXERCISES

Differentiate

1. $y=5x^3$.
2. $y=x^{\frac{1}{2}}$.
3. $y=x^3+2x-3$.
4. $y=\sin 5x$.
5. $y=x \sin x$.

6. Suppose $x^2=a^2+b^2-2ab\cos\theta$, where a and b are constant, find $\frac{dx}{d\theta}$.

7. If $V=\frac{m\cos\theta}{r^2}$, where m is a constant,

find $\left(\frac{dV}{dr}\right)_{\theta=constant}$ and $\left(\frac{dV}{d\theta}\right)_{r=constant}$.

Note. $\left(\frac{dV}{dr}\right)_{\theta=constant}$ means differentiate V with respect to r when θ remains constant.

[**Ans.** 1. $15x^2$. 2. $\frac{1}{2}x-\frac{1}{4}$. 3. $3x^2+2$. 4. $5\cos 5x$.

5. $x\cos x+\sin x$. 6. $\frac{ab\sin\theta}{\sqrt{a^2+b^2-2ab\cos\theta}}$

7. $-\frac{2m\cos\theta}{r^3}$, $\frac{-m\sin\theta}{r^2}$]

Function of a Function. Sin, cos, $\sqrt{\ }$, power, log, $\sin^{-1}$, etc. are symbols of a function. When $y=\sin x$, we say y is the sine function of x. When $y=\sin\sqrt{x}$, we say y is a sine function of $\sqrt{x}$ and $\sqrt{x}$ itself is a function of x. Hence here y is a function of a function. When $y=\sin\cos^{-1}x^2$, we say y is a function of a function of a function of a function. To differentiate such function the principle

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}$$

is applied. Here y is a function of u, u is a function of v and v is a function of x.

Example

$y=a\sin\omega t$ where a and ω are constants.

$$\frac{dy}{dt}=\frac{d}{dt}(a\sin\omega t)=a\frac{d}{dt}(\sin\omega t)=a\frac{d(\sin\omega t)}{d(\omega t)}\cdot\frac{d(\omega t)}{dt}$$

$$=a\cos\omega t.\ \omega=a\omega\cos\omega t.$$

EXERCISES

1. If $y=\sin\sqrt{x}$; find $\frac{dy}{dx}$.

2. If $y = \log \sin x$, find $\frac{dy}{dx}$.

$$\left[\text{Ans. 1. } \tfrac{1}{2}\frac{\cos\sqrt{x}}{\sqrt{x}}. \quad 2.\ \cot x\right]$$

1.6. Geometrical Interpretation of $\frac{dy}{dx}$

Suppose P is a point on the continuous curve of the plot of y against x. The co-ordinates of P are (x, y). Consider another point Q close to P. Let the co-ordinates of Q be $(x+\triangle x,\ y+\triangle y)$. Draw the tangent PT to the curve at P. Let its inclination with the x-axis be φ. Let the line QP cut the x-axis at some point and let its inclination be θ.

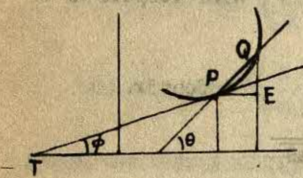

Fig. 1.1

Obviously $\angle QPE = \theta$ and $\tan\theta = \frac{\triangle y}{\triangle x}$.

Now let us bring Q closer and closer to P, i.e., make $\triangle x$ smaller and smaller. As we do so, the inclination θ approaches φ and ultimately when $\triangle x \to 0$, θ would be equal to φ i.e., we may say that as $\triangle x \to 0$, the limiting value of θ is φ.

Hence, $$\tan\varphi = \lim_{\triangle x \to 0} \frac{\triangle y}{\triangle x}.$$

According to definition of differential co-efficient,

$$\frac{dy}{dx} = \lim_{\triangle x \to 0} \frac{\triangle y}{\triangle x}.$$

Hence $$\frac{dy}{dx} = \tan\varphi.$$

Thus geometrically $\frac{dy}{dx}$ is the tangent of the inclination of the tangent to the curve with the x-axis. This is called *slope of the curve.*

1.7. $\frac{dy}{dx}$ as a Rate Measurer

The rate of change of a quantity (y) with respect to another quantity (x) is defined as the ratio of the change of y to the change in x, however small the change in x may be.

If $\triangle y$ be the change in y corresponding to a change $\triangle x$ in x, then according to the definition, the rate of change of y with respect to x is the limiting value of the ratio $\frac{\triangle y}{\triangle x}$ when $\triangle x$ tends to zero.

Thus, the rate of change of y with respect to $x = \lim\limits_{\Delta x \to 0} \frac{\triangle y}{\triangle x} = \frac{dy}{dx}$.

When we simply say rate of change of y, we mean change of y with respect to time. So rate of change of y is $\frac{dy}{dt}$.

Examples

1. *Find the slope of the tangent to the curve $y=3x^2-5$ at the point* (2, 7).

Sol.
$$y=3x^2-5.$$
$$\therefore \quad \frac{dy}{dx}=6x.$$

At the point (2, 7), $\frac{dy}{dx}=6.2=12.$

$$\therefore \quad \tan\varphi=12.$$

2. *Find the rate of change in area of a square of side* 4 *cm when its side is increasing at the rate of* ·01 *cm per second.*

Sol. We have, $s=a^2$ where s = area of the square and a = length of the square on each side. Differentiating with respect to t (time) we have,

$$\frac{ds}{dt}=2a\frac{da}{dt}$$
$$=2.4\,(\cdot 01)=\cdot 08 \text{ cm}^2 \text{ per second.}$$

EXERCISES

1. Find the inclination with the x-axis of the tangent to the curve $y^2=4x$ at (1, 2).

2. A man 2 m tall is walking away from a lamp post 8 m high at the rate 1·5 ms^{-1}. Find how fast his shadow is lengthening.

[Ans. **1.** 45°; **2.** ·5 ms^{-1}]

1.8. Integration is the reverse process of differentiation. In differentiation we find the differential of a given function. In integration the differential of the function is given, one has to find the function itself.

Definition. An integral of a given function is that function whose differential co-efficient is the given function.

You know that the differential co-efficient of $\frac{x^{n+1}}{n+1}$ with respect to x is x^n. Therefore by definition, the integral of x^n with respect to x is $\frac{x^{n+1}}{n+1}$ (when $n \neq -1$) which is symbolically written as $\int x^n dx = \frac{x^{n+1}}{n+1}$ (when $n \neq -1$).

Important Results (standard forms) to be committed to memory.

(i) $\int x^n dx = \frac{x^{n+1}}{n+1}$ except when $n = -1$.

(ii) $\int \frac{dx}{x} = \log x$, or $\int x^{-1} dx = \log x$.

(iii) $\int \sin ax\, dx = -\frac{\cos ax}{a}$. (iv) $\int \cos ax\, dx = \frac{\sin ax}{a}$.

(v) $\int e^{ax} dx = \frac{e^{ax}}{a}$.

(vi) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$.

(vii) $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$.

1.9. Constant of Integration

After each integral one must add a constant. The reason for adding a constant is as follows :

The differential of x^{n+1} is $(n+1)\, x^n dx$. The differential of $(x^{n+1} + C)$ is also $(n+1)\, x^n dx$ because the differential co-efficient of a constant is zero. Hence in general one has to add a constant after performing an integration. This constant is called the *constant of integration.*

RULES FOR INTEGRATION

(i) $\int k\, dx = k\int dx$ where k is a constant.

(ii) $\int (u \pm v)\, dx = \int u\, dx \pm \int v\, dx$

i.e., the integral of a sum = sum of the integrals of the individual terms.

(iii) $\int uv\, dx = u \int v\, dx - \int (\int v\, dx) du$

i.e., the integral of a product = first factor × integral of the second factor − integral of (integral of the second × differential co-efficient of the first).

In the result (i) putting $n=0$, we have $\int dx = x$. Similarly $\int dy = y$. Thus if φ is a function of x then $\int d\varphi = \varphi$. So we find that integral of the differential of a function is the function itself.

Examples

1. *Integrate* : $\int x^2 dx$.

Sol. Here $n=2$. By the result of the standard form (i) we have

$$\int x^2 dx = \frac{x^{2+1}}{2+1} = \frac{x^3}{3} = \tfrac{1}{3}x^3 + C$$

where C is the constant of integration.

2. *Integrate* : $\int \sin^2 x\, dx$.

$$\begin{aligned}\int \sin^2 x dx &= \tfrac{1}{2}\int 2 \sin^2 x dx\\ &= \tfrac{1}{2}\int (1-\cos 2x) dx\\ &= \tfrac{1}{2}\int dx - \tfrac{1}{2}\int \cos 2x dx\\ &= x/2 - \tfrac{1}{2}\,\frac{\sin 2x}{2} = x/2 - \tfrac{1}{4}\sin 2x + C.\end{aligned}$$

Note. To perform an integration, in the first step, the 'form' of the function must be examined carefully. If it is of the 'standard form', perform the integration by applying the important results listed above. If not bring it down to the 'standard form' and then perform the integration by applying standard results. This is illustrated in the above example.

EXERCISES

1. $\int 3x^4 dx$. **2.** $\int \sin 2\theta\, d\theta$.

3. $\int (x^2 + \sin x) dx$. **4.** $\int x \sin x\, dx$.

5. $\int\left(x^2+\frac{1}{x}\right)dx.$ 6. $\int e^{5x}dx.$

[Ans. 1. $\frac{3}{5}x^5+C.$ 2. $-\frac{1}{2}\cos 2\theta+C.$ 3. $\frac{1}{3}x^3-\cos x+C.$
4. $\sin x-x\cos x+C.$ 5. $\frac{1}{3}x^3+\log x+C.$ 6. $\frac{1}{5}e^{5x}+C.$]

1.10. Definite Integrals

A definite integral of a differential function is the difference between the values of the integral for the two given values of the variable. It is denoted by

$$\int_a^b f(x)dx=F(b)-F(a)$$ where $F(x)$ is the integrated value of $f(x)dx$.

Examples

Integrate : $\int_0^l x^2dx.$

Sol. $$\int_0^l x^2dx=\left[\frac{x^3}{3}\right]_0^l=\frac{l^3}{3}-\frac{0^3}{3}=\frac{1}{3}l^3.$$

EXERCISES

1. $\int_0^a x^3dx$ 2. $\int_0^{\frac{\pi}{2}} \sin\theta\, d\theta.$

3. $\int_{-\frac{l}{2}}^{+\frac{l}{2}} x^2dx$ 4. $\int_a^b \frac{dr}{r}.$

[Ans. 1. $\frac{1}{4}a^4$. 2. 1. 3. $\frac{1}{12}l^3$. 4. $\log b/a$.]

1.11. Integral as Summation

Suppose a whole quantity is y and it is divided into large number of small pieces. Let each piece be as small as one can imagine. According to the definition of 'differential' such a small part of it is differential of the quantity denoted by dy. Now if we reverse the process i.e., if we add up all these small pieces together, we shall get back the original quantity y. By the definition of integral ,we have

$\int dy = y$. Thus 'integration' may be considered as '*summation*' of all the small bits of a quantity. *This fact makes 'integration' the most important mathematical tool in Physics.* Let us explain this by an example. The potential at a point due to a point charge is the total work done in bringing unit positive charge from infinity up to that point by an external agent.

Suppose $+Q$ is a point charge situated at O. P is a point at a distance r from O. The potential (V) at P is to be found. Since the potential is the work done in bringing a unit positive charge from infinity up to the point by an enternal agent, we have to find the total work (W) done. An infinitesimally small quantity of it is dW.

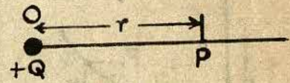

Fig. 1.2

The electric force on unit charge at a distance x from $Q = \frac{1}{4\pi\epsilon_0} \cdot \frac{Q}{x^2}$

where ϵ_0 is a constant. This is given by Coulomb's law of force in electrostatics. Move the unit charge through an infinitesimally small distance dx towards Q.

The small bit of work done (dW) by the agent $= \left(\frac{1}{4\pi\epsilon_0} \cdot \frac{Q}{x^2}\right)(-dx)$

(minus sign is to indicate that the work is done in the decreasing direction of x)

or $$dW = -\frac{Q}{4\pi\epsilon_0} \cdot \frac{dx}{x^2}.$$

To find the total work we have to sum up such small bits of work for which we can adopt integration. Summation must be done from $x=\infty$ to $x=r$.

$$W = \int_{x=\infty}^{x=r} dW = -\frac{Q}{4\pi\epsilon_0} \int_{\infty}^{r} \frac{dx}{x^2}$$

$$= -\frac{Q}{4\pi\epsilon_0} \int_{\infty}^{r} x^{-2} dx = -\frac{Q}{4\pi\epsilon_0} \left[-\frac{1}{x} \right]_{\infty}^{r}$$

$$= -\frac{Q}{4\pi\epsilon_0} \left[\left(-\frac{1}{r}\right) - \left(-\frac{1}{\infty}\right) \right] = \frac{Q}{4\pi\epsilon_0 r}.$$

According to the definition, this is the potential V at P.

$$\therefore \qquad V=\frac{Q}{4\pi\epsilon_0 r}.$$

1.12. Integration as 'Area' (Geometrical Interpretation of Integration)

Suppose $y=f(x)$ be the equation of a continuous curve. Consider two points P and Q on the curve very close to each other. Let the co-ordinates of P be (x, y) and then those of Q are $(x+\triangle x, y+\triangle y)$. The finite small area $\triangle A$ $(=MNQP)$ between x-axis and the curve is obviously greater than the area of the rectangle $MNEP=y\triangle x$ and less than the area of the rectangle $MNQF$ $=\triangle x(y+\triangle y)$.

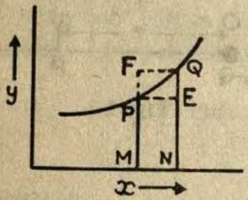

Fig. 1.3

Hence, $\triangle x\ (y+\triangle y)>\triangle A>y.\triangle x$

or $\qquad y+\triangle y>\dfrac{\triangle A}{\triangle x}>y.$ Let Q tend to P so that $\triangle x\to 0$.

Then according to the definition of differential co-efficient,

$$y=\lim_{\Delta x\to 0}\frac{\triangle A}{\triangle x}=\frac{dA}{dx},\ \text{or } dA=y\,dx$$

or $$\int dA=\int y\,dx,\ \text{or}\ A=\int y\,dx.$$

Note. The integral is definite, that is, it should be taken between the end values of the variable.

Similarly the area between the curve and the y-axis is

$$\int x\,dy.$$

Example

1. *Find the area of the ellipse* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ *and hence find the area of a circle of radius* a.

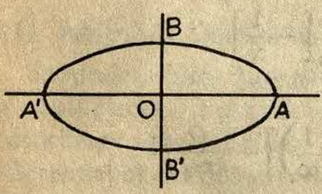

Fig. 1.4

Sol. Suppose that the ellipse represented by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is $ABA'B'A$. Obviously the area of the ellipse is 4 times the area of OAB. By the principle explained above,

$$\text{area } OAB = \int_0^a y\, dx = \int_0^a \frac{b}{a}\sqrt{a^2-x^2}\, dx$$

$$= \frac{b}{a}\int_0^a \sqrt{a^2-x^2}\, dx.$$

Put $x = a \sin\theta$. Then $dx = a\cos\theta\, d\theta$

and when $x=0$, $\theta=0$ and when $x=a$, $\theta=\frac{\pi}{2}$.

$$\therefore \quad \text{Area } OAB = \frac{b}{a}\int_0^{\pi/2} \sqrt{a^2-a^2\sin^2\theta}.\ a\cos\theta\, d\theta$$

$$= ab\int_0^{\pi/2} \cos^2\theta\, d\theta$$

$$= \tfrac{1}{2}ab\int_0^{\pi/2} (1+\cos 2\theta)\, d\theta$$

$$= \tfrac{1}{2}ab\int_0^{\pi/2} d\theta + \tfrac{1}{2}ab\int_0^{\pi/2} \cos 2\theta\, d\theta$$

$$= \tfrac{1}{2}ab\Big[\theta\Big]_0^{\pi/2} + \tfrac{1}{2}ab\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \tfrac{1}{2}ab.\ \pi/2 + \tfrac{1}{2}ab.\ 0 = \tfrac{1}{4}\pi\ ab.$$

$\therefore$ Area of the ellipse $= 4\times$ area of $OAB = 4.\frac{1}{4}\pi ab = \pi ab$. **Ans.**

For a circle, $a=b$. Hence area of a circle $= \pi a^2$.

EXERCISES

1. *Find the area between x-axis and the parabola* $y^2=4ax$ *from* $x=0$ *to* $x=a$.

2. *Find the area between y-axis and the hyperbola* $xy=a$ *from*

$$y=\frac{a}{2} \text{ to } y=a.$$

[**Ans. 1.** $\frac{4}{3}a^2$; **2.** $a \log 2$.]

CHAPTER 2

UNITS AND DIMENSIONS

2.1. Units

Measurement is the most powerful tool in the hands of Scientists to establish the truth about a fact or phenomenon. Our knowledge about any thing is complete only when we can measure and express it. Our faith in any law is established only when we put it to experimental test by measuring various physical quantities involved in the law. Thus measurement is very important in Science. To measure a physical quantity and express its measure we need a standard of that physical quantity. This 'standard' is called the unit of that physical quantity.

The measure of any physical quantity $= nu$, where $n =$ the numerical value of the measure of the quantity, $u =$ the unit of the quantity. If we change the unit of the physical quantity, the numerical value of the measure will change but not the actual measure of the quantity. Suppose the length of a black board is 2 metre. Here 2 is the numerical value and 'metre' is the unit of length. Let us now change the unit to 'cm'. Then measure of the length of the blackboard is 200 cm. Here the numerical value is 200 and cm is the unit of length. Though here the numerical value has increased, the measure of the length of the blackboard has definitely not changed at all. The guiding principles in the selection of a standard (i.e., unit) for a physical quantity :

(i) be '*well defined*' and of suitable size;
(ii) be easily '*reproducible*' at all places;
(iii) be not subject to any secular change (i.e., changes with time);
(iv) be not liable to variations with physical conditions like temperature, pressure, etc. If it varies at all, the exact manner of variation must be accurately known.

2.2. System of Units

The common systems of units are (i) Foot Pound Second (F. P. S.) system, (ii) Centimetre-Gramme-Second (C. G. S.) system,

(iii) Metre-Kilogramme-Second (M. K. S. or Giorgi) system and (iv) the recently developed *Systeme Internationale d' unités* (SI system).

2.3. Base Units

In Science there are certain physical quantities which are very fundamental in nature. There is no way out but to define 'Standard or Units' for such quantities. The units of all other quantities can be derived from the units of these quantities. The units of these quantities are called the 'base units' of the system. In mechanics three base units are required. These are the units of length, mass and time. In heat and thermodynamics, besides these three we require two more quantities, namely, the temperature and the amount of substance. In electricity and magnetism we require besides the three base units in mechanics a standard for current. In light a standard for luminous intensity is required. Thus in any self-consistent and well-developed system of measurements 'seven' base units are required. Recently there is an all-round attempt to develop such a system of measurements. As a result of such attempt a new system has emerged which is known as the *Systéme Internationale d' unités* abbreviated as SI system of units.

The seven base units in SI are :

(i) *the metre* (m), the standard of length
(ii) *the kilogramme* (kg), the standard of mass
(iii) *the second* (s), the standard of time
(iv) *the ampere* (A), the standard of electric current
(v) *the kelvin* (K), the standard of temperature
(vi) *the candela* (cd), the standard of luminous intensity and
(vii) *the mole* (mol), the standard of amount of substance.

DEFINITIONS OF BASE UNITS

The metre (m). This is defined as 1650763·73 times the wavelength of the orange light emitted by $_{36}Kr^{86}$ in transition from $2p_{10}$ to $5d_5$.

The kilogramme (kg). This is simply the mass of a platinum-iridium cylinder kept at Sevres near Paris.

The second (s). This is the time taken by 9192631770 cycles of the radiation from the hyperfine transition in cesium when unperturbed by external fields.

The ampere (A). This is the constant current which, if maintained in each of two infinitely long straight parallel wires of negligible cross-section placed 1 metre apart, in vacuum, will produce between the wires a force of 2×10^{-7} newton per metre length of the wires.

The kelvin (K). In SI, temperatures are measured on the thermodynamic scale with the absolute zero as zero and the triple point of water (i.e., the temperature at which ice, water and water vapour are in equilibrium) as the upper fixed point. The interval is divided into 273·16 divisions and each division is taken as unit temperature. This unit is called the kelvin.

The candela (cd). This is defined as the luminous intensity in the perpendicular direction of a surface of 1/600000 square metre of a full radiator at the temperature of freezing platinum under a pressure of 101325 newton per square metre.

The mole (mol). The mole is the amount of substance which contains as many elementary entities as there are atoms in ·012 kg of the carbon isotope ${}_6C^{12}$.

The cgs system takes the centimetre as the unit of length which is one-hundredth of the distance between two lines at 0°C on a platinum-iridium bar, preserved at the International Bureau of Metric Weights and Measures at Sevres, near Paris; the gramme as the unit of mass which is one-thousandth part of a cylinder of platinum-iridium preserved at Sevres and the second as the unit of time which is $\frac{1}{86,400}$th part of the mean solar day, which is the average value for one year of the solar day (i.e., the time which elapses between two consecutive transits of the sun across the meridian).

The mks system is the system of the engineers, very much akin to the cgs system, in which the units of length, mass and time are the *metre*, the *kilogramme* and the *second* respectively.

The fps system is now an abandoned system and was prevalent is Great Britain and its colonies. In this system the unit of length is the *foot*, the unit of mass is the *pound* and the unit of time is the *second.*

Among the systems, SI is the most systematic and self-consistent one. It not only removed the discrepancies of the old cgs system but also ironed out all irregularities existing in the various expressions and laws in all the branches of Physics, esspecially electricity and

magnetism. When units were first required for the electrical and magnetic quantities it was natural to define them in terms of the three fundamental units, centimetre, gramme and second, which were already commonly used in mechanics and any other quantity in electricity. Electrical phenomena are related to mechanical phenomena by two effects : (a) the force between static electric charges (Coulomb's law) in vacuum

$$\Delta F=\frac{1}{\epsilon_0}\,\frac{\Delta q_1\Delta q_2}{r^2} \qquad \text{.. (2.1)}$$

where ϵ_0 is a constant and (b) the force between the electric current elements (Ampere's law)

$$\Delta F=\mu_0\,\frac{I_1\Delta l_1 I_2\Delta l_2 \sin\theta}{r^2} \qquad \text{.. (2.2)}$$

where μ_0 is another constant; $I_1\Delta l_1$ and $I_2\Delta l_2$ are two parallel current elements separated by a distance r and θ is the inclination of either element with r.

Correspondingly two distinct systems of cgs electrical units were introduced : the cgs electrostatic units (cgse units) and the cgs electromagnetic units (cgsm units). It is very unfortunate that for the same electrical quantity there should be two units in the same system. In SI Ampere's law (after rationalization explained below) is used to define the unit of current and this is taken as a 'base unit' for electrical and magnetic quantities. In SI the units for electrical quantities are unique.

2.4. Rationalization of Laws

It is observed that π invariably occurs in the expression for area, volume, etc. of bodies having cylindrical or spherical symmetry. Hence occurrence of π in the formula for the magnetic and electric fields for bodies having cylindrical or spherical symmetry is to be treated as something 'rational'. If the above expressions (2.1) and (2.2) are used to calculate the electric and magnetic fields respectively for cylindrical and spherical bodies, the results obtained are not found to be 'rational'. To rationalise them, 4π (a constant) is introduced in the denominator in the very beginning so that later they may yield rational results. This is why $1/4\pi$ is called the rationalising factor. Thus after rationalisation Coulomb's law is :

$$\Delta F=\frac{1}{4\pi\epsilon_0}\cdot\,\frac{\Delta q_1\Delta q_2}{r^2} \qquad \text{.. (2.3)}$$

and Ampere's law is :

$$\triangle F=\frac{\mu_0}{4\pi}.\ \frac{I_1\triangle l_1 I_2\triangle l_2 \sin\theta}{r^2}. \qquad .. \quad (2.4.)$$

The cgse units are defined by setting ϵ_0 to 1 in Eq. 2.1 or $\frac{1}{4\pi\epsilon_0}=1$ in 2.3 and cgsm units by setting μ_0 to 1 in Eq. 2.2 or $\frac{\mu_0}{4\pi}=1$ in Eq. 2.4. It was shown by Maxwell in his famous theory of electro–magnetic waves that in a self-consistent system ϵ_0 and μ_0 of Eq. 2.3 and Eq. 2.4 are related to the velocity of electromagnetic waves in vacuum by the relation :

$$c=\frac{1}{\sqrt{\mu_0\epsilon_0}}. \qquad .. \quad (2.5)$$

Thus in a single consistent system $\frac{\mu_0}{4\pi}$ and $\frac{1}{4\pi\epsilon_0}$ cannot be independently set to 1 as it was done in cgs system. In SI μ_0 is set to $4\pi\times10^{-7}$ arbitrarily in Eq. 2.4 and the unit of current is defined and ϵ_0 is calculated from Eq. 2.5. The appropriate value of ϵ_0 is :

$$\frac{1}{4\pi\epsilon_0}=9\times10^9$$

when $c=3\times10^8$ ms^{-1} and $\epsilon_0=8{\cdot}85\times10^{-12}$ farad per metre (Fm^{-1}).

2.5. Standard of Time in SI

Whereas the standards of length and mass of the cgs system have been retained in SI, the standard of time has specially been defined carefully. The second defined in terms of the rotation of the earth is called universal time (UT). This standard of time lacks 'invariability'. Tidal friction between the water and the land, for example, causes a slowing down of the earth's rotation. Also the seasonal motion of the winds introduces a seasonal variation in the rotation. Other variations may be associated with the melting of ice-caps, mass shifted in avalanches. The earth's rotation rate is higher in summer and low in winter and exhibits a steady decrease from year to year. It is because of this variability of the earth's rotation that in 1956 the International Congress of Weights and Measures redefined the second in terms of the earth's orbital motion about the

sun. The second is defined as the fraction $\frac{1}{3{,}15{,}56{,}925{\cdot}9747}$ of the tropical year 1900; the selection of a particular earth orbit in the definition automatically makes the time standard *invariable*. Time defined in terms of the earth's orbital motion is called *ephemeris time* (ET). Nowadays 'atomic clocks' have been developed which keep time with a high degree of accuracy. The 'cesium atomic clock' at Boulder Laboratories of the National Bureau of Standards promises to keep time within 1 second in 3000 years. This is why in SI the standard has been defined carefully in terms of frequency of cesium atoms. In atomic clocks the atoms act like a pendulum in a pendulum clock. Each clock possesses a characteristic frequency that is used to control a time-keeping device. The cesium clock has a frequency of 9,19,26,31,177 vibrations per second of the ephemeris time.

We shall use the SI system throughout the text.

2.6. Derived Units

Units of all quantities may be obtained by combining the base units. These units are called the derived units. Often derived units are given names. For example, the unit of force in SI is $kgms^{-2}$. It has been named the newton (N). The unit of power is kgm^2s^{-3}. It is named watt (W).

2.7. Dimensions

The existence of different systems gives rise to the problem of converting units of one system into those of another. Obviously, a change in the basic units should lead to a change in the derived ones. It is, therefore, desirable to find a relationship that would make it possible to determine how the derived unit of a quantity of interest changes with a change in each of the base units. If a derived unit changes proportional to the *p*th power of a change in the unit of a base, then it is said to have a dimension *p* relative to the unit of that base. If the unit of a certain quantity *A* has the dimensions *p*, *q*, *r* and *s* relative to the unit of length, mass, time and current, then this is symbolically written as

$$[A]=L^pM^qT^rI^s \quad .. \quad (2.6)$$

where the brackets enclosing *A* stand for 'dimension of' and the symbols *L*, *M*, *T* and *I* are generalised designations of the units of

length, mass, time and current without indicating the concrete magnitude of the unit. Formula 2.6 is called the formula of the dimension of the unit of A, or, as is frequently said for brevity, the dimensions of A. Thus, *the dimensions of a derived unit may be defined as the powers to which the base units must be raised to represent it and the dimensional formula as an expression, showing which of the fundamental units enter into the unit of the physical quantity and with what power.*

2.8. Calculation of Dimensional Formulae

To find the dimensional formula of a physical quantity, first write down the formula for the given quantity or the expression in which it occurs. This needs a comprehensive knowledge of the laws and formulae of the different branches of Physics. After writing the formula take the dimensions of the fundamental quantities. For examples,

(*i*) *Dimension of Velocity.* $\text{Velocity} = \dfrac{\text{Distance}}{\text{time}}$

$$\therefore \qquad [\text{Velocity}] = \frac{[\text{Distance}]}{[\text{time}]} = \frac{L}{T} = LT^{-1}.$$

Thus dimensional formula of velocity is LT^{-1}.

(*ii*) *Dimension of Acceleration.* $\text{Acceleration} = \dfrac{\text{Velocity}}{\text{time}}$

$$\therefore \qquad [\text{Acceleration}] = \frac{[\text{Velocity}]}{[\text{time}]} = \frac{LT^{-1}}{T} = LT^{-2}.$$

Thus dimensional formula of acceleration is LT^{-2}.

(*iii*) *Dimension of Force.* Force = mass × acceleration

$$\therefore \qquad [\text{Force}] = [\text{mass}] \times [\text{acceleration}] = MLT^{-2}.$$

(*iv*) *Dimension of gravitational constant.* This constant occurs in Newton's law of gravitation :

$$F = G\frac{m_1 m_2}{d^2}; \qquad \text{or } G = \frac{F.d^2}{m_1 m_2}.$$

$$\therefore \qquad [G] = \frac{[F]\,[d^2]}{[m_1]\,[m_2]} = \frac{MLT^{-2}L^2}{MM} = M^{-1}L^3T^{-2}.$$

(*v*) *Dimension of permittivity* ϵ_0. This constant occurs in

Coulomb's law of force :

$$F=\frac{1}{4\pi\epsilon_0}\cdot\frac{q_1q_2}{d^2}$$

$$\epsilon_0=\frac{q_1q_2}{4\pi Fd^2}$$

$$\therefore \qquad [\epsilon_0]=\frac{[q_1][q_2]}{[F][d^2]}.$$

Charge = current × time

$\therefore$ [Charge] = [current] [time] = IT

$$\therefore \qquad [\epsilon_0]=\frac{IT.IT}{MLT^{-2}L^2}=M^{-1}L^{-3}T^4I^2.$$

(*vi*) *Dimension of thermal conductivity* (λ). This occurs in the standard formula :

$$Q=\frac{\lambda A(\theta_1-\theta_2)t}{d}$$

where Q is the heat conducted through a conductor of length d and cross-sectional area A in t seconds when the temperature difference between its ends is $(\theta_1-\theta_2)$

or $$\lambda=\frac{Q.d}{A(\theta_1-\theta_2)t}$$

$$\therefore \qquad [\lambda]=\frac{[Q][d]}{[A][\theta_1-\theta_2][t]}=\frac{ML^2T^{-2}L}{L^2KT}=MLT^{-3}K^{-1}.$$

(*vii*) *Dimension of magnetic induction field* B. B is defined by the formula :

$\triangle F=I\triangle lB\sin\theta$ where θ is the angle between $\triangle l$ and B.

or $$B=\frac{\triangle F}{I\triangle l\sin\theta}$$

$$\therefore \qquad [B]=\frac{[F]}{[I][\triangle l]}=\frac{MLT^{-2}}{IL}=MT^{-2}I^{-1}$$

($\sin\theta$ is non-dimensional).

It is to be pointed here that there is a certain amount of arbitrariness in the choice of the fundamental quantities. The choice of mass, length and time as the fundamental quantities is not unique. We might equally well choose force, length and time as fundamental

length, mass, time and current without indicating the concrete magnitude of the unit. Formula 2.6 is called the formula of the dimension of the unit of A, or, as is frequently said for brevity, the dimensions of A. Thus, *the dimensions of a derived unit may be defined as the powers to which the base units must be raised to represent it and the dimensional formula as an expression, showing which of the fundamental units enter into the unit of the physical quantity and with what power.*

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$$\epsilon_0=\frac{q_1q_2}{4\pi Fd^2}$$

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where Q is the heat conducted through a conductor of length d and cross-sectional area A in t seconds when the temperature difference between its ends is $(\theta_1-\theta_2)$

or
$$\lambda=\frac{Q.d}{A(\theta_1-\theta_2)t}$$

$$\therefore\qquad [\lambda]=\frac{[Q][d]}{[A][\theta_1-\theta_2][t]}=\frac{ML^2T^{-2}L}{L^2KT}=MLT^{-3}K^{-1}.$$

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($\sin\theta$ is non-dimensional).

It is to be pointed here that there is a certain amount of arbitrariness in the choice of the fundamental quantities. The choice of mass, length and time as the fundamental quantities is not unique. We might equally well choose force, length and time as fundamental

quantities and in fact we do so very often. Let us calculate the dimension formula of a few physical quantities in terms of force (F), length (L), and time (T), as fundamental units.

(*viii*) *Dimension of mass.* We have

$$\text{Force} = \text{Mass} \times \text{Acceleration}$$

$$\therefore \text{ Mass} = \frac{\text{Force}}{\text{Acceleration}}$$

or $$[\text{Mass}] = \frac{[\text{Force}]}{[\text{Acceleration}]} = \frac{F}{LT^{-2}} = FL^{-1}T^{2}.$$

(*ix*) *Dimension of density.*

$$\text{Density} = \frac{\text{Mass}}{\text{Volume}}$$

$$\therefore \quad [\text{Density}] = \frac{[\text{Mass}]}{[\text{Volume}]} = \frac{FL^{-1}T^{2}}{L^{3}} = FL^{-4}T^{2}.$$

Dimension of Young's modulus.

$$\text{Young's modulus} = \frac{\text{Stress}}{\text{Strain}}$$

$$[\text{Young's modulus}] = [\text{Stress}] \quad (\because \text{ strain is dimensionless})$$

$$= \frac{[\text{Force}]}{[\text{Area}]}$$

$$= \frac{F}{L^{2}} = FL^{-2}.$$

Hence unit of Young's modulus in SI is newton per square metre (Nm^{-2}).

For dimension of other physical quantities see the author's *Numerieal Examples in Physics*, pp. 1 to 8.

2.9. Uses of Dimension Formula

The three chief uses of dimension formulae are the following :

(a) to change the value of a physical quantity from one system into another.

(b) to test the correctness of the results already arrived at.

(c) to determine the exact relation among physical quantities which are likely to be inter-related.

Let us consider these in some detail.

(a) We know that the measure of a physical quantity $= nu$ where $n=$ the numerical value of the measure of the quantity and $u=$ the unit of the quantity.

If we change the unit of the physical quantity, the numerical value of the measure will change but not the actual value of the physical quantity. This fact affords us an easy method of changing over from one system of units to another. If n_1 and u_1 be the numerical value and unit of a physical quantity in one system and n_2 and u_2 be that in another system respectively, then by the principle of conversion, we have

$$n_1u_1 = n_2u_2.$$

Let $M^aL^bT^c$ be the dimension formula of the quantity, then

$$u_1 = M^a{}_1L^b{}_1T^c{}_1 \text{ and } u_2 = M^a{}_2L^b{}_2T^c{}_2$$

$$\therefore \quad n_1M^a{}_1L^b{}_1T^c{}_1 = n_2\, M^a{}_2L^b{}_2T^c{}_2. \quad \text{.. (2.7.)}$$

Examples

1. *The value of Young's modulus in cgs is* 2×10^{11} *dynes per sq. cm. What is the value of the same in SI ?*

Sol. Dimension forumula of Young's modulus is $ML^{-1}T^{-2}$.

$\therefore$ Unit of Young's modulus in SI $=$ kgm^{-1}s^{-2}
and unit of Young's modulus in cgs $=$ gm cm^{-1} s^{-2}.

By the principle of conversion

$$2\times10^{11} \text{ gm cm}^{-1}\text{ s}^{-2} = n_2 \text{ kgm}^{-1}\text{ s}^{-2}.$$

$$\therefore \quad n_2 = 2\times10^{11}\left(\frac{\text{gm}}{\text{kg}}\right)\left(\frac{\text{cm}^{-1}}{\text{m}^{-1}}\right)\left(\frac{\text{s}^{-2}}{\text{s}^{-2}}\right)$$

$$\text{or} \quad n_2 = 2\times10^{11}\left(\frac{1\text{ gm}}{1000\text{ gm}}\right)\left(\frac{\text{cm}^{-1}}{100^{-1}\text{cm}^{-1}}\right)$$

$$= 2\times10^{11}\times10^{-3}\times10^{2} = 2\times10^{10}.$$

$\therefore$ Young's modulus in SI $= 2\times10^{10}$Nm^{-2}. **Ans.**

2. *The value of* μ_0 *is* $4\pi\times10^{-7}$ *in SI. What is the value in cgs ? (Given that unit of current in SI (ampere)* $=\frac{1}{10}\times$ *unit of current in cgsm)*

Sol. Dimension formula of μ_0 is $MLT^{-2}I^{-2}$ (see the author's Numerical Examples in Physics, p 5).

$$\therefore \quad 4\pi\times10^{-7} \text{ kgm s}^{-2}\text{ A}^{-2} = n \text{ gm cm s}^{-2}\text{ A}'^{-2}$$

where A stands for unit of current in SI and A′ stands for cgsm

unit of current.

$$\therefore \quad n = 4\pi \times 10^{-7}\left(\frac{\text{kg}}{\text{gm}}\right)\left(\frac{\text{m}}{\text{cm}}\right)\left(\frac{\text{s}^{-2}}{\text{s}^{-2}}\right)\left(\frac{\text{A}}{\text{A}'}\right)^{-2}$$

$$= 4\pi \times 10^{-7}\left(\frac{1000\ \text{gm}}{\text{gm}}\right)\left(\frac{100\ \text{cm}}{\text{cm}}\right)\left(\frac{\text{A}}{10\ \text{A}}\right)^{-2}$$

$$= 4\pi \times 10^{-7} \times 1000 \times 100 \times 100 = 4\pi.$$

Hence the corresponding value of μ_0 is 4π in cgsm. **Ans.**

Note. When μ_0 is 4π, i.e., $\frac{\mu_0}{4\pi}$ is 1 in cgsm, $\frac{\mu_0}{4\pi}$ in SI is 10^{-7}

This is the reason for setting μ_0 to $4\pi \times 10^{-7}$ in SI.

(b) *Checking the results arrived at.* This depends upon what is called the principle of *homogeneity of dimensions.* The two sides of an equation must represent the same physical quantity because it is absurd that two physical quantities of different nature be equal. Hence the dimension of the two sides of an equation already established must be dimensionally the same from both sides. This fact provides us a method to check up the results already arrived at.

Examples

To check, by the method of dimensions, whether the following relations are true—

(i) $t = 2\pi\sqrt{\frac{l}{g}}$; (ii) $\tau = \frac{n\pi r^4}{2l}$

where τ *is the twisting couple twisting a wire of length* l *and radius* r *through* θ *radian,* n *is the modulus of rigidity of the material of the wire.*

Sol. (i) $[\text{R. H. S.}] = \sqrt{\frac{[l]}{[g]}} = \sqrt{\frac{L}{LT^{-2}}} = T$

$[\text{L. H. S.}] = [t] = T.$ Hence the relation is correct.

(ii) $[\text{R. H. S.}] = \frac{[n]\,[r^4]}{[l]}$ (π and 2 are dimensionless)

$$= \frac{ML^{-1}T^{-2}L^4}{L} = ML^2T^{-2}$$

$[\text{L. H. S.}] = [\text{force}] \times [\text{distance}]$

$= MLT^{-2} \times L$

$= ML^2T^{-2}.$

Hence the relation is correct.

(c) *Derivation of correct relationship among different physical quantities.* The principle of homogeneity of dimensions of the two sides of an equation can be extended to deduce a relationship among different physical quantities which are known, by *intuition* or clear *insight* into the problem, to be inter-related.

Examples

1. *To deduce an expression for the time-period of a simple pendulum.*

Sol. Suppose, by intuition we think that the time period of a simple pendulum may depend on the length of the pendulum (l), the acceleration due to gravity g and the mass of the bob (m).

We can write,

$t = kl^a g^b m^c$ where k is a dimensionless constant.

Taking the dimensions of the terms on either side of the equation, we have

$$[t] = [l^a]\,[g^b]\,[m^c]$$

or
$$T = L^a[LT^{-2}]^b M^c = M^c L^{a+b} T^{-2b}.$$

Equating the indices of M, L and T, we have

$$c = 0,\ a + b = 0,\ \text{and}\ -2b = 1$$

whence, $c = 0,\ a = -b = \frac{1}{2}$.

Therefore, $t = kl^{\frac{1}{2}} g^{-\frac{1}{2}}$; or $t = k\sqrt{\dfrac{l}{g}}$.

2. *To deduce an expression for the viscous force on a spherical body moving with velocity v through a viscous medium.*

Sol. Suppose, by intuition, we think that the viscous force on a spherical body may depend on the coefficient of viscosity of the medium, radius of the body and velocity of the body.

We may write, $F = k\eta^a r^b v^c$.

Taking dimension of the terms on either side of the equation, we have

$$MLT^{-2} = [\eta^a]\,[r^b]\,[v^c]$$
$$= [ML^{-1}T^{-1}]^a L^b [LT^{-1}]^c$$

or
$$MLT^{-2} = M^a L^{-a+b+c} T^{-a-c}.$$

Equating the indices of M, L and T we have,

$$a = 1,\ -a + b + c = 1\ \text{and}\ -a - c = -2$$

whence, $a = 1,\ c = 1,\ b = 1.$

Therefore, $F = k\eta\, rv$ where k is a dimensionless quantity.

2.10. Limitations of Dimensional Analysis

The method of dimensional analysis, though very helpful at times, has also certain limitations :

(i) One obvious drawback is that its use is restricted solely to physical relationships which are power functions. Relationships involving exponential and trigonometric functions lie quite outside its purview.

(ii) Another obvious drawback is that it gives no information about the non-dimensional constant of proportionality.

(iii) The method succeeds only when the constant of proportionality is dimensionless. For example, if it is given that force between two masses depends on the masses and the distance between them, we cannot find the relation because the constant of proportionality, namely, the gravitational constant (G) is a dimensional constant.

(iv) Since at best only three equations can be obtained by equating the powers of M, L and T in mechanics and four equations by equating the powers of M, L, T and I in electricity and magnetism, the method is of no avail in deducing the exact form of a physical relation which may depend upon more than three quantities in mechanics and four in electricity and magnetism.

(v) The method needs a trained, subtle and intuitive mind, with a solid background of the subject to decide the factors on which the given physical quantity may possibly depend on other physical quantities with which its relation is sought.

2.11. Supplementary Units

Two supplementary units are defined : the radian and the steradian, which are the units for plane and solid angles respectively :

(*i*) *The radian* (rad). The radian is the plane angle between two radii of a circle which cut off on the circumference an arc equal in length to the radius.

(*ii*) *The steradian* (sr). The steradian is the solid angle which, having its vertex in the centre of a sphere cuts off an area of the surface of the sphere equal to that of a square with sides of length equal to the radius of the sphere.

2.12. S I Prefixes and Multiplication Factors

To obtain multiples and submultiples of units, standard prefixes are used as shown below :

Multiplication factors	*Prefix*	*Symbol*
10^{12}	tera	T
10^{9}	giga	G
10^{6}	mega	M
10^{3}	kilo	k
10^{2}	hecto	h
10^{1}	deca	da
10^{-1}	deci	d
10^{-2}	centi	c
10^{-3}	milli	m
10^{-6}	micro	μ
10^{-9}	nano	n
10^{-12}	pico	p
10^{-15}	femto	f
10^{-18}	atto	a

2.13. Derived Units of Length, Mass and Time

Through common usage, certain multiples and submultiples of the three fundamental units have been given names. A list of the more common ones is given below as they have been in frequent use. None of them is a recognised S I unit.

Length	*Area of cross-section*	*Mass*	*Time*
Micron (μ m) = 10^{-6} m	Are (a) = 100 m^2	Tonne (t) = 1000 kg	Minute (min) = 60 s.
Angstrom ($\mathring{A}$) = 10^{-10} m	Barn (b) = $10^{-28} m^2$		Hour (h) = 60 min
Fermi (fm) = 10^{-15} m			Day (d) = 86400s
			Year (a) = $3{\cdot}1557 \times 10^7$s.

2.14. Measurement of Length and Time (Atomic to Astronomical Range)**

Length

We have a vast range of lengths—from the diameters of subatomic particles to the interstellar distances. Very small distances

such as the size of the nucleus of an atom are termed as 'atomic distances'. The units for expressing such distances are angstrom unit (Å) $= 10^{-10}$ m, X-ray unit (X.U.) $= 10^{-13}$ m. Very large distances such as the distance of the sun from the earth, distances of the stars from the earth are termed as 'astronomical distances'. The units for expressing such distances are 'astronomical unit' (A. U.), parsec, light year. 1 A.U. = average distance of the earth from the sun $= 1{\cdot}496 \times 10^{11}$ m. Parsec = distance at which an astronomical unit would subtend an angle of 1 second of an arc $= 3{\cdot}084 \times 10^{16}$ m. Light year = distance described by light in 1 year $= 9{\cdot}4605 \times 10^{15}$ m.

One single instrument cannot measure the whole range of lengths. The lengths of objects which are accessible and are within the range of common measuring instrument—a measuring tape (Fig. 2.1a), a metre scale (Fig. 2.1b), a slide callipers (Fig. 2.1c), a screw gauge (Fig. 2.1d), and a spherometer (Fig. 2.1e)—are measured directly. Every instrument can measure a certain minimum distance without eye-estimation. This is called its least count. The least count of a measuring tape is 1 cm, that of a metre scale 1 mm, that of a slide callipers $\frac{1}{10}$ mm and that of screw gauge or spherometer is $\frac{1}{100}$ mm or $\frac{1}{200}$ mm according as the pitch of the screw used in this instrument is 1 mm or $\frac{1}{2}$ mm. The pitch of a screw is the distance through which its tip advances on giving one full rotation to it.

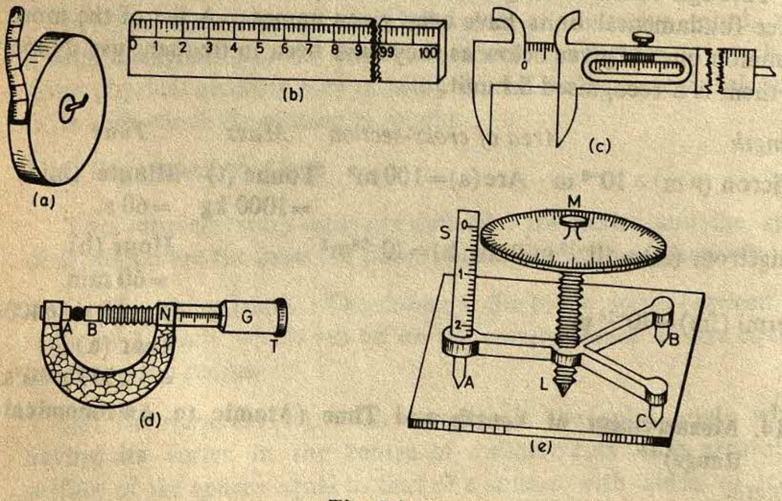


Fig. 2.1

Atomic and astronomical distances are measured by 'indirect methods', that is, such methods are adopted in which all other

quantities connected by a physical relation can be measured by the conventional instruments except the given atomic or astronomical distances. Here are a few illustrations.

(i) Suppose we are required to measure an astronomical distance, say, the distance of the moon from the centre of the earth. This can be done by the method used by a surveyor of land to measure the distance between two widely separated distance, namely, *the method of triangulation*, using the distance AB between two widely separated places A and B on the surface of the earth as the base line. It is assumed that the distance AB is known. A common distant star (say, N) is selected by both the observers at A and B. Since the star is at a very large distance, the rays reaching A and B may be taken practically to be parallel. Both the observers at A and B measure the angle between the direction of the distant star N and the moon M, that is, A measures the angle between AN and AM and B measures the angle between BN and BM. The sum of these two angles is the angle subtended by AB at M. Let the sum be φ. Draw perpendicular BA' from B on AM.

Since the distance of the moon is very large, $BA = BA' = d$ (known). We have from geometry of the figure, $d = r \tan\varphi$

or $r = \dfrac{d}{\varphi}$ ($\because$ φ is small, $\tan\varphi = \varphi$)

where r is the distance of the moon from B. This is also the distance of the moon from its centre of the earth because radius of the earth is negligible in comparison to the distance BM or AM.

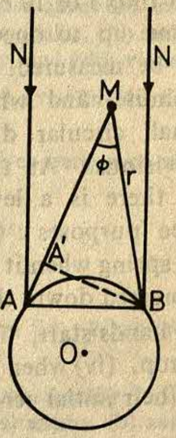


Fig. 2.2

(ii) Next let us take the measurement of an atomic distance, say, the diameter of the gold nucleus. This is done by alpha-particle scattering experiment due to Rutherford. The distance of the 'closest approach' of the alpha-particles to the gold nucleus is measured. The limiting value of the 'closest approach' for the fastest α-particles is the radius of the gold nucleus.

Time

Any phenomenon that repeats itself regularly can be used to measure time. A clock or watch works on this principle—the periodic vibration of a pendulum or coiled spring. All clocks or watches are calibrated to read the ephemeris second. Very long intervals of time and very short intervals of time are measured by indirect methods. For example, the age of the earth which is of the order of 10^{17} seconds is measured by the principle of carbon dating'. The small interval of time such as the time of travel of light through ordinary distances is measured by 'rotating wheel or mirror' method.

A Stop-Watch

In laboratories a stop-watch is used to measure how long an event lasts. It has a long second-hand which moves over a circular dial with 60 equal divisions, each division representing ephemeris second. Each division is generally sub-divided into 5 or 10 equal parts so that time up to one-tenth of a second can be measured. There is a small minute-hand which moves over a small circular dial divided into 60 divisions. At the top of the watch there is a lever which serves three purposes : (i) it winds the main spring when it is rotated, (ii) when pressed down the second and minute hands start, (iii) on pressing for the second time both the hands stop, (iv) when pressed for the fourth time both the hands fly back to their initial zero position.

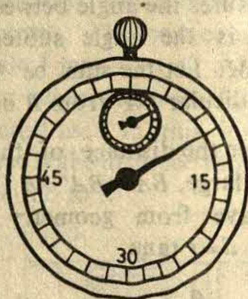


Fig. 2.3

QUESTIONS

(A)

1. $ML^{-1}T^{-1}$ is the dimension formula of (a) surface tension, (b) momentum, (c) coefficient of viscosity, (d) Young's modulus.

2. The dimension formula of torque is (a) ML^2T^{-1}, (b) $M^2L^2T^{-1}$, (c) ML^2T^{-2}, (d) MLT^{-3}.

3. If force, length, and time are taken as fundamental quantities the dimensional formula of power is (a) $FL^{-1}T^{-1}$, (b) FL^2T^{-2}, (c) FLT^{-1}, (d) FLT^{-2}.

4. The most 'invariable' second is (a) ephemeris second, (b) universal second, (c) cesium second, (d) none of these.

5. If the units of length and force be increased three times, the unit of work is increased (a) three times, (b) four times, (c) eight times, (d) nine times.

6. Per sec is the unit of (a) distance, (b) time, (c) velocity, (d) angle.

7. The thickness of a soap film can be measured by (a) spherometer, (b) chronometer, (c) interferometer, (d) micrometer screw.

8. The method suitable to measure astronomical distances is (a) triangulation method, (b) alpha ray scattering method, (c) interferometer method, (d) none of these.

9. The method suitable for measuring nuclear radius is (a) triangulation method, (b) alpha ray scattering method, (c) interferometer method, (d) none of these.

[Ans. : 1. (c). 2. (c). 3. (c). 4. (a). 5. (d). 6. (a). 7. (c). 8. (a). 9. (b)]

(B)

1. Choosing force, length and time as fundamental quantities in mechanics, calculate the dimensions of (i) density, (ii) Young's modulus, (iii) surface tension, (iv) pressure, (v) momentum, (vi) work, (vii) energy, and (viii) viscosity.

[Ans : (i) $FL^{-4}T^2$, (ii) FL^{-2}, (iii) FL^{-1}, (iv) FL^{-2}, (v) FT, (vi) FL, (vii) FL, and (viii) $FL^{-2}T$.]

2. Explain the use of 'dimensional formula' for conversion of one system of units into another with two examples.

3. Point out the limitations of dimensional analysis.

(C)

1. What are base units ? Define the base units in S I. What are conditions that a standard for a physical quantity must satisfy ?

2. What do you mean by dimensions of a physical quantity ? Calculate the dimensions of the following physical quantities—(a) energy, (b) angular acceleration, (c) Young's modulus. Explain the uses of the dimensional formula with one example for each.

3. Distinguish between universal time and ephemeris time. Which one is more realiable so far as 'invariability' is concerned and why ? Is the 'cesium second' more fundamental than the 'ephemeris second' ?

(D)

1. The surface tension of water in cgs is 72 dyne per cm. What is the value in S I ? [Ans : ·072 Nm^{-1}]

2. If the units of length and force be increased four times, show that the unit of energy is increased sixteen times.

3. Test by the method of dimensions the correctness of the following :

(i) $F=6\pi\eta rv$ where F is the viscous force on a spherical body of radius r and moving with velocity v through a viscous liquid of coefficient of viscosity η.

(ii) $\tau=mB \sin\theta$ where τ is the torque on a magnet of moment m and B is the intensity of magnetic field.

(iii) $n=\frac{1}{2l}\sqrt{\frac{T}{m}}$ where n, l, T and m have their usual meaning.

4. Investigate the relation of the frequency of vibration of a stretched string with its length, tension and linear density.

(E)

1. Is the choice of mass, length and time as fundamental quantities unique ?

2. Do you think that a definition of a physical quantity for which no method of measurement is given or known has meaning ?

3. MKS stands for............

4. SI stands for............

5. The second is defined as the fraction............of the tropical year 1900.

6. Can you suggest a method to measure (a) the thickness of soap film, (b) the diameter of an atom, (c) radius of the earth ?

7. When in future man would settle on other planets, what drawbacks would our present standards of length and time have ? What drawbacks would atomic standards have ?

[**Ans : 1.** No. **2.** Yes, it makes no sense. For answers of **3, 4, 5** see the text.]

6. (a) interferometer method, (b) alpha ray scattering method, (c) triangulation method. 7. Standard of length would have no drawback but standard of time would have because it is defined in terms of diurnal or annual motion of the earth. Atomic standards would have no drawbacks.]

CHAPTER 3

VECTORS AND SCALARS

CROSS PRODUCT AND DOT PRODUCT OF VECTORS

3.1. All physical quantities fall under two categories – Scalars and Vectors

Scalars. A scalar quantity is that one which requires only magnitude for its complete specification e.g. mass, time, volume, temperature, speed, energy, magnetic flux, electric current, etc.

Vectors. A vector quantity is that one which requires both magnitude and direction for its complete specification e.g. displacement, velocity, force, acceleration, electric field, magnetic field, current density, etc.

3.2. Geometrical Representation of Vectors

Geometrically a vector is represented by the directed line segment i.e. by a line to which a direction has been assigned with an arrowhead in the direction of the vector and whose length is proportional to the magnitude of the vector.

To represent a vector geometrically first a suitable 'scale', say 1 cm = 'a' units of the vector is selected. Then a line is drawn parallel to the direction of the vector. Cut a length OP from it on that scale and finally put an arrow on the line along the direction of the vector. Now this directed line segment, namely, OP represents the vector in magnitude and direction. It is written as $\vec{OP}$. 'O' is called the 'inital point' of the vector and P, the 'terminal point' of it. The vector $\vec{OP}$ is also written as $\vec{r}$ i.e. we also write $\vec{r} = \vec{OP}$.

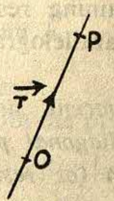


Fig. 3.1

3.3 Definitions

(i) *Modulus of a vector (in short 'mod' of a vector).* The modulus of a vector is simply the magnitude of the vector and is written as $|\vec{r}|$.

The two parallel lines flanking $\vec{r}$ is read as 'mod of'. Obviously mod of a vector is a scalar.

(ii) *Unit vectors.* A vector whose 'mod' is unity is called a unit

vector. $\vec{r}$ is a unit vector if $|\vec{r}|=1$. The unit vector is distinguished from other vectors by putting the symbol $\wedge$ over the vector e.g. $\hat{a}$ stands for unit vector along $\vec{a}$ and is read as 'a-hat' or 'a-caret'. Similarly $\hat{r}$ stands for unit vector along $\vec{r}$ and is read as 'r-caret' or r-hat.

(iii) *Proper vector.* A vector whose 'mod' is not zero is called a proper vector.

(iv) *Null vector.* A vector whose 'mod' is zero is called a null vector.

(v) *Constant vector.* A vector is constant when its components are constant.

3.4. Addition of Vectors (Composition of Vectors)

Vectors are added or composed by the 'principle of physical independence of vectors' according to which the effect of a vector on a body does not depend on the presence of other vectors. Hence we are permitted to consider their effects separately and then add them to find their resultant effect. Suppose that the effect of a vector is to displace a body from O to A (Fig. 3.2) in a certain time and that of another from O to B in the same time. When both act simultaneously, the body is displaced from O to C. Thus OC gives the sum or resultant of the two vectors. Obviously, OC is the diagonal of the parallelogram $OACB$. This principle of obtaining resultant (or vector sum) of two vectors is known as law of parallelogram of vectors. This law states :

If two vectors are represented by the two adjacent sides of parallelogram in magnitude and direction, then the diagonal passing through their meeting point represents their vector sum (or resultant) in magnitude and direction.

The two vectors are called components of the resultant. It follows from this that two given vectors can have only one resultant because only one parallelogram can be constructed with the two given vectors as adjacent sides but a given vector may have infinite components because with the given vector as diagonal infinite number of parallelograms can be constructed.

Let $\vec{a}$ and $\vec{b}$ be any two vectors at inclination θ to each other and suppose that we have to add them. First take a point O as reference

and then draw two lines parallel to the directions of $\vec{a}$ and $\vec{b}$ through O. Choose a proper scale, say, 1 cm $= k$ units of the vectors and then cut off a length, $OA = a/k$ from the line OA drawn parallel to $\vec{a}$

and $OB = b/k$ from the line drawn parallel to $\vec{b}$.

Then $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$.

Complete the parallelogram $OACB$. Then by the theorem of parallelogram of vectors the resultant of $\vec{OA}$ and $\vec{OB}$ is $\vec{OC}$.

$$\therefore \quad \vec{OC} = \vec{OA} + \vec{OB}.$$

Now $$\vec{AC} = \vec{OB}.$$

$$\therefore \quad \vec{OC} = \vec{OA} + \vec{AC}. \qquad .. (3.1)$$

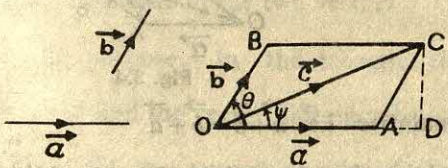


Fig. 3.2

Thus the parallelogram is not needed to find the vector sum of the two vectors. The triangle OAC is sufficient to find the sum of the vectors. In the triangle OAC, $\vec{OA} = \vec{a}$ with 'O' as the initial point and A as the terminal point and $\vec{AC} = \vec{b}$ with A as the initial point and C as the terminal point. Then $\vec{OC}$ is the vector sum of $\vec{a}$ and $\vec{b}$. Let $\vec{c}$ be the vector sum of $\vec{a}$ and $\vec{b}$.

Then $\vec{c} = \vec{a} + \vec{b}$. $\quad \therefore \quad \vec{OC} = \vec{c}$.

From the property of a triangle

$$|\vec{c}| = |\vec{OC}| = \sqrt{OA^2 + AC^2 - 2OA.AC \cos \angle OAC}$$

or $$|\vec{c}| = \sqrt{a^2 + b^2 - 2ab \cos(\pi - \theta)}$$

or $$|\vec{c}| = \sqrt{a^2 + b^2 + 2ab \cos\theta}. \qquad .. (3.2)$$

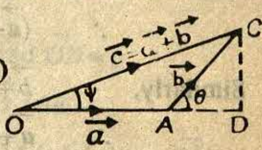


Fig. 3.3

Let the resultant vector make angle ψ with $\vec{a}$.

Then $\tan \psi = \dfrac{CD}{OD} = \dfrac{CD}{OA + AD} = \dfrac{AC \sin\theta}{OA + AC \cos\theta}$

or $$\boxed{\tan \psi = \frac{b \sin\theta}{a + b \cos\theta}}. \qquad .. (3.3)$$

The above principle of finding the vector sum of two vectors is known as the *triangle law of vector addition.* Geometrically Eq. 3.1 represents the triangle law of vector addition.

We can extend this law to 'summing up' of a number of vectors. For this, represent the vectors in order by taking the 'terminal point' of one as the 'initial point' of the next. Then the line joining the initial point of the first vector with the terminal point of the last vector represents the sum of the vectors. In the fig. 3.4,

$\vec{OA}=\vec{a}$, $\vec{AB}=\vec{b}$ and $\vec{BC}=\vec{c}$.

By the triangle law of vector addition

$$\vec{OB}=\vec{OA}+\vec{AB} \text{ and}$$

$$\vec{OC}=\vec{OB}+\vec{BC}.$$

$$\therefore\ \vec{OC}=\vec{OA}+\vec{AB}+\vec{BC}$$

$$\text{or } \vec{OC}=\vec{a}+\vec{b}+\vec{c}.$$

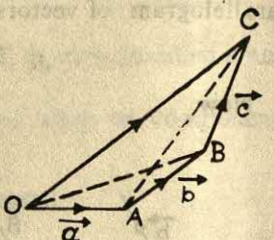


Fig. 3.4

3.5. Vector Addition is Commutative i.e. $\vec{a}+\vec{b}=\vec{b}+\vec{a}$

Refer to the fig 3.2. By applying the triangle law of vector addition to the triangle OAC we have

$$\vec{OC}=\vec{OA}+\vec{AC}=\vec{a}+\vec{b}.$$

Again from triangle OBC

$$\vec{OC}=\vec{OB}+\vec{BC}=\vec{b}+\vec{a}.$$

$$\therefore\ \vec{a}+\vec{b}=\vec{b}+\vec{a}.$$

3.6. Vector Addition is Associative i.e. $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$

Refer to the fig 3.4.

$\vec{a}+\vec{b}=\vec{OA}+\vec{AB}=\vec{OB}$, by the triangle law of vector addition.

$$\therefore\quad (\vec{a}+\vec{b})+\vec{c}=\vec{OB}+\vec{BC}=\vec{OC}.$$

Similarly, $$\vec{b}+\vec{c}=\vec{AB}+\vec{BC}=\vec{AC}.$$

$$\therefore\quad \vec{a}+(\vec{b}+\vec{c})=\vec{OA}+\vec{AC}=\vec{OC}.$$

$$\therefore\quad (\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c}).$$

3.7. Subtraction of Vectors

If there are two vectors $\vec{a}$ and $\vec{b}$, then $\vec{a}-\vec{b}=\vec{a}+(-\vec{b})$ i.e., the

subtraction of $\vec{b}$ from $\vec{a}$ is to be treated as the addition of $-\vec{b}$ with $\vec{a}$.

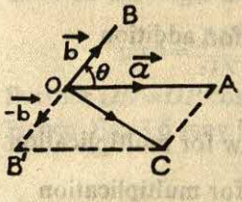


Fig. 3.5

Suppose $\vec{OA}=\vec{a}$ and $\vec{OB}=\vec{b}$. Take $\vec{OB'}=-\vec{b}$ and complete the parallelogram $OB'CA$. Then $\vec{OC}$ is the vector sum of $\vec{OB'}$ and $\vec{OA}$ i.e., $\vec{OC}=\vec{OA}+\vec{OB'}$

$$=\vec{a}+(-\vec{b})=\vec{a}-\vec{b}.$$

Thus $\vec{OC}$ is the vector difference between $\vec{OA}$ and $\vec{OB}$. Here also the parallelogram is not needed to find vector difference of the two vectors. he triangle OAC is sufficient for the purpose. In the triangle OAC, $\vec{OA}=\vec{a}$ and $\vec{AC}=-\vec{b}$. Hence to find the vector difference between $\vec{a}$ and $\vec{b}$, represent $\vec{a}$ and $-\vec{b}$ by the two successive sides of a triangle and apply the triangle law of vector addition to this triangle.

In the fig. 3.5,

$$\vec{OC}=\vec{OA}+\vec{AC}$$

$$=\vec{a}+(-\vec{b})=\vec{a}-\vec{b}.$$

From $\triangle OAC$, $|\vec{OC}|=\sqrt{OA^2+AC^2-2\,OA.\,AC\cos\angle OAC}$

or $\qquad |\vec{OC}|=\sqrt{a^2+b^2-2ab\cos\theta}. \qquad .. \quad (3.4)$

3.8. Representation of a Vector in Terms of Unit Vector along the vector itself

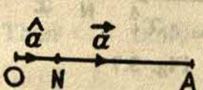


Fig. 3.6

Suppose $\vec{OA}=\vec{a}$ and $\vec{ON}=\hat{a}$

Let $\qquad |\vec{OA}|=|\vec{a}|=a.$

Then obviously $\vec{OA}$ is 'a' times $\vec{ON}$.

$\therefore \qquad \vec{OA}=a\,\vec{ON}$

or $\qquad \vec{a}=|\vec{a}|\,\hat{a}. \qquad .. \quad (3.5)$

3.9. Laws of Vector Algebra

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are vectors, m and n are scalars, then

1. $\vec{a}+\vec{b}=\vec{b}+\vec{a}$. Commutative law for addition
2. $\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}$. Associative law for addition
3. $(m+n)\,\vec{a}=m\,\vec{a}+n\,\vec{a}$. Distributive law
4. $m\,\vec{a}=\vec{a}\,m$. Commutative law for multiplication
5. $m(n\,\vec{a})=mn\,\vec{a}$. Associative law for multiplication

3.10. Representation of a Vector in Terms of Unit Vectors Along the Co-ordinate Axes (Resolution of Vectors)

The reverse process of composition of vectors, i.e., breaking a vector into its components in mutually perpendicular directions is called resolution of vectors.

The unit vectors along x, y and z-axis of a three dimensional rectangular co-ordinate system are denoted by $\hat{i}$, $\hat{j}$ and $\hat{k}$ respectively.

(i) *In the two dimensional co-ordinate system.* Any vector $\vec{r}$ in a plane can be represented with its initial point at the origin O of the two dimensional co-ordinate system. Let (x, y) be the co-ordinates of the terminal point P of the vector.

Then $$\vec{OP}=\vec{r}.$$

Draw perpendicular PN from P on Ox. We have by the triangle law of vector addition

$$\vec{OP}=\vec{ON}+\vec{NP}.$$

But $$\vec{ON}=ON\,\hat{i}=x\,\hat{i}$$

$$\vec{NP}=NP\,\hat{j}=y\,\hat{j}$$

$$\therefore\quad \vec{r}=x\,\hat{i}+y\,\hat{j}. \qquad (3.6)$$

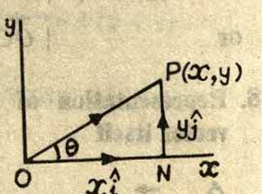


Fig. 3.7

The vectors $x\,\hat{i}$ and $y\,\hat{j}$ are x-component and y-component of the vector respectively.

Clearly, $\left|\vec{r}\right| = OP = \sqrt{ON^2 + NP^2}$

or $\left|\vec{r}\right| = \sqrt{x^2 + y^2}$. .. (3.6 a)

Let θ be the angle made by $\vec{r}$ with the x-axis. Then

$$\tan\theta = \frac{PN}{ON}, \text{ or } \tan\theta = \frac{y}{x} \text{ or } \theta = \tan^{-1}\frac{y}{x}.$$

Further $ON = OP\cos\theta = r\cos\theta$ and $PN = OP\sin\theta = r\sin\theta$.

Hence Eq. 3.6 may be written as

$$\vec{r} = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j}. \quad .. (3.7)$$

Here $r\cos\theta$ and $r\sin\theta$ are the scalar components of the vector along x and y axis respectively.

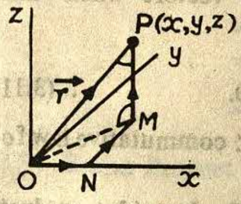


Fig. 3.8

(ii) *In the three dimensional co-ordinate system.* Let the co-ordinates of the terminal point P of the vector be x, y and z.

Draw perpendicular PM from P on xy plane and then MN from M on Ox.

We have by the triangle law of vector addition,

$$\vec{OP} = \vec{OM} + \vec{MP}.$$

Again from triangle ONM, $\vec{OM} = \vec{ON} + \vec{NM}$.

$$\therefore \quad \vec{OP} = \vec{ON} + \vec{NM} + \vec{MP}$$

Now $\vec{ON} = x\hat{i}$, $NM = y\hat{j}$ and $MP = z\hat{k}$

$$\therefore \quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \quad .. (3.8)$$

$$\left|\vec{r}\right| = \sqrt{OP^2} = \sqrt{OM^2 + MP^2} = \sqrt{ON^2 + NM^2 + MP^2}$$

or $\left|\vec{r}\right| = \sqrt{x^2 + y^2 + z^2}$. .. (3.9)

Let θ_z be the angle made by r with z-axis.

Then $\tan\theta_z = \dfrac{OM}{PM} = \dfrac{\sqrt{ON^2 + NM^2}}{PM} = \dfrac{\sqrt{x^2 + y^2}}{z}$

$$\left.\begin{aligned} &\text{or} & \theta_z &= \tan^{-1}\frac{\sqrt{x^2+y^2}}{z} \\ &\text{Similarly,} & \theta_x &= \tan^{-1}\frac{\sqrt{y^3+z^2}}{x} \\ & & \theta_y &= \tan^{-1}\frac{\sqrt{z^2+x^2}}{y}. \end{aligned}\right\} \quad .. (3.10)$$

The cosines of the angles θ_x, θ_y and θ_z are known as 'direction cosines' of the line *OP* and are generally denoted by *l*, *m* and *n* respectively. Then $\vec{r} = r\,(l\hat{i}+m\hat{j}+n\hat{k})$ where $r=\left|\vec{r}\right|=\sqrt{x^2+y^2+z^2}$

and $$\hat{r}=\frac{\vec{r}}{r}=(l\hat{i}+m\hat{j}+n\hat{k}).$$

3.11. Scalar or Dot Product of Vectors

Definition. If $\vec{a}$ and $\vec{b}$ are two vectors, then their scalar or dot product is defined as $\left|\vec{a}\right|\left|\vec{b}\right|\cos\theta$ where θ is the angle $(0<\theta<\pi)$ between the direction of the two vectors when their initial points coincide and is written as $\vec{a}.\ \vec{b}$.

Therefore by definition $\vec{a}.\vec{b} = \left|\vec{a}\right|\left|\vec{b}\right|\cos\theta$. .. (3.11)

It follows from definition that $\vec{a}.\ \vec{b}=\vec{b}.\ \vec{a}$ i.e. commutative law for scalar product holds good.

3.12. Geometrical Representation of Scalar Product (dot product) of two Vectors

Consider two vectors $\vec{a}$ and $\vec{b}$ and represent them by $\vec{OA}$ and $\vec{OB}$ respectively with '*O*' as their common initial point. Drop a perpendicular *BN* from *B* on *OA* and *AM* from *A* on *OB*.

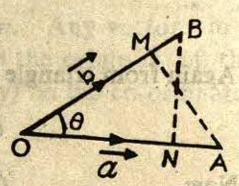


Fig. 3.9

Now, $\vec{a}.\ \vec{b} = a\,b\cos\theta$ when $\left|\vec{a}\right|=a$ and $\left|\vec{b}\right|=b$

$= a(b\cos\theta)$

$= a\,ON = a$ times the projection of $\vec{b}$ on $\vec{a}$

or $\vec{a}.\ \vec{b} = (a\cos\theta)b$

$= OMb = b$ times the projection $\vec{a}$ on $\vec{b}$.

Hence the scalar (dot) product of two vectors is the product of the modulus of either vector and the resolute of the other in its direction.

3.13. Distributive Law for Scalar (dot) Product holds i.e.

$\vec{\mathbf{a}}.\ (\vec{\mathbf{b}}+\vec{\mathbf{c}}) = \vec{\mathbf{a}}.\ \vec{\mathbf{b}} + \vec{\mathbf{a}}.\ \vec{\mathbf{c}}$

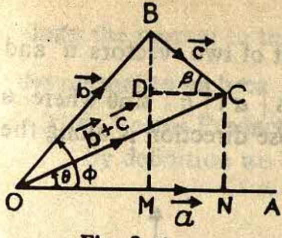


Fig. 3.10

Suppose $\vec{OA}=\vec{a}$, $\vec{OB}=\vec{b}$ and $\vec{BC}=\vec{c}$ so that by the triangle law of vector

$$\vec{OC}=\vec{OB}+\vec{BC}$$
$$=\vec{b}+\vec{c}.$$

Now, $\vec{a}.(\vec{b}+\vec{c})=\vec{a}.\vec{OC}$

$$=a\,OC\cos\theta$$
$$=aON$$
$$=a(OM+MN)$$
$$=a(OM+DC)$$
$$=aOM+aDC$$
$$=ab\cos\varphi+ac\cos\beta$$
$$=\vec{a}.\vec{b}+\vec{a}.\vec{c}.$$

3.14. Scalar Product of Unit Vectors Along Co-ordinate Axes

$$\hat{i}.\hat{i}=|\hat{i}||\hat{i}|\cos 0°=1.1.1.=1.$$

Similarly $\hat{j}.\hat{j}=1$ and $\hat{k}.\hat{k}=1$.

$\therefore$ $\hat{i}.\hat{i}=\hat{j}.\hat{j}=\hat{k}.\hat{k}$ i.e., scalar (dot) product of like unit vectors is 1.

Now, $\hat{i}.\hat{j}=|\hat{i}||\hat{j}|\cos\pi/2=1.1.0=0.$

$\therefore$ $\hat{i}.\hat{j}=\hat{j}.\hat{k}=\hat{k}.\hat{i}=0$

i.e. Scalar (dot) product of unlike unit vectors is zero.

3.15. Scalar Product of Any Two Vectors

Suppose $\vec{a}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$

and $\vec{b}=x_2\hat{i}+y_2\hat{j}+z_2\hat{k}.$

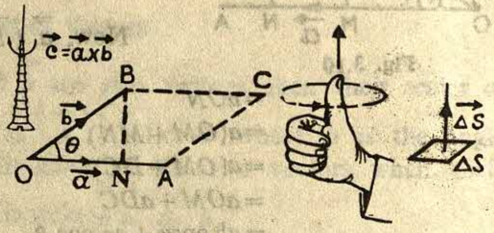
Then
$$\vec{a}.\vec{b}=(x_1\hat{i}+y_1\hat{j}+z_1\hat{k}).(x_2\hat{i}+y_2\hat{j}+z_2\hat{k})$$
$$=x_1x_2\,\hat{i}.\hat{i}+y_1x_2\,\hat{j}.\hat{i}+z_1x_2\,\hat{k}.\hat{i}+x_1y_2\,\hat{i}.\hat{j}+y_1y_2\,\hat{j}.\hat{j}$$
$$+z_1y_2\,\hat{k}.\hat{j}+x_1z_2\,\hat{i}.\hat{k}+y_1z_2\,\hat{j}.\hat{k}+z_1z_2\,\hat{k}.\hat{k}$$
$$=x_1x_2+y_1y_2+z_1z_2$$

($\because$ $\hat{i}.\hat{i}=\hat{j}.\hat{j}=\hat{k}.\hat{k}=1$ and $\hat{i}.\hat{j}=\hat{j}.\hat{k}=\hat{k}.\hat{i}=0$).

$$\therefore\quad \vec{a}.\vec{b}=x_1x_2+y_1y_2+z_1z_2. \qquad \text{(3.12)}$$

3.16. Vector (Cross) Product of Two Vectors

Definition. The vector (or cross) product of two vectors $\vec{a}$ and $\vec{b}$ is defined as a vector $\vec{c}$ whose modulus is $|\vec{a}||\vec{b}|$ sinθ where θ is the angle between the vectors and whose direction is along the direction of translation of a right-handed screw perpendicular to $\vec{a}$ and $\vec{b}$ and rotating from $\vec{a}$ to $\vec{b}$ in the sense in which θ is taken. It is also conveniently given by 'Maxwell's right hand rule (see Fig. 3.11b). It is written as $\vec{a}\times\vec{b}$.



(a) (b) (c)

Fig 3.11

Therefore, by definition, $\vec{a}\times\vec{b}=\vec{c}$ such that

$$|\vec{a}\times\vec{b}|=|\vec{c}|=|\vec{a}||\vec{b}|\sin\theta.$$

or $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\,\hat{n}$ where $\hat{n}$ is the unit vector along the normal to the plane of $\vec{a}$ and $\vec{b}$.

It easily follows from the definition that $\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$ i.e. vector product is not commutative.

3.17. Geometrical Representation of Vector (or Cross) Product of Two Vectors

Refer to the fig. 3.11 (a) where $\vec{OA}=\vec{a}$ and $\vec{OB}=\vec{b}$. Complete the parallelogram *OACB*. Draw a perpendicular *BN* from *B* on *OA*.

By definition, $|\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\sin\theta$

$$=OA.\ OB\sin\theta$$
$$=OA.\ BN=2.\tfrac{1}{2}OA.BN$$
$$=2.\triangle OAB$$
$$=\text{area of the parallelogram } OACB.$$

Thus the cross product of two vectors is a vector whose modulus is the area of the parallelogram formed by the two vectors as the adjacent sides and its direction is along normal to the parallelogram. This fact suggests that an element of area can be represented by a vector of modulus equal to the area, and oriented in space

along the normal to the area. Hence if $\hat{n}$ is the unit vector along the normal to an area $\Delta\vec{S}$ we can write $\Delta\vec{S}=\Delta S\,\hat{n}$.

3.18. Vector Product of Unit Vectors Along the co-ordinate Axes

By definition we have

$$\left|\hat{i}\times\hat{i}\right|=\left|\hat{i}\right|\left|\hat{i}\right|\sin 0^\circ=1.1.0=0.$$

$\therefore\quad \hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=0$ i.e., the oross product of like unit vectors along co-ordinate axes is zero.

By definition

$$\left|\hat{i}\times\hat{j}\right|=\left|\hat{i}\right|\left|\hat{j}\right|\sin\pi/2=1.1.1=1.$$

Thus $\hat{i}\times\hat{j}$ is a vector of unit magnitude in the direction of a right-handed screw perpendicular to $\hat{i}$ i.e., x-axis and $\hat{j}$ i.e., y-axis and rotating from i to j i.e., from x-axis to y-axis through smaller angle (namely $\pi/2$). The direction of translation of such a screw is along the positive direction of z-axis. Hence $\hat{i}\times\hat{j}$ is a vector of unit magnitude along the z-axis. This is exactly $\hat{k}$.

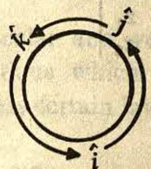


Fig. 3.12

$\therefore\qquad \hat{i}\times\hat{j}=\hat{k}.$

Similarly, $\qquad \hat{j}\times\hat{k}=\hat{i}$

and $\qquad \hat{k}\times\hat{i}=\hat{j}.$

Thus the cross-product of any two unlike unit vectors taken in cyclic order (order being $\hat{i},\hat{j}$ and $\hat{k}$) is equal to the third one.

3.19. Vector (or Cross) Product of any Two Vectors

Suppose
$$\vec{a}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$$
$$\vec{b}=x_2\hat{i}+y_2\hat{j}+z_2\hat{k}.$$

Then $\vec{a}\times\vec{b}=(x_1\hat{i}+y_1\hat{j}+z_1\hat{k})\times(x_2\hat{i}+y_2\hat{j}+z_2\hat{k})$

or
$$\vec{a}\times\vec{b}=x_1x_2\hat{i}\times\hat{i}+y_1x_2\hat{j}\times\hat{i}+z_1x_2\hat{k}\times\hat{i}+x_1y_2\hat{i}\times\hat{j}+y_1y_2\hat{j}\times\hat{j}$$
$$+z_1y_2\hat{k}\times\hat{j}+x_1z_2\hat{i}\times\hat{k}+y_1z_2\hat{j}\times\hat{k}+z_1z_2\hat{k}\times\hat{k}$$
$$=y_1x_2(-\hat{k})+z_1x_2\hat{j}+x_1y_2\hat{k}+z_1y_2(-\hat{i})+x_1z_2(-\hat{j})+y_1z_2\hat{i}$$
$$=(y_1z_2-z_1y_2)\,\hat{i}+(z_1x_2-x_1z_2)\,\hat{j}+(x_1y_2-y_1x_2)\,\hat{k}$$

or $$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ x_1 & y_1 & z_1\\ x_2 & y_2 & z_2\end{vmatrix}.$$

3.20. The Distributive Law for Vector Product Holds

i.e., $$\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}.$$

Let $\vec{a}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$; $\vec{b}=x_2\hat{i}+y_2\hat{j}+z_2\hat{k}$

and $\vec{c}=x_3\hat{i}+y_3\hat{j}+z_3\hat{k}$.

Now $\vec{b}+\vec{c}=(x_2+x_3)\hat{i}+(y_2+y_3)\hat{j}+(z_2+z_3)\hat{k}$.

$$\therefore\quad \vec{a}\times(\vec{b}+\vec{c})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ x_1 & y_1 & z_1\\ x_2+x_3 & y_2+y_3 & z_2+z_3\end{vmatrix}.$$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ x_1 & y_1 & z_1\\ x_2 & y_2 & z_2\end{vmatrix}+\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ x_1 & y_1 & z_1\\ x_3 & y_3 & z_3\end{vmatrix}$$

$$=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}.$$

Hence $$\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}.$$

3.21. Polar and Axial Vectors

A vector which has a natural direction is called a polar vector. Force, velocity, displacement, electric field, magnetic field, current density, etc. are polar vectors.

Vectors like angular velocity, angular acceleration, torque etc. which are associated with rotation of a body about a line (axis of rotation) are called axial vectors. Such vectors are represented by a straight line along the axis and a curved arrow round the axis indicating the direction of rotation. Suppose that a disc is rotating about a vertical line in the anticlockwise direction with angular velocity ω. The direction of the axis is usually shown by an

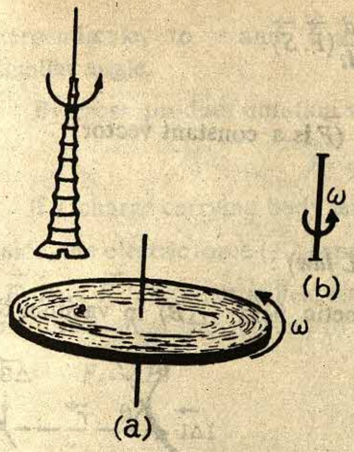


Fig 3.13

arrow beaming in the direction of translation of a right hand screw placed along the axis of rotation and rotating in the direction of rotation of the body. Fig. 3.13 (a) shows the angular velocity and (b) shows its vector representation.

In a scalar product, either both the vectors should be polar or both of them axial. The vector product of two polar vectors is an axial vector e.g. moment of a force and the vector product of a polar vector with an axial vector is a polar vector like $\vec{v}=\vec{\omega}\times\vec{r}$.

3.22. Uses of Scalar (Dot) Product and Vector (Cross) Product

Scalar and vector products of two vectors are very important notations which enable us to generalise many expressions and express certain laws in a compact form.

Examples

1. *Show that, in general work done is given by dot (scalar) product of force and displacement and power (rate of doing work) is given by dot product of force and velocity.*

Suppose that the point of application of a force $\vec{F}$ is displaced through a distance $|\vec{S}|$ and the line of action of the force is inclined at an angle θ to the displacement vector $\vec{S}$.

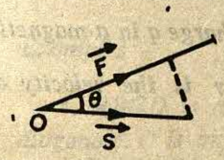


Fig. 3.14

Then work done is given by

W = Force × displacement of the point of application of the force along the force

$$= |\vec{F}|\ |\vec{S}| \cos\theta$$

$$= \vec{F}.\vec{S}.$$

The rate of doing work $= \frac{dW}{dt} = \frac{d}{dt}(\vec{F}.\vec{S})$

$$= \vec{F}.\frac{d\vec{S}}{dt} \quad (\vec{F} \text{ is a constant vector})$$

$$= \vec{F}.\ \vec{v}.$$

2. *Biot-Savart-Laplace's law (BSL law)* :

This law states that the magnetic field $(\triangle B)$ in vacuum due to a current element $(I\triangle l)$ at a point is proportional to $I\triangle l$, inversely proportional to the square of the distance and proportional to the sine of the angle between the element and the line joining it with the point.

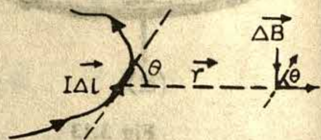


Fig. 3.15

Hence $$\triangle B = \frac{\mu_0}{4\pi} \frac{I \triangle l \sin\theta}{r^2}$$

where r is the distance of the point from the element and $\frac{\mu_0}{4\pi}$ is a constant. The law further adds that the direction of the field is along the direction of translation of a right handed screw perpendicular to the plane containing $\triangle\vec{l}$ and $\vec{r}$ and rotating from $\triangle\vec{l}$ to $\vec{r}$ through the smaller angle. Here θ is the smaller angle between $\triangle\vec{l}$ and $\vec{r}$ $(0<\theta<\pi)$.

If we adopt vector product notation we can express this law in the compact form :

$$\triangle\vec{B} = \frac{\mu_0}{4\pi} \frac{I\triangle\vec{l}\times\hat{r}}{r^2}.$$

3. *The force (F) experienced by a moving charge q in a magnetic field B is given by $F=q\ v\ B \sin(\vec{v}, \vec{B})$ where v is the velocity of the charge.*

The direction of the force is given by a right-handed screw

perpendicular to $\vec{v}$ and $\vec{B}$ and rotating from $\vec{v}$ to $\vec{B}$ through the smaller angle.

By cross product notation we can write

$$\vec{F}=q\vec{v} \times \vec{B}.$$

If a charge carrying body is projected into electric and magnetic fields, the electric force $(\vec{F_e})$ experienced by it due to the electric field is $q\vec{E}$ along $\vec{E}$. The electric field force does not depend on the magnitude and direction of the velocity of the body.

$$\vec{F_e}=q\vec{E}.$$

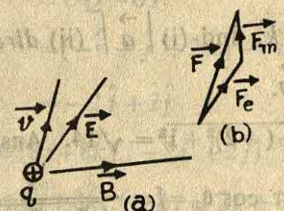


Fig. 3.16

The magnitude and direction of the force $(\vec{F_m})$ due to the magnetic field is $q\vec{v} \times \vec{B}$ i.e, $qvB \sin(\vec{v}, \vec{B})$ in the direction of translation of a right-hand screw perpendicular to $\vec{v}$ and $\vec{B}$ and rotating from $\vec{v}$ to $\vec{B}$ through the smaller angle.

$\therefore$ The total force ($\vec{F}$) experienced by a moving charge

$$=\vec{F_e}+\vec{F_m}$$

$$\vec{F}=q\vec{E}+q\vec{v}\times\vec{B}$$

or $$\vec{F}=q(\vec{E}+\vec{v}\times\vec{B}).$$

This is called Lorentz force.

Examples

1. *ABCD is a parallelelogram and P is the intersection of the diagonals. O is any point. Show that*

$$\vec{OA}+\vec{OB}+\vec{OC}+\vec{OD}=4\vec{OP}.$$

Sol. By the triangle law of vector addition

$\vec{OA}=\vec{OP}+\vec{PA}=\vec{OP}+\frac{1}{2}\ \vec{CA}$

$\vec{OB}=\vec{OP}+\vec{PB}=\vec{OP}+\frac{1}{2}\ \vec{DB}$

$\vec{OC}=\vec{OP}+\vec{PC}=\vec{OP}+\frac{1}{2}\ \vec{AC}$

$\vec{OD}=\vec{OP}+\vec{PD}=\vec{OP}+\frac{1}{2}\vec{BD}$

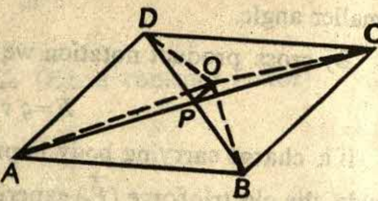


Fig. 3.17

$\therefore\ \vec{OA}+\vec{OB}+\vec{OC}+\vec{OD}=4\ \vec{OP}+\frac{1}{2}\ (\vec{CA}+\vec{AC})+\frac{1}{2}(\vec{DB}+\vec{BD})$

$=4\ \vec{OP}$. **Proved.**

2. *If* $\vec{a}=3\hat{i}-2\hat{j}+\hat{k}$, find (i) $|\vec{a}|$, (ii) *direction cosines of* $\vec{a}$, (iii) *unit vector along* $\vec{a}$.

Sol. $|\vec{a}|=\sqrt{3^2+(-2)^2+1^2}=\sqrt{14}$. **Ans.**

$\tan\ \theta_x=\frac{\sqrt{z^2+y^2}}{x}$ or $\cos\ \theta_x=l=\frac{x}{\sqrt{x^2+y^2+z^2}}=\frac{3}{\sqrt{14}}$.

Similarly, $m=\frac{y}{\sqrt{x^2+y^2+z^2}}=\frac{-2}{\sqrt{14}}$

and $n=\frac{1}{\sqrt{14}}$.

$\therefore$ Direction cosines of the vector are $\left(\frac{3}{\sqrt{14}},\frac{-2}{\sqrt{14}},\frac{1}{\sqrt{14}}\right)$

$$\hat{a}=\frac{3}{\sqrt{14}}\hat{i}-\frac{2}{\sqrt{14}}\hat{j}+\frac{1}{\sqrt{14}}\hat{k}\ .\ \textbf{Ans.}$$

3. *Find the angle between the directions of the vectors*

$$\vec{a}=3\ \vec{i}+4\ \vec{j}+5\ \vec{k}\text{ and }\vec{b}=\hat{i}\ +\hat{j}+\hat{k}.$$

Sol. $\vec{a}.\ \vec{b}=3.\ 1+4.\ 1+5.\ 1=3+4+5=12$

$|\vec{a}|=\sqrt{3^2+4^2+5^2}=\sqrt{50}$

$|\vec{b}|=\sqrt{1^2+1^2+1^2}=\sqrt{3}$.

We have. $\vec{a}.\vec{b}=|\vec{a}|\ |\vec{b}|\cos\theta$

$$12=\sqrt{50}\,\sqrt{3}\cos\theta, \text{ or } \cos\theta=\frac{12}{\sqrt{150}}=\cdot 9798$$

$$\theta=11°48'. \textbf{ Ans.}$$

4. The point of application of a force $\vec{F}=6\hat{j}+8\hat{k}$ is displaced from $P(1,-1,2)$ to $Q(-1,1,2)$. find the work done.

Sol.
$$\vec{OP}=\hat{i}-\hat{j}+2\hat{k}$$
$$\vec{OQ}=-\hat{i}+\hat{j}+2\hat{k}$$

$\vec{S}$ (displacement vector) $=\vec{PQ}=\vec{PO}+\vec{OQ}$

$$=\vec{OQ}-\vec{OP}$$
$$=(-\hat{i}+\hat{j}+2\hat{k})-(\hat{i}-\hat{j}+2\hat{k})$$
$$=-2\hat{i}+2\hat{j}$$

$$W=\vec{F}.\vec{S}=(6\hat{j}+8\hat{k}).(-2\hat{i}+2\hat{j})$$
$$=(0\hat{i}+6\hat{j}+8\hat{k}).(-2\hat{i}+2\hat{j}+0\hat{k})$$
$$=0(-2)+6.\,2+8.\,0$$
$$=12 \text{ units. } \textbf{Ans.}$$

QUESTIONS

(A)

1. $\vec{a}+\vec{b}=\vec{c}$ and $c=\sqrt{a^2+b^2}$. The angle between $\vec{a}$ and $\vec{b}$ is (a) $\pi/2$, (b) π, (c) $\pi/3$, (d) $\pi/4$.

2. If $\vec{A}+\vec{B}=\vec{A}-\vec{B}$, then (a) $\vec{A}=0$, (b) $\vec{B}=0$, (c) $\vec{A}$ and $\vec{B}$ are simultaneously zero, (d) $\vec{A}+\vec{B}=0$.

3. The angular velocity of a body is a (a) polar vector, (b) axial vector, (c) constant vector, (d) none of these.

4. The cross product of two vector $\vec{a}$ and $\vec{b}$ is a vector (a) perpendicular to $\vec{a}$ but not to $\vec{b}$, (b) perpendicular to $\vec{b}$ but not to $\vec{a}$, (c) perpendicular to both $\vec{a}$ and $\vec{b}$, (d) none of these.

5. The cross product of two vectors $(\vec{a}\times\vec{b})$ is a vector (a) along the direction of translation of a right-hand screw perpendicular to $\vec{a}$ and $\vec{b}$ and rotating from $\vec{a}$ to $\vec{b}$ through the smaller angle, (b) along the direction of transla-

tion of a right-hand screw perpendicular to $\vec{a}$ and $\vec{b}$ and rotating from $\vec{b}$ to $\vec{a}$ through the smaller angle, (c) parallel to the plane of $\vec{a}$ and $\vec{b}$, (d) none of these.

6. If $\vec{a}.\ \vec{b}=0$, then $|\vec{a}\times\vec{b}|$ is equal to (a) zero, (b) $|\vec{a}|\,|\vec{b}|$, (c) unity, (d) $\sqrt{|\vec{a}|\,|\vec{b}|}$.

7. Which one of the following is a scalar quantity ?
(a) velocity, (b) current, (c) magnetic flux density, (d) force.

8. Which one of the following is a vector quantity ?
(a) work, (b) power, (c) electric potential, (d) momentum.

9. If $\vec{a}\times\vec{b}=0$, then $\vec{a}.\ \vec{b}$ is (a) zero, (b) unity, (c) $|\vec{a}|\,|\vec{b}|$, (d) none of these.

[**Ans.** **1.** (a), **2.** (b), **3.** (b), **4.** (c), **5.** (a), **6.** (b), **7.** (b), **8.** (d), **9.** (c).]

(B)

1. Show that, in general, work done is given by the dot product of force and displacement.

2. Show that the distributive law for vector product holds.

3. State and prove the distributive law for a scalar product.

4. Calculate the vector product and dot product of two vectors in terms of their scalar components along the co-ordinate axes.

5. Two vectors can have only one resultant but a given vector can have many component vectors. Explain why.

(C)

1. What are scalars and vectors ? Give examples. Explain the triangle law of vector addition.

2. Define dot and cross product of two vectors and explain their significance with suitable illustrations.

3. Explain the significance of dot product and cross product with reference to work done, Biot-Savart-Laplaces law and Lorentz force.

4. What do you mean by polar vector and axial vector ? Give examples. Show that in rotational motion

$$\vec{v}=\vec{\omega}\times\vec{r}.$$

(D)

1. Prove that for a vector $\vec{a}$ defined by $\vec{a}=a_x\,\hat{i}+a_y\,\hat{j}+a_z\,\hat{k}$ the scalar components are given by $a_x=\hat{i}.\ \vec{a}$, $a_y=\hat{j}.\ \vec{a}$ and $a_z=\hat{k}.\ \vec{a}$.

[**Hints :** $\hat{i}.\ \vec{a}=|\hat{i}|\,|\vec{a}|\cos(\hat{i}.\ \vec{a})=1.a.\cos\theta=a\cos\theta=a_x$.]

2. Forces P, Q act at O and have resultant R. If any transversal cuts their lines

of action at A, B and C respectively, show that

$$\frac{P}{OA}+\frac{Q}{OB}=\frac{R}{OC}.$$

3. If $\vec{a}=3\hat{i}-2\hat{j}+\hat{k}$, $\vec{b}=2\hat{i}-4\hat{j}-3\hat{k}$ and $\vec{c}=-\hat{i}+\hat{j}+2\hat{k}$, find the magnitude of (i) $\vec{a}+\vec{b}+\vec{c}$, (ii) $2\vec{a}-3\vec{b}-5\vec{c}$.

[Ans. (i) $\sqrt{41}$, (ii) $\sqrt{35}$]

4. Whether a force $\vec{F}$ varies in magnitude and direction or remains constant, the work done is given by $W=\int\vec{F}.\,\vec{dr}$. Show from this that in general,

$$W=\tfrac{1}{2}mv^2-\tfrac{1}{2}mv_0{}^2.$$

i.e. work done by the force=change in kinetic energy.
This is called Work-Energy theorem.

[Hints :
$$W=\int\vec{F}.\,d\vec{r}=\int(F_x\hat{i}+F_y\hat{j}).\ (\hat{i}\,dx+\hat{j}\,dy)$$

or
$$W=\int(F_xdx+F_ydy)$$
$$=\int m\frac{dv_x}{dt}dx+\int m\frac{dv_y}{dt}dy.\left(\because F_x=m\frac{dv_x}{dt},\ \ F_y=m\frac{dv_y}{dt}\right)$$

5. Show that $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}$.

6. Show that $\vec{a}.(\vec{b}\times\vec{c})=a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-b_1c_3)+a_3(b_1c_2-b_2c_1)$

7. Show that $\vec{a}\times(\vec{b}+\vec{c})+\vec{b}\times(\vec{c}+\vec{a})+\vec{c}\times(\vec{a}+\vec{b})=0$.

8. Show that $\vec{a}.(\vec{b}\times\vec{c})=\vec{c}.(\vec{a}\times\vec{b})$.

9. Show that $|\vec{a}\times\vec{b}|^2+(\vec{a}.\vec{b})^2=(\vec{a}.\vec{a})(\vec{b}.\vec{b})$.

10. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are the position vectors of three collinear points A, B and C with respect to the origin O, show that $\vec{a}\times\hat{p}=\vec{b}\times\hat{p}=\vec{c}\times\hat{p}$ where $\hat{p}$ is the unit vector along the line ABC.

(E)

1. Is $\vec{a}.(\vec{b}.\vec{c})$ defined ?
2. If $\vec{c}=\vec{a}\times\vec{b}$, then $|\vec{c}|=$............
3. Is the cross-product of two vectors axial or proper ?
4. Is current density a scalar or a vector ?
5. If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, then x, y and z components of the vector $\vec{a}$ are........
6. Can two vectors of different magnitude be combined to give a zero resultant ? Can three vectors ?
7. Can a vector be zero if one of its components is not zero ?

8. Does it make any sense to call a quantity a vector when its magnitude is zero ?

9. Can a scalar product be a negative quantity ?

10. Do the commutative law and associative law apply to vector subtraction ?

11. We can order events such as past, present and future. Hence there is a sense of time, distinguishing past, present and future. Is time a vector therefore ? If not, why not ?

[Ans. 1. No. 2. $\left|\vec{a}\right|\left|\vec{b}\right|\sin\theta$. 3. Axial. 4. Vector. 5. $a_1\hat{i}$, $a_2\hat{j}$, $a_3\hat{k}$. 6. No, Yes. 7. No. 8. Yes. This is called a null vector. 9. Yes. 10. Yes. 11. No, because they (times) do not add, subtract or multiply according to rules of vector algebra.]

CHAPTER 4

KINEMATICS : RELATIVE VELOCITY : NEWTON'S LAWS OF MOTION : MOMENT OF FORCES : EQUILIBRIUM OF BODIES

4.1 Displacement:Velocity : Instantaneous, Uniform and Average

The shortest distance specified completely in magnitude and direction from the origin of the frame of reference in which motion is considered is called the 'position vector' of the particle and the shortest distance specified in magnitude and direction between the final position after a certain interval of time and the initial position of the particle in the beginning of that interval is called the 'displacement' of the particle during that interval. If $\vec{r_i}$ and $\vec{r_f}$ are the initial and final position vectors during an interval of time $\triangle t$, then $(\vec{r_f}-\vec{r_i})$ is the 'displacement' of the particle during $\triangle t$.

The velocity of a particle is the rate at which its position changes with time.

The position of a particle in a frame of reference is given by its *position vector* drawn from the origin of the frame to the particle. Let $\vec{r}$ be the position vector of a particle at the instant t and $\triangle\vec{r}$ be the displacement in a small interval $\triangle t$, following the instant t, then the velocity of the particle at time t, is

$$\vec{v} = \lim_{\triangle t \to 0} \frac{\triangle \vec{r}}{\triangle t} = \frac{d\vec{r}}{dt} \text{ (by calculus).} \qquad \text{.. (4.1)}$$

This velocity is called *Instantaneous Velocity*. If the instantaneous velocity of a particle is the same at every instant, its velocity is said to be uniform. The average velocity $(\bar{v})$ is the total displacement divided by the total time in which the displacement takes place. If $\vec{r_1}$ is the position vector at time t_1 and $\vec{r_2}$ is the position vector at time t_2, then,

$$\vec{\bar{v}} = \frac{\vec{r_2} - \vec{r_1}}{(t_2 - t_1)}.$$

Obviously the average velocity is also a vector, for it is obtained by dividing a vector by a scalar.

The magnitude v of the instantaneous velocity is called the *Speed* and is simply the absolute value of $\vec{v}$. That is,

$$v = \left| \vec{v} \right| = \left| \frac{d\vec{r}}{dt} \right|.$$

Graphically the velocity of the particle is given by the slope of the plot of position vector of the particle against time. According to the geometrical interpretation of the differential co-efficient, $\frac{d\vec{r}}{dt}$ represents the slope of the plot, i.e., tangent of the angle of inclination of the plot at the given point.

4.2. Acceleration : Instantaneous, Uniform and Average

The acceleration of a particle is the rate of change of its velocity with time.

If $\vec{v}$ is the velocity of a particle at time t and $\Delta\vec{v}$ is the change in velocity in the next short interval Δt, then the acceleration at time t is

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}. \text{ (by calculus)} \qquad .. \quad (4.2)$$

This acceleration is called *instantaneous acceleration.* If the instantaneous acceleration of a particle is the same at every instant, then the acceleration of the body is said to be *uniform.* The average acceleration during an interval is defined as the net change in velocity divided by the total time in which the change takes place.

If $\vec{v}_1$ is the instantaneous velocity at time t_1 and $\vec{v}_2$ is that at t_2

then $$\bar{\vec{a}} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1}.$$

Obviously the average acceleration is also a vector, for it is obtained by dividing a vector by a scalar. Graphically the acceleration of the particle is given by the slope of the plot of the velocity of the particle against time.

Example

1. *A train is moving at an essentially constant speed of 60 kph eastward for 40 minutes, then in the direction 45° east of north for 20 minutes, and finally westward for 50 minutes. What is the average velocity of the train during this run ?*

Sol. Let us take the eastward direction as x-axis and northward direction as y-axis.

The first displacement is $60\times\frac{40}{60}=40$ km. eastward.

$$\therefore \qquad \vec{S}_1 \text{ (the first displacement)} = 40\,\hat{i}.$$

The next displacement is $60\times\frac{20}{60}=20$ km in the direction 45° east of north.

$$\therefore \qquad \vec{S}_2 \text{ (the second displacement)} = 10\sqrt{2}\,\hat{i}+10\sqrt{2}\,\hat{j}.$$

The last displacement is $60\times\frac{50}{60}=50$ km westward.

$$\therefore \quad \vec{S}_3=-50\,\hat{i}.$$

$$\therefore \quad \vec{S}=\vec{S}_1+\vec{S}_2+\vec{S}_3=40\hat{i}+\ 0\sqrt{2}\,\hat{i}+10\sqrt{2}\,\hat{j}-50\,\hat{i}$$
$$=4.14\,\hat{i}+14.14\,\hat{j}.$$

$\therefore$ The net displacement $=\sqrt{4.14^2+14.14^2}=\sqrt{217}=14{\cdot}73$ km.
The total time $=(40+20+50)$ minutes $=110$ minutes

$$=\frac{110}{60}=\frac{11}{6} \text{ hour.}$$

$\therefore$ The average velocity during the interval $=\dfrac{14{\cdot}73}{\frac{11}{6}}=7{\cdot}85$ kph

in the direction $\tan^{-1}\frac{14.14}{4.14}=72°20'$ north of east. **Ans.**

4.3. Kinematic Equations for a Uniformly Accelerated Body along a Straight Line

Suppose that a particle is moving with uniform acceleration along a straight line. Let S be the distance of the particle from an arbitrary origin on the line. Then the instantaneous velocity of the particle is

$$v=\frac{dS}{dt} \qquad \text{.. (i)}$$

and the acceleration is $a=\dfrac{dv}{dt}$. .. (ii)

Proceeding with (ii) we have,

$$dv=a\,dt$$

or $\int dv=a\int dt+c$, where c is a constant

or $v=at+c$ ($\because$ a is constant here)

Let $v=v_0$ at $t=0$, then $v_0=c$.

$$\therefore \quad \boxed{v=v_0+at} \qquad \text{.. (4.3)}$$

Proceeding with (i) we have

$$dS=vdt=(v_0+at)dt$$

$\therefore$ $\int dS=\int(v_0+at)dt+c$ (a constant)

or $S=\int v_0dt+\int atdt+c$

or $S=v_0t+a.\dfrac{t^2}{2}+c.$

When $t=0$, let $S=S_0$ (say).

Then $S_0=c$

$\therefore$ $S=v_0t+\frac{1}{2}at^2+S_0.$

If the particle be at the origin in the beginning then, $S_0=0$ and then

$$\boxed{S=v_0t+\tfrac{1}{2}at^2}\,. \qquad \text{.. (4.4)}$$

Eliminating t from Eq. 4.4 with the help of Eq. 4.3 we have,

$$\boxed{v^2-v_0{}^2=2\,aS} \qquad \text{.. (4.5)}$$

4.4. Equations of Motion in Free Fall

Near the surface of the earth a body falls freely under gravity with uniform acceleration. If we select a reference frame rigidly attached to the earth then the acceleration due to gravity will be a vector pointing downward i.e. in the negative direction of y-axis. Denoting the height of the particle projected vertically upwards with velocity v_o from the origin of the frame by h, we have

$$\left.\begin{aligned} v&=v_0-gt \\ h&=v_0t-\tfrac{1}{2}gt^2 \\ \text{and} \qquad v^2-v_0^2&=-2gh \end{aligned}\right\} \qquad \ldots (4.6)$$

by replacing 'a' by $-g$ in Eqs. 4.3, 4.4 and 4.5. Note carefully that these equations are true only when the distance of the particle above the surface of the earth is negligible in comparison to the radius of the earth.

If the particle is dropped from a certain height and the frame of reference is fixed at the point from where it is dropped then the distance described and the acceleration due to gravity are both negative.

We have then,

$$v=-gt$$
$$h=-\tfrac{1}{2}gt^2$$
$$v^2=2(-g)(-h)=2gh$$

by replacing 'a' by $-g$ and S by $-h$ in Eqs. 4.3, 4.4 and 4.5 and further setting $v_0=0$ because the body is dropped. The minus sign in the first two equations indicates that v and h are downward.

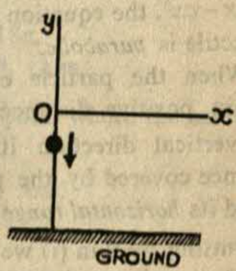


Fig. 4.1

4.5. Projectiles

An example of two-dimensional motion with constant acceleration is the *projectile motion.* The motion of projectile is one of constant acceleration g, directed downward and no horizontal acceleration. Let a particle be projected with velocity v_0 at inclination θ_0 with the horizontal from the point O. Let us erect a frame of

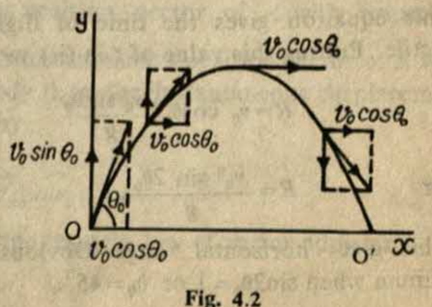


Fig. 4.2

reference at O with x-axis along the horizontal direction and y-axis along the vertically upward direction. Then initial velocities along x-direction and y-direction are respectively $v_0\cos\theta_0$ and $v_0 \sin\theta_0$. The above kinematic Eqs. 4.3, 4.4 and 4.5 hold good independently for every direction.

Since there is acceleration in the downward direction we have from Eq. 4.4 for vertical direction,

$$y=(v_0 \sin\theta_0)t-\tfrac{1}{2}gt^2 \quad \text{.. (i)}$$

$$(\because \quad a=-g \text{ in the } y\text{-direction})$$

and for horizontal direction we have,

$$x=(v_0 \cos\theta_0)t \quad \text{.. (ii)}$$

$$(\because \quad a=0 \text{ in the } x\text{-direction}).$$

Taking value of t from (ii) and putting in (i) we have,

$$y=x \tan \theta_0-\tfrac{1}{2}g\frac{x^2}{v_0{}^2 \cos^2\theta_0}.$$

This relates y to x and is the equation of the trajectory of the projectile. Since v_0 and g are constants, this equation is of the form, $y=bx-cx^2$, the equation of a parabola. Hence the trajectory of a projectile is *parabolic*.

When the particle comes down to the ground it would cover certain positive distance along the horizontal direction, but along the vertical direction its displacement component is zero. This distance covered by the particle along the horizontal direction is called its *horizontal range* or simply *range* denoted by R.

Putting $y=0$ in (i) we have,

$$(v_0 \sin \theta_0)t-\tfrac{1}{2}gt^2=0.$$

$$\because \quad t\neq 0,$$

$$\therefore \quad t=\frac{2v_0 \sin\theta_0}{g}. \quad \text{.. (iii)}$$

This equation gives the time of flight from O to O' by the projectile. Putting this value of t in (ii) we have,

$$R=v_0 \cos\theta_0\frac{2v_0 \sin\theta_0}{g}$$

or

$$R=\frac{v_0{}^2 \sin 2\theta_0}{g}. \quad \text{(iv)}$$

This gives horizontal range. Obviously the horizontal range is maximum when $\sin2\theta_0=1$ or $\theta_0=45°$.

Examples

1. *A balloon is ascending at the rate of* 12 ms^{-1} *at a height* 80 *m above the ground when a package is dropped. How long does it take the package to reach the ground ?*

Sol. Let us fix up a reference frame on the level of balloon and take the downward direction as positive.

Here $v_0 = -12 \text{ ms}^{-1}$ and $h = 80$ m.

$\therefore$ $80 = -12t + \frac{1}{2} \times 9{\cdot}8t^2 \quad (\because g = 9{\cdot}8)$

or $80 = -12t + 4{\cdot}9\, t^2$

or $4{\cdot}9\, t^2 - 12\, t - 80 = 0$ or $t = 5{\cdot}445$ s. **Ans.**

2. *A car moving with constant acceleration covers the distance between two points 54 m apart in 6 s. Its speed, as it passes the second point, is* 15 ms^{-1}. (*a*) *What is its speed at the first point ?* (*b*) *What is its acceleration ?* (*c*) *At what prior distance from the first point was the car at rest ?*

Sol. We have from Eq. 4.3, $15 = v_0 + 6a$
and from Eq. 4.4,

$$54 = 6v_0 + \tfrac{1}{2}a\, 6^2 = 6v_0 + 18a.$$

Hence, $v_0 = 3 \text{ ms}^{-1}$; $a = 2 \text{ ms}^{-2}$. **Ans.**

If S is the prior distance from the first point where the car was at rest, then by Eq. 4.5,

$$3^2 = 2 \times 2\ S \text{ or } S = 2{\cdot}25 \text{ m. } \textbf{Ans.}$$

4.6. Relative Velocity

The relative velocity of a body B with respect to another body A is the rate of change of displacement of B relative to A and similarly the rate of change of displacement of A relative to B is called the Relative Velocity of A relative to B.

If $\vec{r}_A$ is the instantaneous position vector of A with respect to some fixed body and $\vec{r}_B$ is the instantaneous position vector of B with respect to the same fixed body then the instantaneous displacement of B relative to A is given by

$$\vec{r}_{BA} = \vec{r}_B - \vec{r}_A. \quad \text{.. (i)}$$

This follows easily from the triangle law of vector addition.

Let $\overrightarrow{OA}=\vec{r}_A$ and $\overrightarrow{OB}=\vec{r}_B$ in Fig. 4.3 (a).

Then by the triangle law of vector addition,

$$\overrightarrow{AB}=\overrightarrow{AO}+\overrightarrow{OB}=-\overrightarrow{OA}+\overrightarrow{OB}$$
$$=\overrightarrow{OB}-\overrightarrow{OA}$$

or $$\vec{r}_{BA}=\vec{r}_B-\vec{r}_A.$$

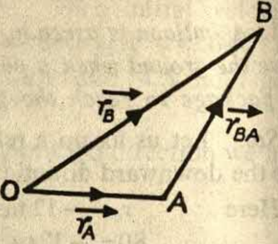


Fig. 4.3 (a)

Differentiating with respect to t, $\dfrac{d\vec{r}_{BA}}{dt}=\dfrac{d\vec{r}_B}{dt}-\dfrac{d\vec{r}_A}{dt}$.

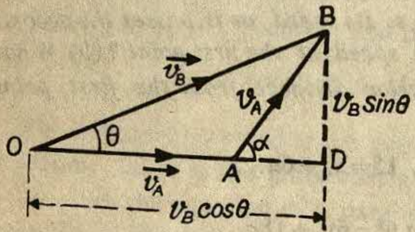


Fig. 4.3 (b)

By definition, $\dfrac{d\vec{r}_{BA}}{dt}$ is the relative velocity of B relative to A and $\dfrac{d\vec{r}_B}{dt}$ and $\dfrac{d\vec{r}_A}{dt}$ are the actual velocities of B and A respectively.

$$\therefore \quad \vec{v}_{BA}=\vec{v}_B-\vec{v}_A \quad \text{or} \quad \vec{v}_B+(-\vec{v}_A) \qquad .. \quad (4.7)$$

Thus analytically the relative velocity is the *vector difference* between the actual velocity of the body and the velocity of the referred body or it is velocity of the body summed with the reversed velocity of the referred body as if it is at rest.

To find the relative velocity graphically, represent $\vec{v}_A$ by $\overrightarrow{OA}$ and $\vec{v}_B$ by $\overrightarrow{OB}$ (Fig. 4.3 b). Join A with B. Then AB represents the relative velocity of B relative to A.

From the property of a triangle if $|\overrightarrow{AB}|$ is the absolute value of the side AB and θ is the angle between $\vec{v}_A$ and $\vec{v}_B$, then

$$|\vec{v}_{BA}|=\sqrt{v_A^2+v_B^2-2v_A v_B\cos\theta}. \qquad .. \quad (4.7a)$$

If α is angle made by the relative velocity of B relative to A with the direction of motion of A, then

$$\tan\alpha=\frac{v_B\sin\theta}{v_B\cos\theta-v_A}.$$

RELATIVE VELOCITY IN TERMS OF COMPONENTS OF VELOCITIES.

Suppose $\vec{v}_A = v_{xA}\hat{i} + v_{yA}\hat{j}$

and $\vec{v}_B = v_{xB}\hat{i} + v_{yB}\hat{j}$

then, $\vec{v}_{BA} = \vec{v}_B - \vec{v}_A = (v_{xB} - v_{xA})\hat{i} + (v_{yB} - v_{yA})\hat{j}$

$$\therefore \left|\vec{v}_{BA}\right| = \sqrt{(v_{xB} - v_{xA})^2 + (v_{yB} - v_{yA})^2} \quad .. \quad (4.7\ b)$$

and its direction with x-axis is

$$\tan \theta_x = \frac{v_{yB} - v_{yA}}{v_{xB} - v_{xA}}.$$

Examples

1. *To a man walking at the rate of* 3 *kph the rain appears to fall vertically downward. When he increases his speed to* 6 *kph, it appears to meet him at an angle of* 30° *with the downward vertical. Find the real direction and the speed of the rain.*

Sol. Let us solve it first graphically and then analytically. Let $\vec{OA}$ represent the velocity of the man and $\vec{OB}$ represent the actual velocity of rain. Then $\vec{AB}$ is the relative velocity of the rain relative to the man. According to the question $\angle OAB = 90°$. In the second figure $\left|\vec{OA}\right| = 6$ and $\vec{AB}$ is the relative velocity of rain. By the question $\angle OAB = 60°$. Let $\left|\vec{OB}\right| = u$. Then from Fig. 4.4(a), $3 = u \cos\theta$ where θ is the angle made by the direction of the rain with the horizontal.

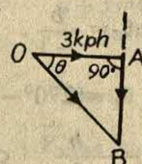


Fig. 4.4 (a)

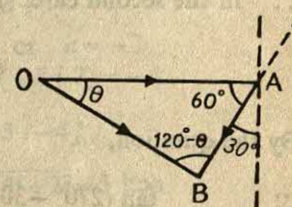


Fig. 4.4 (b)

In Fig. 4.4(b) by the sine property of a triangle,

$$\frac{OB}{\sin 60°} = \frac{OA}{\sin(120° - \theta)}, \text{ or } \frac{u}{\frac{\sqrt{3}}{2}} = \frac{6}{\sin(120° - \theta)}.$$

Solving, we have $\theta = 60°$ and $u = 3 \sec 60° = 6$ kph.

Thus the actual velocity of the rain is 6 kph at an angle 60° with

the horizontal or 30° with the dawnward vertical. The rain strikes the man from behind.

Let us take the horizontal direction to the right as x-axis and the vertically upward direction as y-axis.

In the first case $\vec{V}_{man}=3\hat{i}$.

Let us assume that $\vec{V}_{rain}=a\hat{i}+b\hat{j}$.

Then $\vec{V}_{rain-man}=(a-3)\hat{i}+b\hat{j}$

$\therefore \quad \tan\theta_x=\dfrac{b}{a-3}$.

By the question $\theta_x=270°$; $\therefore \quad a-3=0$ or $a=3$.

$\therefore \quad \vec{V}_{rain}=3\hat{i}+b\hat{j}$.

In the second case,

$$\vec{V}_{man}=6\hat{i}.$$

$\therefore$ In the second case $\vec{V}_{rain-man}=-3\hat{i}+b\hat{j}$

$$\therefore \quad \tan\theta_x=\frac{b}{-3}.$$

By the question, $\theta_x=(270°-30°)$

$\therefore \quad \tan(270°-30°)=\dfrac{b}{-3}$ or $b=-3\sqrt{3}$

$\therefore \quad \vec{V}_{rain}=3\hat{i}-3\sqrt{3}\,\hat{j}$

$\therefore \quad |\vec{V}_{rain}|=\sqrt{3^2+(3\sqrt{3})^2}=\sqrt{36}=6$ kph.

Let α be the angle made by the direction of the rain with the x-axis,

Then $\tan\alpha=\dfrac{-3\sqrt{3}}{3}$. Since the perpendicular is negative and the base is positive, α is an angle in the fourth quadrant. Therefore, $\alpha=300°$. Thus the rain is of speed 6 kph. It strikes the man at an angle of 60° with the horizontal from behind. **Ans.**

2. *To a man running at 7 kph due west, the wind appears to blow from the north west, but when he walks at 3 kph due west, the wind appears to blow from the north. What is the actual direction and velocity of the wind ?*

Sol. Let us take eastward as x-axis and y-axis towards the north.

In the first case $\qquad \vec{V}_{man} = -7\hat{i}.$

Let $\qquad \vec{V}_{wind} = a\hat{i} + b\hat{j}$

Then, $\qquad \vec{V}_{wind-man} = (a+7)\,\hat{i} + b\,\hat{j}$

$$\tan\theta_x = \frac{b}{a+7} = \tan(360° - 45°) = -1$$

or
$$b = -a - 7 = -(a+7)$$

$$\therefore \quad \vec{V}_{wind} = a\,\hat{i} - (a+7)\,\hat{j}.$$

In the second case, $\qquad \vec{V}_{man} = -3\,\hat{i}$

$$\therefore \quad \vec{V}_{wind-man} = (a+3)\,\hat{i} - (a+7)\,\hat{j}$$

$$\therefore \quad \tan\theta_x = \frac{-(a+7)}{(a+3)} = \tan 270°$$

$$\therefore \quad a + 3 = 0 \quad \text{or} \quad a = -3$$

$$\therefore \quad b = -(-3+7) = -4$$

$$\therefore \quad \vec{V}_{wind} = -3\,\hat{i} - 4\,\hat{j}$$

$$\therefore \quad \left|\vec{V}_{wind}\right| = \sqrt{4^2 + 3^2} = 5 \text{ kph}$$

$$\tan\theta_x = \frac{-4}{-3}.$$

Since both the perpendicular and the base are negative, θ_x is an angle in the third quadrant. Therefore, $\theta_x = 180° + 53°8'$. Thus the wind is of velocity 5 kph in the direction 53° 8′ north of east. **Ans.**

4.7. Momentum

Momentum is a measure of 'quantity of motion' of a body. Suppose a small iron sphere and a big one are set in motion with the same speed on the floor. It is a matter of common experience that the smaller one can be stopped with little effort, whereas to stop the bigger one we have to make a greater effort and so we say that the quantity of motion in the bigger body is greater. The quantity of

'mass' in the bigger body is also greater. Thus the quantity of motion is directly proportional to the 'mass' of the body. Now if the balls are of the same mass and are set in motion with different speeds then the one having the greater velocity will have the greater amount of motion. Thus *Momentum of a body in motion is the physical quantity possessed by virtue of its 'mass' and 'velocity'.*

If p is the momentum of a body of mass 'm' and velocity v then,

$p=kmv$ where k is a constant.

By taking momentum of a body of unit mass moving with unit velocity as unit momentum the constant is reduced to 1. Since velocity is a vector, momentum is also a vector.

$$\therefore \quad \boxed{\vec{p}=m\vec{v}} \ \mathrm{kg\ m\ s^{-1}}. \qquad \text{.. (4.8)}$$

The unit of momentum in SI is kg m s^{-1} because mass is in kg and velocity is in metre per second (m s^{-1}).

4.8. Newton's Laws of Motion

In kinematics we study the motion of a particle with the emphasis that motion (uniform or accelerated) of the particle does exist without bothering as to what 'causes' the motion. The concept of 'cause' of motion was given by Newton through the precise definition of the term 'force'. In our everyday language force is associated with a push or pull exerted by our muscles but precisely what is 'force' was given by the ever great scientist Sir Isaac Newton through his famous laws of motion.

The three laws are :

First Law. Every body continues in its state of rest or uniform motion in a straight line unless it is compelled to change that state by a force impressed upon it.

Second Law. The rate of change of momentum is proportional to the impressed force and the change takes place in the direction in which force acts.

Third Law. To every action there is an equal and opposite reaction.

4.9. The First Law

The first part of the law asserts something about the behaviour of material bodies. It tells us that material bodies on their self-instigation cannot change their state of rest or state of uniform motion in a straight line unless a force acts on them. This fact that material bodies stay at rest or retain their uniform motion in a straight line in the absence of a force or forces is often described by assigning a property to matter called Inertia. Thus Newton's first law of motion may also be termed as the *law of inertia.*

There are two implications in the law. In the first place this law must be taken as a statement about reference frames. For, in general, the motions of a body depends on the reference frame relative to which it is measured. The first law tells us that if no force acts, the body will either be at rest or in uniform linear motion. If a frame of reference is fixed in the body, it will appear at rest and if it is observed from a frame moving with uniform velocity the body will appear to move in a straight line. In both the frames the law of inertia i.e. 'no force no acceleration' holds. If a body is at rest, its acceleration is zero and if it is in motion in a straight line with uniform speed then also its acceleration is zero. By the first law of motion, a force is the cause of change of motion in a straight line or state of rest and hence acceleration which is the rate of change of velocity must be due to force. Thus the law of inertia i.e. the first law of motion may be put in this way—'no force, no acceleration'. All the frames in which this law holds good are called inertial frames. Inertial frames are the frames at rest or moving with uniform velocity.

The second implication of the law is that it makes no distinction between the absence of all forces and the presence of forces whose resultant is zero. Hence the law of inertia should correctly read like this—'If no net force acts on a body, its acceleration is zero'.

It is interesting to cite a few common examples of the law of inertia—(i) When a train suddenly starts, passengers lean opposite to the direction in which the train starts to move. This is due to the inertia of the body. Due to the sudden start of the train the lower part of the body in contact with the train comes in motion whereas the upper part due to inertia for rest stays at rest. Thus there is relative displacement of the two parts of the body and consequently the passengers lean to the opposite direction.

(ii) To remove dirt from a coat it is hit by a stick. On hitting the coat by the stick it comes in motion but dirt due to inertia remains at rest and so dirt gets detached from the coat and falls off.

(iii) While taking a long jump an athlete runs for a while and then takes a jump. By running for a while he gains inertia of motion and that helps him to take a longer 'jump'.

(iv) While alighting from a slowly moving train one must run for a short while in the direction of the train and then let off the train. When he sets his foot on the ground, the lower part of his body comes to rest instantaneously but the upper part of the body continues to move in the direction of the train. Due to relative displacement he is liable to fall forward and hurt himself. If he runs for a short while all the parts of the body will be in the same state and hence there will be no relative displacement of the different parts of the body.

4.10. Force : Environment

Newton's first law of motion precisely defines force. *Force is that external cause which acting on a body changes or tends to change its state of rest or uniform motion in a straight line.* A force on a body arises due to the presence of other bodies called its *environment*. The force exerted by the environment may be direct when the body and its environment are in contact ** or may be from a distance called 'field force'. In general, only nearby objects need to be considered in the environment, the effects of distant objects usually are negligible. The number of forces acting on a body will be as many as the number of bodies in the environment. When a body lies on a table there are two bodies in the environment—one is the table and other is the earth. The table will exert a direct force on it and the earth will exert a field force, namely, the gravitational force. The motion of a body depends on the resultant of all the forces exerted by the bodies in its environment and *not* on the force exerted by it on

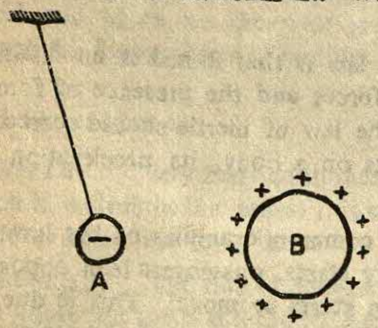


Fig. 4.5

** Called the 'contact force'.

bodies in the environment. An artificial satellite has only one body in its environment and that is 'the earth' and hence the only force acting on it is the gravitational field force. A charged spherical ball (A) is suspended by a thread near another heavy, charged body (B). Here there are three bodies in the environment of the body A—the earth, the thread and the charged body. Note that the ceiling is not* in the environment of the body A. It is the environment of the string. So there are three forces on A—the tension of the string, the electric field force, and the gravitational field force.

Tension, Push, Pull : Tension is the force exerted by one part over the other part of a body subjected to an external force. Push and Pull are respectively the forces directed towards or away from the interior of a body.

4.11. The Second Law

Newton's second law of motion gives a measure of 'force'. Suppose that a force $\vec{F}$ acts on a body of mass m and its instantaneous velocity is $\vec{v}$. Then by the definition of momentum

$$\vec{p}=m\,\vec{v}.$$

The rate of change of momentum is given by,

$$\text{rate of change of momentum}=\frac{d\vec{p}}{dt}=\frac{d}{dt}\,(m\,\vec{v})=m\frac{d\vec{v}}{dt}=m\,\vec{a}$$

$$\left(\text{since } \vec{a}=\frac{d\vec{v}}{dt}\right).$$

By the second law $\quad \vec{F} \propto m\vec{a}$

or $\quad \vec{F}=km\vec{a}.$

By selecting unit force as the force producing unit acceleration on a body of unit mass, the constant k is reduced to unity,

$$\therefore \quad \vec{F}=m\vec{a} \qquad \text{.. (4.9)}$$

$$\text{or} \quad \vec{F}=d\vec{p}/dt. \qquad \text{.. (4.9a)}$$

In words, $\boxed{\text{force}=\text{mass}\times\text{acceleration.}}$

In SI unit, the unit of force is kg m s^{-2} because the mass is in kg and the acceleration is in m s^{-2}. This is called a newton (abbreviated

* Will be considered as its environment if we want to take into account the gravitational field force of the ceiling. This is *however negligibly small and so it is not its environment.*

by N) after the name of Sir Isaac Newton. Thus the force of one newton *is that force which acting on a body of mass of one kilogramme produces an acceleration of one metre per second per second.*

This law gives us by implication another fundamental principle and that is the Physical Independence of Forces. The law tells us that a change in momentum takes place in the direction in which the force acts and it remains completely silent about the presence of other forces. Hence we may conclude that the effect of a force on a body will be independent of the presence of other forces. This is called *the physical inaependence of forces.* A horse-rider in a circus jumps up and again falls back on the horse. It seems to the audience that he plays a trick. In fact he plays no trick. He falls back on the horse due to a physical truth and that is the physical independence of forces. When he jumps up the only force acting upon on him vertically downward is the gravitational pull which does not affect the horizontal force (zero in magnitude) due to the physical independence of forces. Consequently his horizontal velocity, which was the same as the horse's remains unaltered. Since the two move in space with the same velocity there will be no relative displacement between the horse and its rider and so the rider falls back on the horse. For the same reason a stone dropped from the top of the mast of a ship falls at the foot of the mast irrespective of the motion of the ship.

4.12. Newton's Third Law of Motion

Forces on a body arise from other bodies that make up its environment. The body itself is an environment of the other bodies and so the exertion of a force is a mutual affair. If a body A exerts a force on another body B, B also will exert the same force on A. In the third law of motion 'action' and 'reaction' refer to these two forces. The force that B exerts on A is the action of B on A and that exerted by A on B is the reaction force or vice versa. Either force may be considered the 'action' and the other 'reaction'. According to this law forces must arise in pair. A single isolated force is an impossibility according to this law. *Though action and reaction are equal and opposite they never neutralise each other because they act on two different bodies.* Two forces balance each other when they are equal, opposite and act on the same body. 'Action' is effective on one body, 'reaction' on the other body and hence they can never neutralise each other.

When we kick open a door we exert a force on the door. This is the action of the foot on the door and is effective on the door. The door opens; at the same time, the door exerts an equal and opposite force on the foot which is effective on the foot.

In our attempt to walk we exert force on the earth in an inclined way. This is our action on the earth and is effective on the earth, which being a large body remains unaffected by the tiny force. The earth exerts the same force on our body in the opposite direction. The horizontal component of this force is effective in producing a horizontal motion and the vertical component balances our weight.

Examples

1. *A train weighing* 450 *metric ton is travelling at the rate of* 80 *kph. The speed of the train is reduced to* 24 *kph in* 30 *s. Find the average retarding force on the train.*

Sol. 450 metric ton $=45\times10^4$ kg.

$$80 \text{ kph}=\frac{80\times1000}{60\times60}=\frac{200}{9}\text{ ms}^{-1}$$

$$24 \text{ kph}=\frac{24\times1000}{60\times60}=\frac{20}{3}=\frac{60}{9}\text{ ms}^{-1}$$

$$v=v_0+at$$

$$\frac{60}{9}=\frac{200}{9}+a.\ 30, \text{ or } a=-\frac{14}{27}\times\text{ms}^{-2}$$

$\therefore$ Retarding force $=45\times10^4\times\frac{14}{27}=23{\cdot}33\times10^4$ newton. **Ans.**

2. *Find the thrust exerted on a vertical wall by the water from a fire-house which delivers water with a horizontal velocity of* 18 ms^{-1} *from a circular nozzle of* 5 *cm diameter. Assume the water not to rebound.*

Sol. Consider any interval of time $\triangle t$.

The mass of water flowing in $\triangle t=\pi\ (5/200)^2\times18\triangle t\times1000\ kg$

($\because$ Density of water $=1000$ kgm^{-3})

$$\text{The initial momentum}=\pi\left(\frac{5}{200}\right)^2\times18\ \triangle t\times1000\times18$$

The final momentum $=0$

$\therefore$ The change in momentum in $\triangle t$ sec.

$$=\pi\,(5/200)^2\,\triangle t\times 1000\times 18^2$$

$\therefore$ Thrust = rate of change of momentum

$$=\frac{\pi(5/200)^2\,\triangle t\times 1000\times 18^2}{\triangle t}$$

$$=\pi\,(5/200)^2\times 18^2\times 1000$$

$$=\frac{\pi 25\times 324}{40000}\times 1000$$

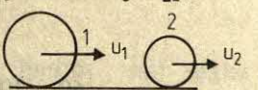
$$=\frac{\pi 25\times 324}{40}=636{\cdot}2 \text{ newton. } \textbf{Ans.}$$

4.13. Principle of Conservation of Linear Momentum and Newton's Laws of Motion

"Under the mutual action and reaction of two or more than two bodies, free from external forces, the algebraic sum of the linear momenta of bodies in any assigned direction remains conserved". This is a principle which is the direct outcome of Newton's laws of motion.

Let us consider the simple case of collision of two bodies. When two bodies collide one exerts force on the other. Let us denote the force exerted by 1 on 2 by F_{12} and that by 2 on 1 by F_{21}.

By the third law of motion



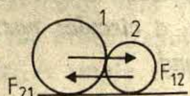
$$F_{12}=-F_{21}. \quad \text{.. (i)}$$

Now F_{12} is effective on 2 and F_{21} is effective on 1. Since no other forces act on the bodies we have, by the second law,

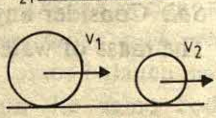
$$F_{12}=\frac{m_2v_2-m_2u_2}{\triangle t}.$$

where v_2 and u_2 are respectively velocities of 2 after and before collision and $\triangle t$ is the duration of collision and

$$F_{21}=\frac{m_1v_1-m_1u_1}{\Delta t}.$$



Substituting F_{12} and F_{21} in (i)



$$\frac{m_2v_2-m_2u_2}{\triangle t}=-\frac{m_1v_1-m_1u_1}{\triangle t}$$

or
$$m_1u_1+m_2u_2=m_1v_1+m_2v_2$$

i.e. momentum before collision = momentum after collision.

Now let us consider a *system* of *n* particles. We assume that the

particles may interact with each other and external forces may also act on the system. According to Newton's second law of motion, $\vec{F}=\frac{d\vec{p}}{dt}$ where $\vec{F}$ is the total force on the system and $\vec{p}$ is the total momentum of the system at any instant.

Now, $\vec{F}=\vec{F}_{external}+\vec{F}_{internal}$.

Again, $\vec{F}_{internal}=\vec{f}_{12}+\vec{f}_{21}+\vec{f}_{13}+\vec{f}_{31}+\vec{f}_{23}+\vec{f}_{32}+\ldots\ldots$
$=0$ because by the third law of motion action and reaction are equal and opposite. That is,

$$\vec{f}_{12}=-\vec{f}_{21}, \vec{f}_{13}=-\vec{f}_{31} \text{ and so on.}$$

$$\therefore \quad \vec{F}_{external}=\frac{d\vec{p}}{dt}.$$

But if no external force acts, then

$$\frac{d\vec{p}}{dt}=0 \text{ or } \frac{dp}{dt}=0, \text{ or } \vec{p}=\text{a constant.}$$

Thus in general if no external forces act on a system of particles, under the mutual action and reaction of the particles, the momentum of the system in any direction remains conserved.

When a shot is fired, the cannon recoils. This is an example of the principle of conservation of momentum. Here the system (shot + cannon) is at rest with respect to the frame of reference fixed to the earth. When a shot is fired, it gains a certain velocity, say, v and by the principle of conservation of momentum the cannon must also acquire some velocity in the opposite direction so that the algebraic sum of the momenta becomes zero. Therefore if m and M are masses of shot and cannon respectively, we have

$$mv+MV=0, \text{ or } mv=-MV, \text{ or } V=-mv/M.$$

The minus sign shows that V must be necessarily opposite to v. This velocity is called the 'velocity of recoil'.

When we throw a ball up from the earth and catch it again the conservation of momentum appears to be confusing.

Consider the earth and the ball as a system and the man throwing the ball to be a part of the earth as he does not loose contact with it. The force exerted by man on the ball and that of the ball on the man may be considered as mutual action and reaction between the earth and the ball. When the ball is in space the earth pulls the ball and the ball also attracts the earth towards it. So if we neglect the air

resistance, the system may be considered free from any external force.

$\therefore \quad 0 = m_b v_b + m_e v_e$ where 'b' stands for the ball and 'e' stands for the earth

or, $$m_b v_b = -m_e v_e$$

where v_b and v_e are the velocities of the ball and the earth respectively in a frame in which the system is at rest. So by the principle of conservation of momentum an observer in the frame will see the earth recoiling away from the ball. Owing to the enormous mass of the earth in comparison to the ball, the recoil speed of the earth is negligibly small. As the ball and the earth, separate out, the gravitational pull decelerates their motion until they cease separating and their motion is accelerated towards each other, the ball moving much more rapidly than the earth. As the ball is caught, its momentum is neutralised by the momentum of the earth.

The modern 'jet planes' and 'rockets' work on the third law of motion.

4.14. 'Free-body Diagram' and Solution of Problems

Newton's third law of motion often creates confusion to average students in solving problems. To remove this a scheme is available which is called the 'Free-body diagram'. In solving any problem by this scheme, follow the following steps—1. First fix up the body whose motion or equilibrium position you want to consider. 2. Next see carefully what are the objects in '*the environment*' because these objects (planes, springs, cords, the earth etc.) exert forces on the body. 3. Now make a separate diagram of the body alone, showing all the forces acting **on** the body and dropping the forces exerted by the body on the bodies in its environment. 4. Finally apply Newton's second law of motion.

ILLUSTRATIONS

(a) *Suppose a block 'A' of mass m_1 is placed on a horizontal (smooth) surface and pulled by a string attached to a block B of mass m_2 hanging over a pulley.*

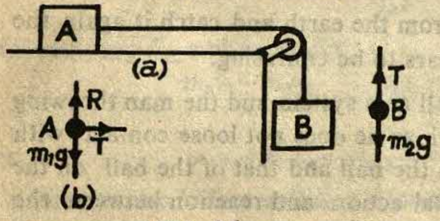


Fig. 4.6

Suppose we choose A as body in point (step 1). The bodies in the environment are (i) the string pulling A to the right, (ii) the 'plane' exerting a force R on A upward, (iii) the earth exerting a force m_1g

on A (step 2). In the third step consider the free body diagram of A. This is shown in the fig. 4 6 (b). Under the action of these three forces the body is moving to the right with an acceleration a_1 (say).

$\therefore$ $\quad R-m_1g=0$ ($\because$ there is no motion along the vertical)

or, $\quad R=m_1g$

and $\quad T=m_1a_1$.

Next take the body B. The bodies in the environment of B are : (i) the string exerting force T upward, (ii) the earth exerting force m_2g downward. Under the action of these two forces B is going down with the acceleration, say, a_2.

$$\therefore \quad m_2g-T=m_2a_2.$$

If the string is inextensible then $a_1=a_2=a$ (say).

$\therefore$ $\quad T=m_1a$

and $\quad m_2g-T=m_2a.$

Adding we have $\quad a=\dfrac{m_2g}{(m_1+m_2)}$

and $\quad T=\dfrac{m_1m_2g}{(m_1+m_2)}.$

(b) *Consider two unequal masses connected by a string which passes over a frictionless and massless pulley.*

By the process explained above the free-body diagrams of the two bodies are shown in Fig. 4.7 (b) and Fig. 4.7 (c). By Newton's second law of motion,

$$T-m_1g=m_1a_1$$

and $\quad m_2g-T=m_2a_2.$

When the string is inextensible

$$a_1=a_2=a.$$

$\therefore$ $\quad T-m_1g=m_1a$

$$m_2g-T=m_2a.$$

Adding we have,

$$(m_2-m_1)g=(m_1+m_2)a$$

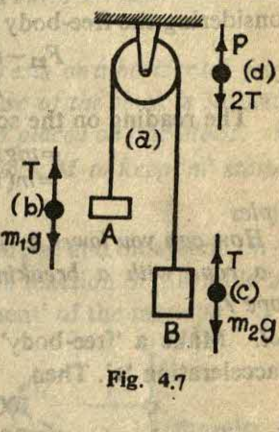


Fig. 4.7

or $\quad a=\left(\dfrac{m_2-m_1}{m_1+m_2}\right)g.$

Substituting the value of a in either equation we have,

$$T=\frac{2m_1m_2}{m_1+m_2}\,g$$

Fig. 4.7 (d) shows the 'free-body' diagram of the pulley. Applying the second law of motion to the pulley

$$P-2T=0$$

$$P=2T=\frac{4m_1m_2}{m_1+m_2}\,g.$$

(c) *The Apparent weight of a body in a lift (elevator).*

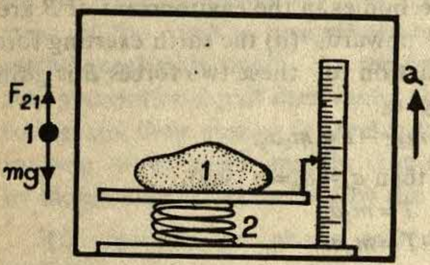


Fig. 4.8

Suppose that an elevator is moving vertically upward with an acceleration 'a' and a stone of m kg is placed on the platform of a weighing machine placed on the floor of the elevator. One point that must be clearly understood is that the scale reading depends on the force with which the spring of the scale is compressed.

Suppose F_{12} is the force exerted by the stone on the spring and F_{21} is the force by the spring on the stone.

The reading on the scale $=F_{12}$.

By Newton's third law, $F_{12}=F_{21}$ (numerically).

Considering the free-body diagram of the stone we have,

$$F_{21}-mg=ma$$

or

$$F_{21}=mg+ma.$$

$\therefore$ The reading on the scale (apparent weight) $=mg+ma$

$=mg(1+a/g)$ newton

$=m(1+a/g)$ kgf (kilogramme force).

Examples

1. *How can you lower a 100 kg body from the roof of a house using a rope with a breaking strength of 80 kg without breaking the rope ?*

Sol. Make a 'free-body' diagram of the body. Let it go down with acceleration 'a'. Then

$$100\text{ g}-T=100a.$$

Here T can be at most $80g$ newton.

$$\therefore \quad 100g-80g=100a.$$

or

$$a=\frac{g}{5}=9{\cdot}8/5=1{\cdot}96\text{ ms}^{-2}.\ \textbf{Ans.}$$

2. *A* 10 *kg monkey is climbing a massless rope attached to a* 15 *kg mass after it is passed over a frictionless pulley. Explain quantitatively how the monkey can climb up the rope so that he can raise the* 15 *kg mass off the ground. If after the mass has been raised off the ground, the monkey stops climbing and holds on to the rope, what will be his acceleration and the tension in the rope ?*

Sol. Make a free-body diagram of the monkey and the mass. For 'just lifting of the mass', its acceleration is zero.

$\therefore\ T-15g=0$, or $T=15g$

and $T-10g=10a$, or $15g-10g=10a$

or $\quad a=\dfrac{g}{2}=4{\cdot}9\ \text{ms}^{-2}$.

Thus the monkey must climb up the rope with an acceleration of $4{\cdot}9\ \text{ms}^{-2}$. When he holds on to the rope he will go up and the mass will go down with a common acceleration given by,

$$a=\frac{15-10}{15+10}g=\frac{9{\cdot}8}{5}=1{\cdot}96\ \text{ms}^{-2}$$

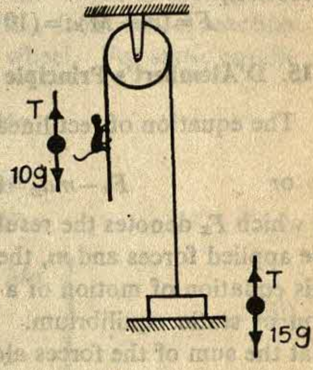


Fig. 4.9

and $$T=\frac{2\times15\times10}{15+10}\times9{\cdot}8=117{\cdot}6\ \text{N. Ans.}$$

3. *A frictionless block of mass* 10 *kg rests on a horizontal frictionless table. The inclination of the hypotenuse of the block is* 30° *with the table. A cubical block of mass* 1 *kg is placed on the block. What horizontal force must be applied on the block M to keep 'm' stationary on the block ?*

Sol. Make 'free-body diagrams of M, m and the system $(m+M)$. R is the reaction of m on M and N is the reaction of the table on M. Note that the table is not the 'environment' of the mass 'm'.

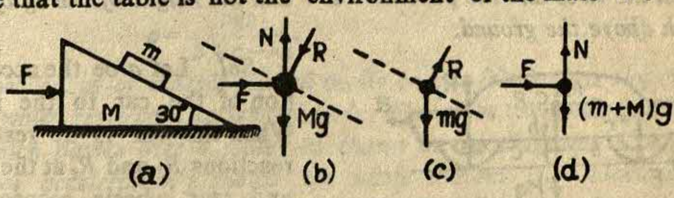


Fig. 4.10

Considering the motion of 'm' along the horizontal direction and the vertical direction,

$$R\cos 30° - mg = 0 \text{ (as there is no vertical acceleration of } m)$$

and $$R\sin 30° = ma;$$

$$\therefore \quad \frac{R\cos 30°}{R\sin 30°} = \frac{mg}{ma}, \text{ or } a = g.\frac{1/2}{\frac{\sqrt{3}}{2}} = \frac{9{\cdot}8}{\sqrt{3}} = 5{\cdot}66 \text{ ms}^{-2}.$$

Considering the motion of the whole system along the horizontal direction,

$$F = (m+M)a = (10+1)\times 5{\cdot}66 = 62{\cdot}26 \text{ newton. Ans.}$$

4.15. D'Alembert's Principle

The equation of rectilinear motion of a particle is in the form

$$F_x = ma_x$$

or $$F_x - ma_x = 0, \text{ or } F_x + (-ma_x) = 0$$

in which F_x denotes the resultant, in the direction of the x-axis, of all the applied forces and m, the mass of the particle. We see at once that this equation of motion of a particle is of the same form as an equation of static equilibrium. The condition for static equilibrium is that the sum of the forces along a given direction must vanish. Thus the above equation suggests that equations of motion could be written as equilibrium equations simply by introducing a fictitious force $(-ma_x)$ called the 'Inertia Force' of the particle. This idea is known as D'Alembert's principle and is very useful in the solution of many difficult problems in dynamics. This principle enables us to reduce a problem of a body in a moving frame to a body in a stationary frame simply by introducing inertia forces in addition to the *real forces* acting on the system.

Examples

1. *Find the maximum acceleration along a level road that a rear-wheel drive automobile can attain if the coefficient of friction between the tires and the pavement is* μ *and the height of the cg of the car is h above the ground.*

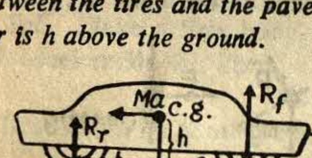


Fig. 4.11

Sol. Let a be the acceleration of the car to the right. When the car is at rest, the reactions R_f and R_r at the front and rear wheels respectively balance the weight of the car.

$$\therefore \quad Mg = R_f + R_r.$$

Also $R_f \times c = R_r \times b$ where b is the horizontal distance of the cg from the rear wheel and c is the distance of the cg from the front wheel.

$$\therefore \quad R_f = \frac{b}{b+c} Mg \quad \text{and} \quad R_r = \frac{c}{b+c} . Mg. \quad \quad .. \quad (i)$$

When the car is in motion with an acceleration a, we can still treat it to be at rest by introducing the inertia force $(-Ma)$ at the centre of gravity of the car.

The real forces acting on the car are : the weight of the car Mg vertically downwards, rear wheel reaction R_r, front wheel reaction R_f' and the pull F of the engine on the rear wheel. For static equilibrium we have,

$$F - Ma = 0, \text{ or } F = Ma$$

and
$$Mg = R_f' + R_r'.$$

Also taking moment about the rear wheel

$$R_f'(b+c) + Mah = Mgb$$

$$\left.\begin{aligned} \therefore \quad R_f' &= \frac{b}{b+c} Mg - \frac{h}{(b+c)} Ma \\ \text{and} \quad R_r' &= \frac{c}{b+c} Mg + \frac{h}{b+c} Ma \end{aligned}\right\} . \quad \quad .. \quad (ii)$$

Comparing Eq. (i) and Eq. (ii) we see that owing to the acceleration of the car the reaction on the front wheels is decreased and that on the rear wheels increased, both by the same amount. From the fact that a large driving force F is dependent upon having a large reaction on the rear wheels, we conclude that a rear wheel drive is more efficient than a front-wheel drive. By the same reasoning, front-wheel brakes will be more efficient than rear-wheel brakes.

When slipping is prevented, $F = \mu R_r'$

$$\text{or} \quad Ma = \mu \left[\frac{c}{b+c} Mg + \frac{h}{b+c} Ma \right]$$

$$\text{or} \quad a = \frac{\mu gc}{b+c-\mu h} . \quad \quad .. \quad \text{(iii) Ans.}$$

2. *Two cubes of masses m_1 and m_2 lie on two frictionless slopes of block A which rests on a horizontal table. The cubes are connected by a string which passes over a pulley as shown in the diagram. To what horizontal acceleration f should the whole system (i.e. block and cubes) be subjected so that the cubes do not slide down the planes ?*

(I. I. T '78)

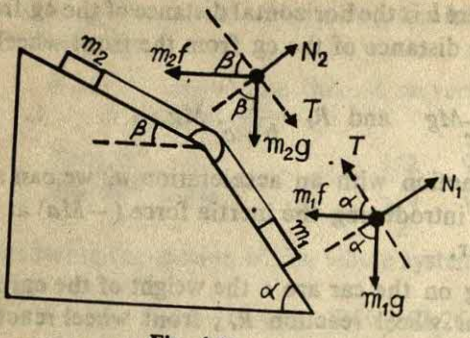


Fig. 4.12

By D' Alembert's principle we can treat the block stationary by simply adding the inertia forces on m_1 and m_2 in addition to the real forces acting on them. Make free-body diagrams of m_1 and m_2. Projecting all forces (including the inertia force) along and perpendicular to the planes we obtain

$$\left.\begin{aligned} N_1 &= m_1 f \sin\alpha + m_1 g \cos\alpha \\ T + m_1 f \cos\alpha &= m_1 g \sin\alpha \end{aligned}\right\} \quad \text{.. (i)}$$

and

$$\left.\begin{aligned} N_2 &= m_2 f \sin\beta + m_2 g \cos\beta \\ T + m_2 g \sin\beta &= m_2 f \cos\beta \end{aligned}\right\}. \quad \text{.. (ii)}$$

Eliminating T we obtain, $f = \dfrac{m_1 \sin\alpha + m_2 \sin\beta}{m_1 \cos\alpha + m_2 \cos\beta} \cdot g$. **Ans.**

4.16. Impulse and Impulsive Force

A great force acting for a very short interval of time on a body is called an impulsive force. There are numerous examples of impulsive forces. When we kick a ball, the force on the ball is an impulsive force. The force experienced by the coil of a moving coil galvanometer due to sudden flow of current is an impulsive force. According to Newton's second law,

$\bar{F} = m\bar{a} = \dfrac{m(v-u)}{\triangle t}$ where $\bar{F}$ is the average force during the short interval $\triangle t$ and $\bar{a}$ is the average acceleration produced. *The impulse of an impulsive force is the great change in momentum produced by it.*

Thus the impulse of an impulsive force = change in momentum

$$= \bar{F} \triangle t.$$

In fact an impulsive force is a variable force. The plot of an

mpulsive force is a curve which is zero in the beginning and the end of the interval showing a peak value at some intermediate instant.

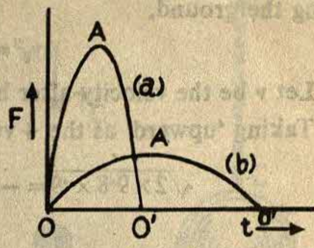


Fig. 4.13

The total impulse of the force during the interval for which it acts is given by the summation of the impulses from instant to instant. At the instant t the elementary impulse is $F.dt$.

$$\therefore \quad \text{total impulse} = \int_0^{\Delta t} F dt$$

$$= \text{area } OAO'$$

$$= \text{change in momentum.}$$

When a cricket ball is hit by a bat, it exerts an impulsive force on the ball and the impulse of the force on the ball is so great that it moves in the opposite direction with tremendous speed. If the same ball is to be stopped in the same interval Δt by a fielder he must apply the same average impulsive force of Fig. 4.13(a). The force being very large he (the fielder) will feel severe pain, particularly in bare hands. If he lowers his hand so as to lengthen the duration of the force, the same change of momentum will need a comparatively smaller force (Fig. 4.13 b) and so he will not feel the pinch of catching the ball.

For the same reason when a man jumps from a high wall on the side concrete road he hurts himself more seriously than when he jumps from the same wall on a heap of sand.

Examples

1. *A tennis ball of mass* 200 *gm is dropped onto the floor from a height of* 1·2 *m. It rebounds to a height of* ·9 *m. If the ball was in contact with the floor for* ·01 *s, what was the average impulsive force and what was its impulse ?*

Sol. Let v_0 be the velocity of the ball immediately before reaching the ground.

$$v_0^2 = 2 \times 9{\cdot}8 \times 1{\cdot}2.$$

Let v be the velocity after bouncing. $v^2 = 2 \times 9{\cdot}8 \times {\cdot}9.$

Taking 'upward' as the +ve direction

$$\sqrt{2 \times 9{\cdot}8 \times {\cdot}9} = -\sqrt{2 \times 9{\cdot}8 \times 1{\cdot}2} + \bar{a} \times {\cdot}01$$

(from $v = v_0 + at$)

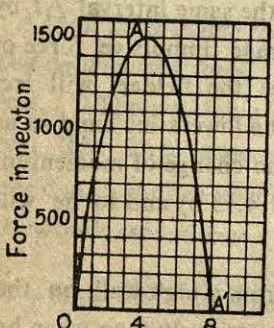
where a is the average acceleration (upward) during the interval.

or $$\bar{a} = \frac{4{\cdot}2 + 4{\cdot}85}{{\cdot}01} = \frac{9{\cdot}05}{{\cdot}01} = 905 \text{ m s}^{-2}.$$

$$\therefore \quad \bar{F} = m\bar{a} = \frac{200}{1000} \times 905 = 181 \text{ newton.}$$

$$\text{Impulse} = \bar{F}.\Delta t = 181 \times {\cdot}01 = 1{\cdot}81 \text{ km s}^{-1}. \text{ Ans.}$$

2. *A croquet ball (·5 kg) is struck by a mallet, receiving the impulse shown in the graph (fig. 4.14). What is the ball's velocity just after the force has become zero ?*



Time in milli second

Fig. 4.14

Sol. Each square represents $100 \times 1 \times 10^{-3} = {\cdot}1$ **N s.**

There are in all $42 \times 2 = 84$ squares (approximately) within OAA'.

$\therefore$ Area of OAA'

$= 84 \times {\cdot}1 = 8{\cdot}4$ newton second

= impulse experienced

= change in momentum

$= {\cdot}5 \times v - {\cdot}5 \times 0$

$= {\cdot}5\ v;$

$$\therefore \ v = \frac{8{\cdot}4}{{\cdot}5} = 16{\cdot}8 \text{ m s}^{-1}. \text{ Ans.}$$

4.17. Some Interesting Facts on Newton's Laws of Motion

(a) A sphere of mass m is supported by a cord C and an identical cord D is attached to the bottom of it. If you give a sudden

jerk to D it will break, but if you pull on D steadily C will break.

Let T_b be the breaking tension of the cords. Considering the free-body diagram of the sphere we have

$T_D + mg - T_C = ma$ where 'a' is the sudden large acceleration produced.

$\therefore \quad T_D - T_C = m(a - g).$

Since a is very large,

$\therefore \quad a > g$ and $T_D > T_C$.

Hence T_D will reach T_b before T_C can reach it and hence D will break when a jerk is given to D. When pulled steadily there is no acceleration.

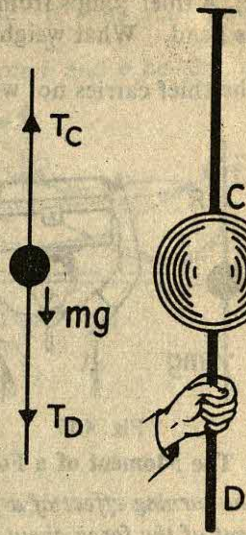


Fig. 4.15

$\therefore \quad T_D + mg - T_C = 0.$

$mg = T_C - T_D.$

$\therefore \quad T_C > T_D$. Hence now T_C will reach T_b before T_D can reach it. Therefore C will break.

(b) Two 2 kg weights are attached to a spring balance as shown in the Fig. 4.16. Does the scale read 0 kg, 2 kg or 4 kg ?

Considering the free-body diagram of the spring we have

$T - T' = 0$

or $\quad T = T'$

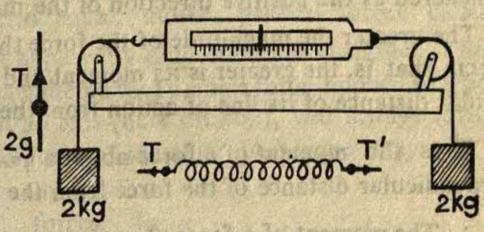


Fig. 4.16

and considering the free-body diagram of either body we have

$2g - T = 0$

or $\quad T' = T = 2g$ newton $= 2$ kg.

Honce the scale will read 2 kg.

(c) A thief jumps from the first floor of a building with a trunk on his head. What weight does he carry during fall ?

The thief carries no weight during fall. Actually the thief normally carries the force exerted by the box on his head. Let R be the action of box on the head of the thief. By Newton's third law the head of the thief exerts the same force on the box upward. Considering the 'free-body' diagram of the box (Fig. 4.17), $mg-R=mg$, or $R=0$.

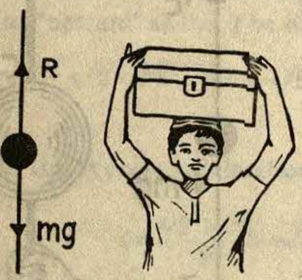


Fig. 4.17

4.18. The Moment of a Force

The turning effect of a force about a point or a line is called the moment of the force about that point or line called the axis of rotation.

The turning effect of a force is dependent on the magnitude of the force and the perpendicular distance of its line of action from the axis of rotation. The force will turn or tend to turn the body either in clockwise or anti-clockwise direction. The direction of the moment of the force is fixed by the direction in which the force turns or tends to turn the body. Generally the anticlockwise direction is considered as the positive direction of the moment of a force.

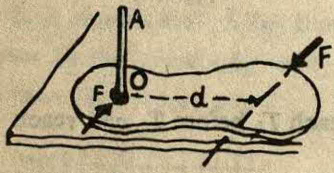


Fig. 4.18

The greater the magnitude of the force the greater is its turning effect, that is, the greater is its moment and also the greater perpendicular distance of its line of action from the axis of rotation.

Thus the moment of a force about a point or line $\propto F$, $\propto d$, the perpendicular distance of the force from the point or line.

$\therefore$ The moment of a force about a line or point$=kFd$ where k is a constant depending on the unit of moment. Taking the moment of a unit force from unit distance as unit moment, the constant of proportionality is reduced to 1.

$\therefore$ The moment of the force$=F.d$ newton metre (N m). In words, the moment of a force$=$the force$\times$its perpendicular distance from the axis of rotation.

The moment of a force is an axial vector. If $\vec{r}$ be the position vector of the point of application of the force $\vec{F}$ and θ be the angle between $\vec{r}$ and $\vec{F}$, then the perpendicular distance of the force is $r \sin\theta$.

$\therefore$ The moment of the force $= Fr \sin\theta$

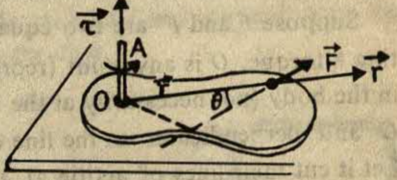


Fig. 4 19 (a)

or vectorially $\vec{\tau} = \vec{r} \times \vec{F}$. .. (4.9)

4.19. The Couple or Torque

Two equal unlike parallel forces form a couple or torque.

Any body which is free to rotate about a fixed point will experience a torque when a force is applied on it. In the Fig. 4. 19 (b) the external force applied on the body is F. An cqual amount of force is called into play at the fixed rod about which the body is free to rotate to balance the applied force. This is necessitated because the body has no translational motion. Thus the body automatically experiences two equal, unlike, parallel forces called a couple or torque, though apparently only one force is applied. Therefore it is always to be remembered that a single force, applied to a body capable of rotating about an axis (here the rod), will always form a torque with the reactional force of the axis. A torque may also be constituted by two externally applied equal unlike parallel forces. In Fig. 4.19 (c) a couple constituted by two externally applied equal unlike parallel forces is shown. The reactional forces of the two applied forces at the fixed rod (the axis of rotation) are equal and opposite and so they cancel out.

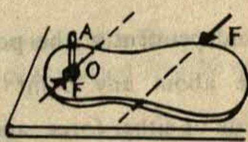


Fig. 4.19 (b)

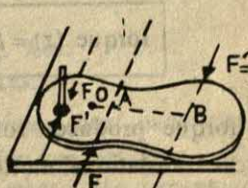


Fig 4.19 (c)

The effect of applying a torque on a body is always to rotate the body because a couple will always have an unbalanced moment about any point or axis of rotation.

Suppose F and F' are two equal unlike parallel forces constituting a torque. O is any point (represented by the dot in the fig. 4.19c) in the body (not necessarily at the axis itself). Draw a line through O and perpendicular to the line of action of F and F' [fig 4.19 (c)]. Let it cut their lines of action at A and B.

The moment of F about $O = F.OA$ (anticlockwise).

The moment of F' about $O = F'.OB$ (clockwise)

$= F.OB.$

$\therefore$ The net moment of the torque $= F.OA$ (anticlockwise)

$+ F.OB$ (clockwise)

$= F(OB - OA)$ (clockwise)

$= F.AB.$

This is independent of the position of O. Hence the moment of a torque about any point is the same and it is the product of the magnitude of either force and the perpendicular distance between the forces, sometimes referred to as *the arm of the torque*. A torque and the moment of the forces of the torque about any point which is a constant are treated synonymous. Therefore, when two equal unlike parallel forces each of magnitude F and distance d apart from each other act on a body we will say either a torque of moment $F{\cdot}d$ or simply torque of magnitude $F.d$ is acting on the body.

$\boxed{\text{torque } (\tau) = F.d}$ newton metre (N m).

A torque produces rotational motion in a body just as a force produces translational motion in it when it is free to move. This will be discussed in detail in the chapter on 'moment of inertia'.

Examples

1. *A door 2 m high and 1 m wide weighs 20 kg. A hinge 30 cm from the top and another 30 cm from the bottom each support half the door's weight. Assuming that the weight of the door acts at its geometrical centre, determine the horizontal and vertical force components exerted by each hinge on the door.*

Sol. The upper hinge will pull the door and the lower will push it.

Considering the horizontal and vertical motion (no acceleration along the two directions) $H-H'=0$

or $$H=H'$$

and $$V+V'=20g.$$

It is given that $V=V'=10g.$

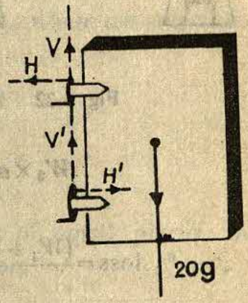


Fig. 4.20

Taking moments about the lower hinge,

$$20g\times{\cdot}5=H(2-{\cdot}6)$$

or $$H=\frac{10g}{1{\cdot}4}=70 \text{ newton.}$$

Ans. : At the upper hinge $V=98$ newton, $H=70$ newton.
At the lower hinge $V'=98$ newton and $H'=70$ newton.

2. *An automobile weighing 1350 kg has a wheel base of 3 m. Its centre of gravity is located 1·75 m behind the front axle. Determine the force exerted on each of the front wheels and the force exerted on each of the back wheels by the level ground.*

Sol. We consider the vertical motion of the car.

$2R_1+2R_2=1350g$ (because there is no vertical acceleration)

or $R_1+R_2=675g.$ .. (i)

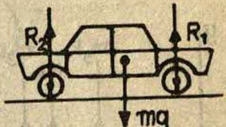


Fig. 4.21

Taking moments about the front wheels,

$$2R_2\times 3=1350g\times 1{\cdot}75..$$

$\therefore$ $R_2=393{\cdot}75g$ newton $=393{\cdot}75$ kg. **Ans.**

$R_1=675-393{\cdot}75=281{\cdot}25$ kg. **Ans.**

3. *A tradesman sells his articles weighing equal quantities alternately from the two arms of a balance having unequal arms. If the ratio of the lengths of the arms be 1·025, what is the percentage loss or gain ?*

Sol. A common balance is purely a 'moment affair' about the axis passing through the fulcrum of its beam. Let a and b be the lenghts of the arms of the balance. Let W be the weight and W_1 be the quantity of matter.

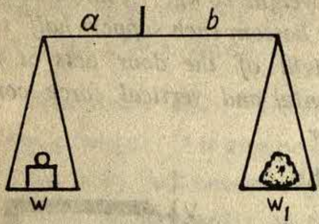


Fig. 4.22

Taking moments about the turning point, $W \times a = W_1 \times b$

or $W_1 = Wa/b$.

When interchanged,

$$W_2 \times a = W \times b, \text{ or } W_2 = \frac{Wb}{a}.$$

$$\therefore \quad \% \text{ loss} = \frac{(W_1 + W_2) - 2W}{2W} \times 100 = \frac{1}{2}\left(\frac{a}{b} + \frac{b}{a} - 2\right) \times 100$$

$$= \frac{1}{2}(1{\cdot}025 + \frac{1}{1{\cdot}025} - 2) \times 100$$

$$= {\cdot}03\%. \textbf{ Ans.}$$

4.20. Centre of Mass and Centre of Gravity

The centre of mass of a body is a point where the whole mass of the body may be supposed to be concentrated so far the action of a system of parallel forces acting on the elementary masses is concerned. The concept of centre of mass helps us to reduce the problem of a 'body' to the problem of a 'particle'.

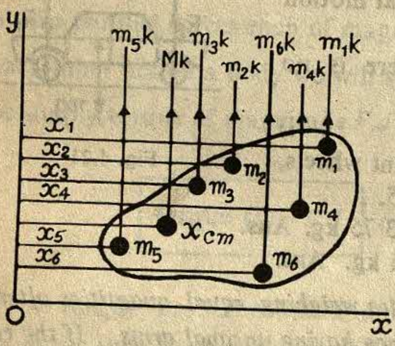


Fig. 4.23

Suppose $m_1, m_2, m_3 \ldots$ are the elementary masses of a body of mass M. Then obviously, $M = \Sigma m_i$.

Let $x_1, x_2, x_3 \ldots\ldots$ be the distances of the elementary masses from the y-axis of a frame of reference. Suppose that a system of parallel forces m_1k, m_2k, $m_3k \ldots$ act on the elementary particles parallel to the axis.

The resultant of the forces $= m_1k + m_2k + m_3k \ldots\ldots = k\Sigma m_i = Mk$.

Since the algebraic sum of the moments of the component forces about any point or line is equal to the moment of the resultant about the same point or line, the algebraic sum of the moments of the forces $m_1k, m_2k, m_3k \ldots\ldots$ about the origin is equal to the moment of their resultant.

$\therefore \quad Mk\, x_{cm} = m_1\, kx_1 + m_2kx_2 + m_3kx_3 + \ldots\ldots$ where cm stands for 'centre of mass'

or $$Mx_{cm} = m_1x_1 + m_2x_2 + m_3x_3 + \ldots\ldots$$

or $$x_{cm} = \left(\frac{\Sigma m_i x_i}{\Sigma m_i}\right) = \frac{\Sigma m_i x_i}{M}.$$

When the body is divided into large number of small pieces, the above summation may be replaced by integration.

$$\therefore \quad x_{cm} = \frac{\int x dm}{M} = \frac{1}{M}\int x dm.$$

Similarly $$y_{cm} = \frac{\Sigma m_i y_i}{\Sigma m_i}, \quad \text{or} \quad y_{cm} = \frac{1}{M}\int y dm;$$

$$z_{cm} = \frac{\Sigma m_i z_i}{\Sigma m_i}, \quad \text{or} \quad z_{cm} = \frac{1}{M}\int z dm.$$

Now, $\vec{r}_{cm} = x_{cm}\vec{i} + y_{cm}\vec{j} + z_{cm}\vec{k}$ where $\vec{i}, \vec{j}$ and $\vec{k}$ are unit vectors along the axes

$$= \left(\frac{\Sigma m_i x_i}{\Sigma m_i}\right)\vec{i} + \left(\frac{\Sigma m_i y_i}{\Sigma m_i}\right)\vec{j} + \left(\frac{\Sigma m_i z_i}{\Sigma m_i}\right)\vec{k}$$

$$= \frac{\Sigma m_i(x_i\vec{i} + y_i\vec{j} + z_i\vec{k})}{\Sigma m_i} = \frac{\Sigma m_i \vec{r}_i}{\Sigma m_i} = \frac{\Sigma m_i \vec{r}_i}{M}$$

or $$M\vec{r}_{cm} = \Sigma m_i \vec{r}_i = m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \ldots.$$

Differentiating with respect to time

$$M\frac{d\vec{r}_{cm}}{dt} = m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + m_3\frac{d\vec{r}_3}{dt} + \ldots\ldots$$

or $$M\vec{v}_{cm} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \ldots.$$

Differentiating with respect to time again,

$$M\frac{d\vec{v}_{cm}}{dt}=m_1\frac{d\vec{v}_1}{dt}+m_2\frac{d\vec{v}_2}{dt}+m_3\frac{d\vec{v}_3}{dt}+\ldots.$$

or $$M\vec{a}_{cm}=m_1\vec{a}_1+m_2\vec{a}_2+m_3\vec{a}_3+\ldots.$$

$$=\vec{f}_1+\vec{f}_2+\vec{f}_3+\ldots\ldots$$

The forces here include both internal forces and external forces. However, from Newton's third law, the internal forces will occur in pairs of equal and opposite forces, so that they contribute nothing to the sum. Hence internal forces can be removed from the problem of the body. The sum on the right hand side of the equation represents the sum of only the external forces acting on all the particles. We can rewrite the above quation as simply

$$M\vec{a}_{cm}=\vec{F}_{external}. \quad \ldots \quad (4.10)$$

Thus it is shown that the centre of mass of a system of particles moves as though all the mass of the system were concentrated at the centre of mass and all external forces were applied at that point.

If no external force acts on a system of particles, then,

$M\vec{a}_{cm}=0$ or $\vec{a}_{cm}=0$. This means acceleration of the centre of mass of the system is zero. Therefore, if the system is free from external force, under the mutual action and reaction of its constituent particles, the centre of mass of the system will remain stationary if it is already at rest, though there may be redistribution of the particles among themselves.

A sailboat cannot be propelled by air blown at the sails from a fan attached to the boat. Here the boat is 'the system' and sails and the fan are its components. Their 'action' and 'reaction' do not contribute to the sum of all the external forces acting on the system. Since the system is free from external force, its centre of mass will remain stationary.

Suppose a man is on a flat boat in still water and he walks on the boat away from the shore. Will the centre of mass of the boat move ? The centre of mass of the system (boat + man) will not move but the centre of mass of the boat will move towards the shore. Here boat and man are the constituent bodies of the system. When the man walks on the boat, he exerts a force on the boat and so the boat

moves towards the shore. The boat also exerts the same force on the man in the opposite direction. The two components individually experience force and hence they suffer displacement but the two bodies when considered as a compound body (the system) will experience no force and hence its (of the system) centre of mass will not move.

One very pertinent question here. If only an external force can change the state of motion of the centre of mass of a body how does it happen that the internal force of the brakes can bring a car to rest?

It is true that the internal force of the brakes cannot bring the car to rest. In fact the brakes bring the wheel to rest when enormous friction arises between the wheel and the ground. This frictional force of the ground on the wheel is the external force on the car which ultimately brings the car to rest.

Centre of Gravity :

The centre of gravity of a body or a system of particles rigidly connected together is a point where the whole mass of the body or the system may be supposed to be concentrated so far gravity (force of attraction due to the earth) on the constituent particles of the body or the system is concerned.

It follows from the law of gravitation that every particle of a body near or upon the surface of the earth is attracted towards the centre of the earth. The vector sum of all the attractive forces on the particles is the total force with which the body is attracted towards the centre of the earth. This force is called the weight of the body and the point of application of this force is called the centre of gravity of the body or system.

For big bodies the centre of gravity and the centre of mass are two distinctly different points but for small bodies these two are coincident points. The radius of the earth is large (6400 km) in comparison to small bodies that we have around us. The forces of attraction of the earth (i.e., weights) on the constituent particles may, therefore, be considered parallel. For big bodies like a mountain, an iceberg, these forces are converging at the centre of the earth. The point of application of the resultant of a parallel system of forces proportional to the elementary masses is called centre of mass. Hence for small bodies centre of mass and centre of gravity are coincident points.

Examples

1. *Find the centre of mass of a two particle system.*

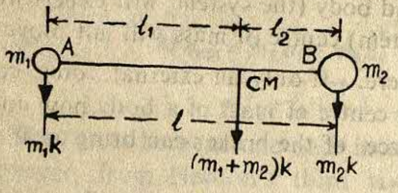


Fig. 4.24

Sol. Suppose m_1 and m_2 are two masses forming a system. Let CM be their centre of mass at a distance l_1 from m_1 and l_2 from m_2. Imagine a system of parallel forces m_1k and m_2k acting on m_1 and m_2 perpendicular to the line joining the two masses. Taking moments about CM, we have

$$m_2kl_2+(-m_1kl_1)=0$$

[This result is based on the fact that the algebraic sum of the moments of the component forces about *any* point is equal to the moment of their resultant about the same point.]

or $$m_2l_2=m_1l_1, \text{ or } \frac{m_1}{m_2}=\frac{l_2}{l_1}$$

or $$\frac{m_1}{m_1+m_2}=\frac{l_2}{l_2+l_1}$$

or $l_2=\dfrac{m_1l}{m_1+m_2}$ $\quad(\because\ l=l_1+l_2$, distance between the masses)

and $l_1=\dfrac{m_2l}{m_1+m_2}$. **Ans.**

Alternatively, using the formula $x_{cm}=\dfrac{m_1x_1+m_2x_2}{m_1+m_2}$ we have with A as moment centre $l_1=\dfrac{m_10+m_2l}{m_1+m_2}=\dfrac{m_2l}{m_1+m_2}$.

Similarly with B as moment centre $l_2=\dfrac{m_1l+m_20}{m_1+m_2}=\dfrac{m_1l}{m_1+m_2}$.

2. *Find the centre of mass of a homogenous semi-circular plate of radius 'a'.*

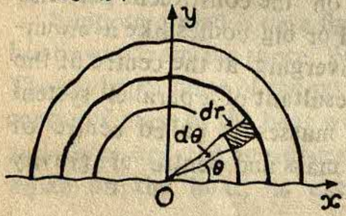


Fig. 4.25

Sol. Obviously the centre of mass will lie on the line OY (taken as y-axis in the fig.) dividing the plate into two equal halves. Consider an elementary area at a point of polar co-ordinates (r, θ).

Area of the element $=r\,d\theta\,dr$.

The mass of the element $= r\,d\theta\,dr\rho$ where $\rho =$ mass per unit area.
The mass of the plate $= \frac{1}{2}\pi a^2\rho$.

Here, $y = r\sin\theta$.

$$y_{cm} = \frac{\int y\,dm}{M} = \int_{r=0}^{r=a}\int_{\theta=0}^{\theta=\pi} r\sin\theta\; r\,d\theta\,dr\rho \Big/ \tfrac{1}{2}\pi a^2\rho$$

or $$y_{cm} = \frac{2}{\pi a^2}\int_{r=0}^{r=a} r^2 dr \int_{\theta=0}^{\theta=\pi} \sin\theta\, d\theta$$

$$= \frac{2}{\pi a^2}\cdot\frac{a^3}{3}\cdot 2 = \frac{4a}{3\pi}.\ \textbf{Ans.}$$

3. *A circular hole is punched in a uniform circular disc with the radius of the disc as diameter of the hole. Find the centre of mass of the remainder.*

Sol. Let a be the radius of the disc and x be the distance of the c. g. of the remainder from the centre of the disc.

Considering moments of the components about the centre of the disc, we have

$$\left\{\pi\left(\frac{a}{2}\right)^2.\rho\right\}\times\frac{a}{2} = \left\{\pi a^2 - \pi\left(\frac{a}{2}\right)^2\right\}\rho\times x$$

or $$x = a/6.\ \textbf{Ans.}$$

4.21. Equation of Motion of a System of Continuously Varying Mass (e.g. rocket)

Let us consider a system of instantaneous mass M and instantaneous velocity v. Let the system lose mass by $\triangle M$ in $\triangle t$ second and thereby let its velocity increase by $\triangle v$. Let the detached mass $\triangle M$ move with a velocity u backward. We have in general,

$$F_{external} = \frac{dp}{dt}$$ where $p =$ instantaneous momentum of the system

or $$F_{external} = \lim_{\Delta t\to 0}\frac{\Delta p}{\Delta t}.$$

Now, $\triangle p = p_f - p_i = [(M-\triangle M)(v+\triangle v) - \triangle Mu] - Mv$

$$= Mv - \triangle Mv + M\triangle v - \triangle M\triangle v - \triangle Mu - Mv$$

$$= M\triangle v - \triangle M(u+v+\triangle v)$$

or $$\frac{\triangle p}{\triangle t} = M\frac{\triangle v}{\triangle t} - \frac{\triangle M}{\triangle t}(u+v+\triangle v)$$

or $$\lim_{\triangle t\to 0}\frac{\triangle p}{\triangle t} = \lim_{\triangle t\to 0} M\frac{\triangle v}{\triangle t} - \lim_{\triangle t\to 0}\frac{\triangle M}{\triangle t}(u+v+\triangle v)$$

or $$\frac{dp}{dt} = M\frac{dv}{dt} - \frac{dM}{dt}(u+v)$$

($\triangle v$ being negligible in comparison to v).

$$\therefore \quad F_{external} = M\frac{dv}{dt} - (u+v)\frac{dM}{dt}.$$

In a rocket, $u+v$ = the relative velocity of the leaving mass relative to the main mass and is called the exhaust velocity.

$$\therefore \quad F_{ext.} = M\frac{dv}{dt} - v_{exhaust}\frac{dM}{dt}$$

or $$\left(F_{ext.} + v_{exhaust} \times \frac{dM}{dt}\right) = M\frac{dv}{dt}. \qquad (4.11)$$

The force $\left(v_{exhaust} \times \frac{dM}{dt}\right)$ is the 'thrust' produced by the leaving mass on the main body.

Examples

1. *A 6000 kg rocket is set for vertical firing. If the exhaust speed is 1000 ms^{-1}, how much gas must be ejected per second to, supply the thrust needed (a) to overcome the weight of the rocket, (b) to give the rocket an initial upward acceleration of 19·6 ms^{-2} ?*

Sol. Thrust produced $= v_{exhaust}\dfrac{dM}{dt}$

$$F_{external} = Mg \text{ (downward)}.$$

$\therefore$ Effective upward force $= v_{exhaust}\dfrac{dM}{dt} - Mg.$

$$\therefore \quad v_{exhaust}\frac{dM}{dt} - Mg = M\frac{dv}{dt}$$

In case (a), $\frac{dv}{dt}=0$; $\therefore$ $Mg = v_{exhaust}\frac{dM}{dt}$

or $$\frac{dM}{dt}=\frac{Mg}{v_{exhaust}}=\frac{6000\times 9{\cdot}8}{1000}=58{\cdot}8\ \text{kg s}^{-1}.\quad \textbf{Ans.}$$

In case (b), $\frac{dv}{dt}=19{\cdot}6\ \text{m s}^{-2}$ (upward).

$\therefore$ The effective upward force = thrust generated − weight

$$= v_{exhaust}\frac{dM}{dt}-Mg$$

$$\therefore \quad v_{exhaust}\frac{dM}{dt}-Mg=M\frac{dv}{dt}$$

or $$1000\frac{dM}{dt}-6000\times 9{\cdot}8=6000\times 19{\cdot}6$$

or $$\frac{dM}{dt}=\frac{6000\times 29{\cdot}4}{1000}=176{\cdot}4\ \text{kg s}^{-1}.\quad \textbf{Ans.}$$

2. *A jet airplane is travelling at a speed of 200 ms^{-1}. The engine takes in air having a mass of 32 kg each second. The air is used to burn 3 kg of fuel each second. The energy generated is used to compress the products of combustion and to eject them at the rear of the plane at 500 ms^{-1} relative to the plane. Find the thrust of the jet engine and its power.*

Sol. $v_{exhaust}=500\ \text{m s}^{-1}$, $\frac{dM}{dt}=32+3=35\ \text{kg s}^{-1}$.

$\therefore$ Forward thrust $= v_{exhaust}\frac{dM}{dt}=500\times 35=17500$ newton.

$$v_{sucking}=200-0\quad (\because \text{ air is at rest})$$
$$=200\ \text{m s}^{-1}$$

and $$\frac{dM}{dt}=32\ \text{kg s}^{-1}.$$

When air is sucked in by the engine, it experiences a backward thrust.

$\therefore$ The backward thrust $= v_{sucking}\dfrac{dM}{dt} = 200 \times 32$

$= 6400$ newton.

$\therefore$ The net forward thrust produced $= 17500 - 6400$

$= 11100$ newton. **Ans.**

Power generated $=$ force $\times$ velocity

$= 11100 \times 200$ watt

$= 2220000$ watt $= 3000$ H. P.

($\because$ 746 watt $= 1$ H.P.)

3. *Sand drops from a stationary hopper at the rate of 5 kg s^{-1} on to a conveyer belt moving with constant velocity 2 m s^{-1}. What is the force required to keep the belt moving and what is the power delivered by the motor moving the belt ?*

Sol. This is a problem on exertion of tangential momentum due to gain of mass. Here the gain in momentum along the horizontal direction by the belt is due to gain in mass.

Rate of increase of momentum $= v\dfrac{dM}{dt} =$ force exerted tangentially by the falling mass on to the belt tending to stop the belt. The same force is therefore needed to keep the belt moving.

$\therefore$ Force required to run the belt $= v\dfrac{dM}{dt}$

$= 2 \times 5 = 10$ newton.

Power needed $=$ force $\times$ velocity

$= 10 \times 2 = 20$ watt. **Ans.**

4. *A machine gun is mounted on a flat rail road car. The gun is firing bullets at the rate of 10 bullets per second each of mass 10 gm. The bullets come out with a velocity 500 m s^{-1} relative to the car. Calculate the acceleration of the car at the instant when its mass is 200 kg. Also calculate the force at that instant.*

Sol. This is also a problem on a system of variable mass.

$$F_{ext.} = M\frac{dv}{dt} + (u + v)\frac{dM}{dt}.$$

Forward thrust generated $=(u+v)\dfrac{dM}{dt}=500\times(10\times10\times10^{-3})$

$$=50 \text{ newton.}$$

The system is free from external force along the horizontal direction.

$$\therefore \quad 50=200\,a$$

or $$a=\cdot 25 \text{ m s}^{-2}. \quad \textbf{Ans.}$$

4.22. Equilibrium of Bodies and Theorems on Equilibrium of Three Forces

When a body under the action of several forces neither moves in a straight line nor rotates about a point, it is said to be in equilibrium.

Since Newton's first law of motion makes no difference between the state of rest or uniform motion in a straight line so far as acceleration is concerned (because in either case the acceleration is zero), the equilibrium of a body means that its linear acceleration as well as angular acceleration must be zero.

Conditions for 'equilibrium' :

(i) The vector sum of all forces acting on the body along any assigned direction must be zero.

$$\Sigma \vec{f}=0.$$

This is necessary for 'no linear' motion i.e., for translatory equilibrium.

(ii) The algebraic sum of moments (torques) of all forces acting on the body about any assigned point or line must vanish.

$$\Sigma \vec{\tau}=0.$$

This is a must for no angular acceleration i.e., for rotational equilibrium.

Triangle of Forces :

This is a theorem regarding equilibrium of three forces. It states—"*If three forces acting on a particle be capable of being represented in magnitude and direction by the three sides of a triangle taken in order, they produce equilibrium.*"

As stated above the condition for equilibrium is that the resultant (vector sum) of all forces must be zero. Hence if there are only three forces involved in the production of equilibrium the resultant of any two must be equal and opposite to the third.

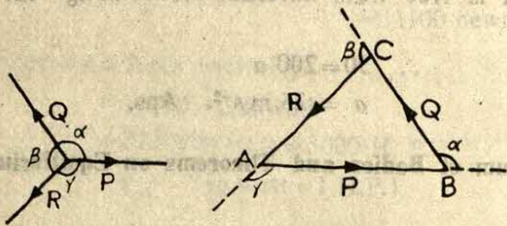


Fig. 4.26

Suppose P, Q and R are three forces acting on a particle. Let $\vec{AB}=\vec{P}$ and $\vec{BC}=\vec{Q}$. Then by the triangle law of vector addition we have

$$\vec{AB}+\vec{BC}=\vec{AC}.$$

By the condition of equilibrium

$$\vec{AC}=-\vec{R}$$

or $$\vec{CA}=\vec{R}.$$

Thus when three forces are in equilibrium we have $AB=kP$, $BC=kQ$ and $CA=kR$ where k is a constant depending on the scale of representation.

$$\therefore \quad \frac{AB}{P}=\frac{BC}{Q}=\frac{CA}{R}.$$

Lami's theorem :

This is another theorem regarding equilibrium of three forces. It states—"If three forces are in equilibrium, then each force is proportional to the 'sine' of the angle between the other two."

Suppose α, β and γ are the angles between P, Q and R in cyclic order (Fig. 4.26). Lt $AB=kP$ and $BC=kQ$, then by the triangle law of forces $CA=kR$. By the properties of the triangle we have

$$\frac{AB}{\sin\angle ACB}=\frac{BC}{\sin\angle CAB}=\frac{CA}{\sin\angle ABC}$$

or $$\frac{P}{\sin(\pi-\beta)}=\frac{Q}{\sin(\pi-\gamma)}=\frac{R}{\sin(\pi-\alpha)}$$

or $$\frac{P}{\sin\beta}=\frac{Q}{\sin\gamma}=\frac{R}{\sin\alpha}.$$

4.23. Different States of Equilibrium : Universal Rule for Stable Equilibrium

There are three states of equilibrium : stable, unstable and neutral equilibrium.

When a body is in equilibrium in such a way that a slight displacement from this position will result in a restoring force tending to return the body to the previous equilibrium, it is said to be in stable equilibrium. In stable equilibrium the body possesses minimum potential energy.

When a body is in equilibrium in such a way that any displacement from this position will result in a force tending to push the body farther from the equilibrium position, it is said to be in unstable equilibrium. In this equilibrium position the body possesses maximum energy.

When the body is in such a state of equilibrium that on displacing slightly from that position it experiences neither a restoring force nor a deflecting force, it is said to be in neutral equilibrium.

In this equilibrium position the potential energy of the body is constant.

In the gravitational field a body is in stable equilibrium when its c.g. lies as low as possible. A conical funnel resting on its rim is in stable equilibrium. A hydrometer having lead shots at the bottom floats in stable equilibrium because lead shots bring down the c.g. of the hydrometer to the lowest possible position. If there are no lead shots it is found to float in the tilted position. From these examples and many more not possible to cite them here we conclude that a body is in stable equilibrium when its potential energy is a minimum. This fact aquired by experience is adopted as the universal principle for stability of equilibrium. The universal principle is thus '*a system is in stable equilibrium when its potential energy is a minimum*'. This is only one aspect of the stability of equilibrium so far as the potential energy of the system is concerned. The position of the centre of gravity

of the system relative to the base of the body also plays a very important role in the determination of the stability of equilibrium. Let us illustrate this by a specific example of a wooden cube resting on a table.

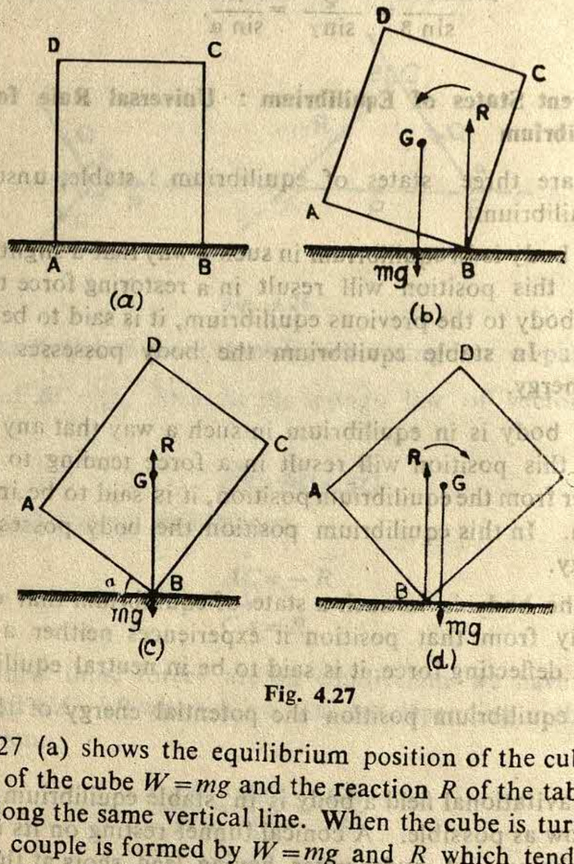


Fig. 4.27

Fig. 4.27 (a) shows the equilibrium position of the cube. Here the weight of the cube $W = mg$ and the reaction R of the table on the cube are along the same vertical line. When the cube is turned about the edge B, couple is formed by $W = mg$ and R which tends to turn the cube to the position shown in (a). This state of affairs continues till the cube is turned to the position in fig. (c) when the vertical line through c.g. just ceases to pass through the base (AB) of the cube. A little further increase of the angle of rotation takes the vertical line through the centre of gravity outside the base and the couple formed by $W = mg$ and R will overturn the cube. Thus for stability of equilibrium the second condition is : *the vertical line through the centre of gravity of the body must pass through the base of the body.* Obviously therefore the more extensive the base of the body the more stable it will be.

Illustrations : There is a tall tower in Pisa which is so inclined that to an observer it appears that it may fall down at any moment. But it never falls. In fact it has been surviving for centuries. The reason for its stability is that the vertical line passing through the centre of gravity lies within the base of the tower.

2. A double-deeker bus is in danger of overturning if more passengers are seated on the upper deck. If there are more passengers on the upper deck, the centre of gravity of the system (bus + passengers) will be shifted upward and the stability of equilibrium will be reduced.

3. A man leans outward while carrying a bucketful of water. The man leans outward to attain a stable equilibrium position. When he leans outward the vertical line through the centre of gravity of the system (man + bucket) passes through the base of the system (i.e. the space between his feet).

Examples

1. *A heavy weight (45 kg) is hung by strings as shown in the fig. 4.28. Calculate the tensions of the strings.*

Sol. Make a free-body diagram of the knot and apply Lami's theorem to it.

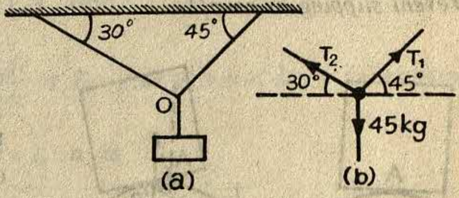


Fig. 4.28

$$\frac{T_1}{\sin(90°+30°)}=\frac{T_2}{\sin(90°+45°)}=\frac{45}{\sin(180°-75°)}$$

or $\quad T_1=\dfrac{\cos 30°}{\sin 75°}\times 45=\dfrac{\cdot 866}{\cdot 966}\times 45=40{\cdot}34$ kg. **Ans.**

And $\quad T_2=\dfrac{\cos 45°}{\sin 75°}=\dfrac{\cdot 707}{\cdot 966}\times 45=32{\cdot}93$ kg. **Ans.**

2. *An 18 m ladder weighing 45 kg rests against a wall at a point 15 m above the ground. The centre of gravity of the ladder is one-third the way up. A man (70 kg) climbs halfway up the ladder. Assuming that the wall is frictionless but not the ground, find the forces exerted by the system on the ground and the wall.*

Sol. Considering the horizontal and vertical equilibrium of the system,

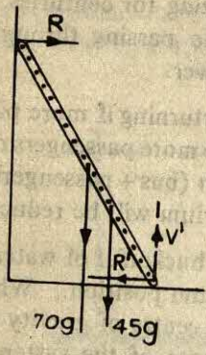


Fig. 4.29

$$R = R'$$

and $$V' = 70 + 45 = 115.$$

The inclination of the ladder against the wall is given by

$$\cos\theta = \frac{15}{18} = \frac{5}{6}.$$

Taking moment about the point of contact with the ground we have,

$$R.\ 15 = 70.9 \sin\theta + 45.6 \sin\theta$$
$$= 900 \sin\theta;$$

$$\therefore \quad R = 60.\ \frac{\sqrt{6^2 - 5^2}}{6} = 10\sqrt{11} = 33{\cdot}17 \text{ kg. } \textbf{Ans.}$$

3. *A cube of uniform density and edge* **a**. *is balanced on a cylindrical surface of radius r as shown in fig.* 4.30. *Show that the criterion for stable equilibrium of the cube, assuming that friction is sufficient to prevent slipping, is $r > a/2$.*

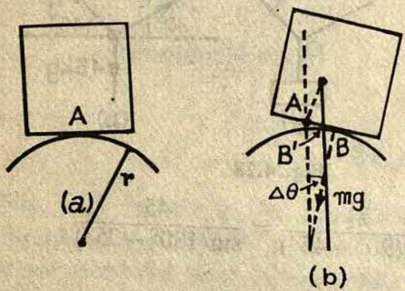


Fig. 4.30

Sol. Let us turn the cube through a very small angle $\triangle\theta$. The distance by which the point of contact is displaced along the surface of the cylinder is $r\triangle\theta$. The vertical line through the centre of gravity of the cube is displaced by $a/2\ \triangle\theta$. Equilibrium is stable so long as the vertical line through the centre of gravity is not beyond the point of contact, i.e. $a/2\ \triangle\theta < r\triangle\theta$

or $$r > a/2. \textbf{ Proved.}$$

Examples

1. *The displacement x of a particle, moving in one dimension under the action of a constant force, is related to the time t by the equation $t = \sqrt{x} + 3$ where x is in metres and t in seconds. Find the displacement of the particle when its velocity is zero.* **(I. I. T. 1979)**

Sol. **Here** $t=\sqrt{x}+3$, or $x=t^2-6t+9$;

$$\therefore \quad \frac{dx}{dt}=2t-6. \text{ When } \frac{dx}{dt}=0, \text{ we have } t=3 \text{ s.}$$

Putting $t=3$ in the expression for x,

$$x=3^2-6.3+9=0. \textbf{ Ans.}$$

2. *A particle starts from rest at time $t=0$ and undergoes acceleration, as shown in the fig.* (4.31 *a*).

(i) *Draw a neat sketch showing the velocity of the particle as a function of time during the interval* 0 *to* 4 *s, indicating each second on the abscissa.*

(ii) *Draw a neat sketch showing the displacement of the particle as a function of time during the same interval.* **(I. I. T. 1977)**

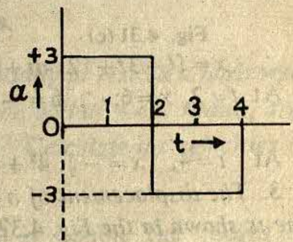


Fig. 4.31 (a)

Sol. (i) We have $a=\frac{dv}{dt}$ or $dv=a\,dt$ or $v=a\int dt+c$ (a constant).

In between $t=0$ and $t=2$, a is fixed at 3.

$\therefore \quad v=3\int dt+c$ or $v=3t+c$.

When $t=0$, $v=0$ (given);

$\therefore \quad c=0. \quad \therefore \quad v=3t$.

At $\quad t=2, v=3.2=6$.

In the range $t=2$ and $t=4$, a is fixed at -3.

$\therefore \quad v=-3t+c$.

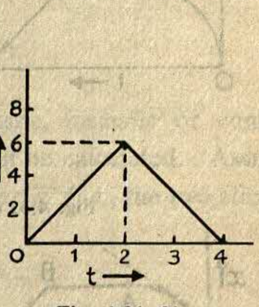


Fig. 4.31 (b)

When $t=2$, $v=6$; $\therefore \quad 6=-3.2+c$, or $c=12$;

$$\therefore \quad v=-3t+12.$$

(ii) We have $\qquad v=\frac{dx}{dt}$ or $dx=vdt$

or $$x=\int v\,dt+c \text{ (a constant)}.$$

In the range $t=0$ and $t=2$, $v=3t$;

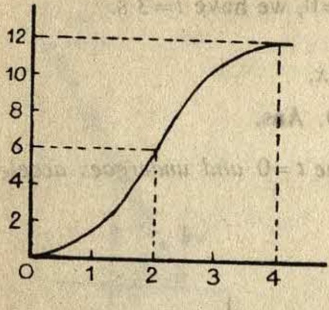


Fig. 4.31 (c)

$$\therefore\ x=\int 3t.dt+c=\tfrac{3}{2}t^2+c.$$

Let us take $x=x_0$ when $t=0$.

$\therefore\ c=x_0$;

$\therefore\ x=x_0+3/2\ t^2$.

Further we may take $x_0=0$ if we please.

Then $x=3/2\ t^2$.

$\therefore$ At $t=2$, $x=6$.

In the range $t=2$ and $t=4$,

$$v=-3t+12;$$

$$\therefore\ x=\int(-3t+12)\,dt+c=-\tfrac{3}{2}t^2+12t+c.$$

At $t=2$, $x=6$; $\therefore\ 6=-\frac{3}{2}.\ 2^2+12.2+c$, or $c=-12$;

$$\therefore\ x=-\tfrac{3}{2}t^2+12t-12.$$

At $t=4$, $x=-\frac{3}{2}.\ 4^2+12.4-12=12$.

3. *The displacement of a particle along the x-axis as a function of time is shown in the Fig.* 4.32 (*a*). *Find the direction of the velocity and acceleration of the particle between the following points* : (i) *between* 0 *and A*, (ii) *between A and B*, (iii) *between C and D*. **(I. I. T. '73)**

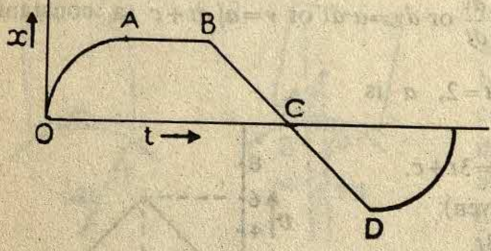


Fig. 4.32 (a)

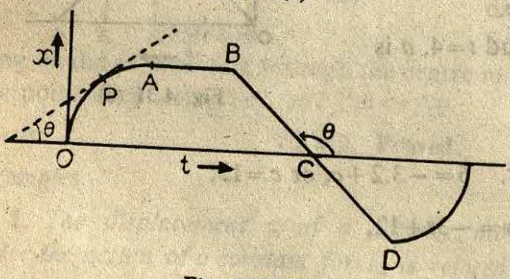


Fig. 4.32 (b)

Sol. We have

$$v=\frac{dx}{dt}.$$

Geometrically $\frac{dx}{dt}$ is the slope of the plot of x against t. The slope of a line is the tangent of the angle of inclination of the tangential line to the plot at the specified point with the x-axis, counted always in the anticlockwise direction. Consider any point P in between O and A (Fig. 4.32 b).

$v_{at_P} = \left(\frac{dx}{dt}\right) = \tan\theta$. Since θ is less than 90°, tanθ is +ve and hence in between O and A velocity is '+ve'. **Ans.**

The slope gradually decreases from O to A, hence there is retardation between O and A. That is, the acceleration in between O and A is negative. **Ans.**

In between A and B, the slope is zero and hence velocity and acceleration are *zero.* In between B and D the slope is constant but negative because $\theta > 90°$. Hence in between B and D, the velocity is negative and the acceleration is zero. **Ans.**

4. *An iron ball of radius 2 cm is initially at rest on a horizontal surface. It is struck head on by another iron ball of 4 cm radius, travelling with a velocity of 81 cm per second. Calculate the velocities of the two balls after collision.*

Sol. Since the radius of the bigger ball is double the radius of the smaller ball, its mass is 8 times greater.

Let m be the mass of the smaller ball.

Then by equating momentum before collision to momentum after collision we have,

$$(m\,0 + 8m.81) = m.v_1 + 8m\,v_2$$

or
$$648 = v_1 + 8v_2.$$

Since the nature of collision (whether elastic, inelastic or completely inelastic) is not given, v_1 and v_2 cannot be calculated. Assuming that the collision is 'completely inelastic' (i.e., the two stick together after collision),

we may take $v_1 = v_2 = v$ (say).

$$\therefore \quad v = \frac{648}{9} = 72 \text{ m s}^{-1}. \textbf{ Ans.}$$

5. *A wheel of radius 40 cm rests against a step of height 20 cm as shown in the fig. 4.33. What is the minimum horizontal force which, if applied perpendicular to the axle, will make the wheel climb the step ? The mass of the wheel is 2 kg.* **(I. I. T. '76)**

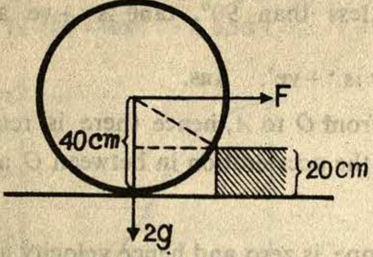


Fig. 4.33

Sol. The moment of the force F about the step $= F.(40-20) = F \times 20$.

The moment of the weight about the edge of the step

$$= 2g \times \sqrt{(40^2 - 20^2)}$$

$$= 2g \times \sqrt{1200}.$$

The wheel will climb the step when

$$F \times 20 \geqslant 2g \times \sqrt{1200}, \text{ or } F \geqslant \sqrt{12g}.$$

$\therefore \quad F_{minimum} = \sqrt{12 \times 9{\cdot}8} = 34$ newton. **Ans.**

6. *A block slided down a frictionless incline making an angle* θ *with an elevator floor. Find its acceleration relative to the incline in the following cases : (a) The elevator ascends with an acceleration a, (b) the elevator descends with an acceleration a, (c) the elevator cable breaks.*

Sol. (a) By *D'* Alembert's principle we may treat the elevator stationary by introducing inertia forces in addition to the real forces. The forces acting on the block are : its weight mg vertically downwards, reaction N of the incline on to the block and ma (inertia force) downward.

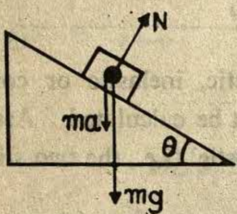


Fig. 4.34

Projecting the forces along and perpendicular to the plane

$$N - (mg\cos\theta + ma\cos\theta) = 0$$

(since there is no acceleration along the normal to the plane).

The unbalanced force down the plane

$$= mg\sin\theta + ma\sin\theta.$$

If f is the acceleration down the plane then

$$mf = mg \sin\theta + ma \sin\theta$$

or $\qquad f = (g+a) \sin\theta.$ **Ans.**

(b) Replacing a by $-a$ in (i),

$$f = (g-a) \sin\theta. \quad \textbf{Ans.}$$

(c) Replacing a by g in (ii)

$$f = (g-g) \sin\theta = 0. \quad \textbf{Ans.}$$

7. *Two men stand facing each other on two boats floating in still water at a short distance apart. A rope is held at its ends by both. The two boats are found to meet always at the same point whether each man pulls separately or both pull together. Why? Will the time taken to meet be different in the two cases? (Neglect friction)*

(I. I. T. '74)

Sol. The system (the two men+boats) is free from external force and hence there will be no displacement of the centre of mass of the system. The time taken to meet when both pull together will be just half the time when each man pulls. **Ans.**

8. *A hemispherical cup of radius r and having its centre of gravity at C rests on the top of a spherical surface of radius R. Assuming that there is sufficient friction to prevent slipping and the centre of gravity at a distance c from the flat surface, establish the criterion of stability of the cup in the position shown in fig. 4.35.*

Sol. If the cup is stable, its potential energy must be a minimum i.e. the centre of gravity C must rise slightly for any infinitesimal virtual displacement defined by the angle $\triangle\varphi$ as shown in Fig. 4.35 (b).

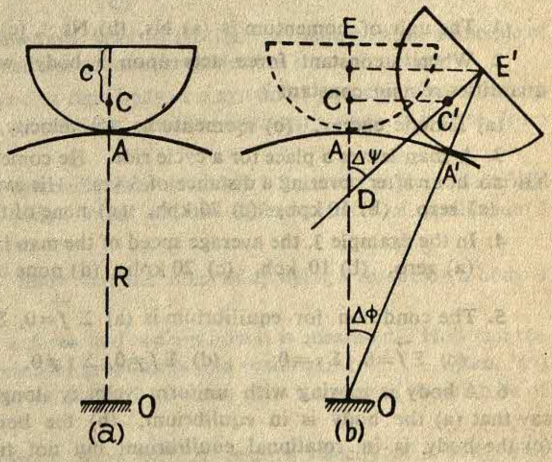


Fig. 4.35 (a) Fig. 4.35 (b)

In Fig. 4.35 (a) the height of c.g. above the datum plane at O is $(R+r-c)$. In the displaced position (fig. 4.35 a), the height of the c.g. above the same datum plane is $(R+r)\cos\triangle\varphi - c\cos\triangle\psi$ where $\triangle\psi$ is the inclination of the line $E'C'$ with the vertical.

For stability,

$$(R+r)\cos\triangle\varphi - c\cos\triangle\psi > R+r-c,$$

$$\text{or} \quad (R+r)\left(1-\frac{\Delta\varphi^2}{2}\right) - c\left(1-\frac{\Delta\psi^2}{2}\right) > (R+r-c)$$

$$\left(\cos\theta = 1-\frac{\theta^2}{2}+\dots\right),$$

or $$c\triangle\psi^2 > (R+r)\triangle\varphi^2.$$

Let $E'C'$ cut the vertical line EAO at D.

Then $EE' = (R+r)\triangle\varphi$, from $\triangle EE'O$.

Also $EE' = ED.\triangle\psi$, from $\triangle EE'D$,

$= r\triangle\psi$ $(\because\ ED \cong r)$.

$(R+r)\triangle\varphi = r\triangle\psi$.

$\therefore$ The required condition is

$$c\triangle\psi^2 > (R+r)\ \frac{r^2}{(R+r)^2}\cdot\triangle\psi^2$$

or $$c(R+r) > r^2 \text{ or } R > r\left(\frac{r}{c} - 1\right).\ \textbf{Ans.}$$

QUESTIONS

(A)

1. The unit of momentum is (a) Ns, (b) Ns^{-1}, (c) $N^{-1}s$, (d) Ns^{-2}.

2. When a constant force acts upon a body, which one of the following quantities remain constant ?

(a) Kinetic energy, (b) momentum, (c) velocity, (d) acceleration.

3. A man leaves a place for a cycle ride. He comes back to that place after half an hour after covering a distance of 5 km. His average velocity is

(a) zero, (b) 10 kph, (c) 20 kph, (d) none of these.

4. In the example 3, the average speed of the man is

(a) zero, (b) 10 kph, (c) 20 kph, (d) none of these.

5. The condition for equilibrium is (a) $\Sigma\vec{f}=0$, $\Sigma\vec{\tau}\neq 0$; (b) $\Sigma\vec{f}\neq 0$, $\Sigma\vec{\tau}=0$;

(c) $\Sigma\vec{f}=0$, $\Sigma\vec{\tau}=0$; (d) $\Sigma\vec{f}\neq 0$, $\Sigma\vec{\tau}\neq 0$.

6. A body is moving with uniform velocity along a straight path. We may say that (a) the body is in equlibrium, (b) the body is not in equilibrium, (c) the body is in rotational equilibrium but not translational, (d) none of these.

7. A body falling freely under gravity moves with uniform

(a) momentum, (b) velocity, (c) acceleration, (d) speed.

8. The dimension of the impulse of an impulsive force is

(a) ML^2T^{-1}, (b) $M^2L^2T^{-2}$, (c) $ML^{-1}T^{-1}$, (d) MLT^{-1}.

9. When a system consisting of several bodies is free from external force, under the mutual action and reaction of the bodies (a) the centre of mass of the system will be displaced but not of the constituent bodies, (b) the centre of mass of the system will not be displaced and the centre of mass of the constituent bodies also will not be displaced, (c) the centre of mass of the system will not be displaced but those of the constituent bodies will be displaced, (d) none of these,

10. The horizontal range of a projectile is maximum for a given velocity of projection when angle of projection is

(a) 30°, (b) 60°, (c) 45°. (d) 90°.

Ans. : **1**. (a). **2**. (d). **3**. (a). **4**. (b). **5**. (c). **6**. (a). **7**. (c). **8**. (d). **9**. (c). **10**. (c).

(B)

1. Show that force = mass × acceleration.

2. State Newton's Laws of motion.

3. Define centre of mass and centre of gravity. Does the centre of mass of a solid body necessarily within the body ? If not, give examples.

What is the importance of the concept of 'centre of mass' ?

4. What is D' Alembert's Principle ? What is its importance ?

5. What is a 'free-body' diagram ? Explain the use of free-body diagram in solving problems in dynamics.

6. Define 'equilibrium' of a body. Explain the different types of equilibria.

(C)

1. State and explain Newton's laws of motion. Explain how the principle of conservation of linear momentum follows from Newton's law of motion.

2. Define centre of mass of a rigid body or a system of particles. Show that the centre of mass of a system of particles remains stationary when it is free from external force.

3. What do you mean by equilibrium of a body ? State and explain the conditions for equilibrum of a body. What are the different states of equilibrium ? Explain them with examples.

4. What is a torque ? Show that the effect of applying a torque on a body is always to rotate the body.

5. Define moment of a force and explain how it is measured. How can the moment of a force be represented graphically and vectorially ? **(Ran. '69)**

(**Hint** : The moment $= Fr = 2.\frac{1}{2}\ Fr = 2\triangle$, where $\triangle$ is the area of the triangle of base length F and height r.)

6. What is relative velocity ? Show that the relative velocity of a body with respect to another body is just the 'vector difference' between the actual velocity of the body and the actual velocity of the referred body.

(D)

1. A train is moving due north at a speed of 20 kph and wind is blowing due east at 15 kph. Find the velocity with which the wind appears to blow to a traveller sitting in the train. **(Ans.** 25 kph at 53° north of west**)**

2. A bullet is fired with a velocity of 300 ms^{-1}, its mass is 10 gm. It hits a block of wood of mass 200 gm, kept on a horizontal table, and is embedded in it. Calculate the velocity of the block of wood. **(Ans.** 14·3 m s^{-1}**)**

3. A constant force acts on a mass 8 kg for 3 s and then ceases to act. During the next 3 s the body describes 81 m. Find the magnitude of the force.

(Ans. 72 newton)

4. A body of mass 2 kg is placed at one end of a uniform rod of length 1 m and of mass 12 kg. At what place must a body of mass 5 kg. be placed on the rod so that the centre of gravity of the system is 2 cm away from the rod ?

(Ans. ·124 m)

5. A man weighing 70 kg is standing on a lift moving downward with an acceleration of $\cdot 2g$. Calculate the thrust of the man on the floor of the lift.

(Ans. 56 kg)

*6. The position of a particle moving along the x-axis depends on the time according to the equation $x=3t^2-t^3$.

(a) At what time does the particle reach its maximum positive x-position ? (b) What total length of path does the particle cover in the first 4 s ? (c) What is the dispacement during the first four seconds ? (d) What is the particle's speed at the end of each of the first four seconds ? (e) What is the particle's acceleration at the end of each of the first four seconds ?

[Ans. : (a) 2 s; (b) 24 m; (c) -16 m; (d) 3 m, 0, -9, -24 m s^{-1};
(e) 0, -6, -12, -18 ms^{-2}.]

7. If a body travels half its total path in the last second of its fall from rest, find the time and height of its fall.

(Ans. 3·415 s; 57 m)

8. The plot of acceleration of a body with time is shown in Fig. 4.36. Assuming the body to start from rest, plot a graph of 'velocity-time' and 'displacement-time' graph.

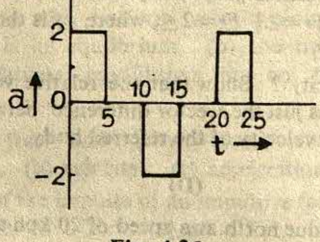


Fig. 4.36

[Ans. : In the interval 0-5 s, $v=2t$, a straight line and $x=t^2$.
In the interval 5-10 s, $v=10$ (constant), $x=10t-25$ (a st. line).
In the interval 10-15 s, $v=30-2t$
$x=30t-t^2-125$.
In the interval 15-20 s, $v=0$.
$x=100$ (For graphs see Fig. 4.37, p. 115.)]

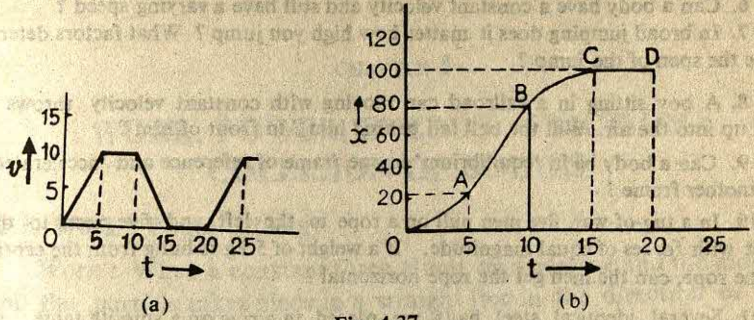


Fig. 4.37

9. Show that the rocket speed is equal to the exhaust speed when the ratio $\frac{M_0}{M}$ is equal e (naperian base) where M_0 = mass of rocket in the beginning and M = instantaneous mass of the rocket. Show also that the rocket speed is twice the exhaust speed when

$$\frac{M_0}{M} \text{ is } e^2.$$

[Hint : $F_{external}=0$; $\therefore \; M\frac{dv}{dt}=-v_{exhaust}\;\frac{dM}{dt}$; hence $M=M_0 e^{-v/v_{exhaust}}$

10. Two cars are moving in the same direction with the same speed (30 kph). They are seperated by a distance of 5 km. What is the speed of a car moving in the opposite direction if it met these two cars at an interval of 4 minutes ? **(I. I. T. '75)**

(Ans. 45 kph)

11. A car covers the first half of the distance between two places at a speed of 40 kph and the second half at 60 kph. What is the average speed of the car ? **(I. I. T. '74)**

(Ans. 48 kph)

(E)

1. A block slides down a smooth inclined plane when released from the top, while another falls freely from the same point. Which one of them will strike the ground (i) earlier, (ii) with greater velocity, (iii) with great impact ? **(I. I. T. '74)**

2. A lorry and a car moving with the same kinetic energy are brought to rest by the application of brakes which provide equal retarding forces. Which of them will come to rest in a shorter distance ? **(I. I. T. '73)**

3. Can a body have zero velocity and still be accelerating ?

4. Can a body have a constant speed and still have a varying velocity ?

5. Can an object have an eastward velocity while experiencing a westward acceleration ?

6. Can a body have a constant velocity and still have a varying speed ?

7. In broad jumping does it matter how high you jump ? What factors deter mine the span of the jump ?

8. A boy sitting in a railroad car moving with constant velocity throws a ball up into the air. Will the ball fall behind him ? In front of him ?

9. Can a body be in 'equilibrium' in one frame of reference and 'accelerated' in another frame ?

10. In a tug-of-war, five men pull on a rope to the left and five men to the right with forces of equal magnitude. If a weight of 5 kg is hung from the centre of the rope, can the men get the rope horizontal ?

11. Several identical steel balls are placed in a row on a smooth table. If one, two, three balls hit them from one side with the same velocity, the same number of balls move out from the other end. Why is this so ?

12. In a train moving with a constant speed if the brakes are applied and the brakes produce the same effective retarding force on all the bogies, will tail or head of the train stop first ?

[Ans. 1. (i) The one falling freely, $t=\sqrt{\frac{2h}{g}}\operatorname{cosec}\alpha$, $t=\frac{2h}{g}$; (ii) same; (iii) the one falling freely because of its normal incidence. 2. They come to rest within the same distance : $S=\frac{K(\text{kinetic energy})}{F(\text{retarding force})}$. 3. Yes, example S. H. M. 4. Yes, example circular motion. 5. Yes, example S. H. M. 6. No. 7. Yes, the angle at which you take the jump. 8. Right into his hand due to the physical independence of forces. 9. Yes, a body in circular motion is in equilibrium when observed from a frame rotating with the same speed but the same body is 'accelerated' to an observer in an inertial frame. 10. No. 11. This is necessitated for compliance with both the principle of conservation of linear momentum and the principle of conservation of energy. 12. As the tension of the rear compartment is the least, the tail of the train will be brought to rest first.

CHAPTER 5

WORK, POWER, ENERGY : FRICTION : COLLISION OF BODIES

5.1. Work : Power : Energy

Work : When a constant force F acts on a particle and the motion of the particle takes place in a straight line in the direction of the force, the work done by the force is defined as the product of the magnitude of the force F and the distance x through which the particle moves. Thus by the simple definition of work we have

$$W = Fx. \quad \text{.. (5.1)}$$

However, the force acting on the particle may not act in the direction in which the particle moves. In this case the work done by the force is defined as the product of the component of the force along the line of motion of the particle and the distance through which the particle moves. If θ is the angle made by F with the line of motion of the particle, then by the definition of 'work' we have

$$W = (F\cos\theta).x \text{ or } F.(x\cos\theta) \quad \text{.. (5.2)}$$

or, work done by a force = force × displacement along the force.

This expression reminds us of the definition of the scalar product of two vectors. Hence we may write "work done" as

$$W = \vec{F}.\vec{x}. \quad \text{.. (5.3)}$$

Let us now consider the case when the force is not constant but varies with distance. In such case we first find the elementary work done by the force on the particle at a distance x and then sum it up for all values of x. Thus we have in general

$$W = \int_{x_1}^{x_2} \vec{F}.d\vec{x}. \quad \text{.. (5.4)}$$

The unit work is the work done by a unit force in moving a particle through a unit distance in the direction of the force. *In SI the unit of work is newton-metre called joule (J).*

Examples

1. *A 20 kg box is dragged by 20 m on a railway platform by 10 newton horizontal force.* (*a*) *What is the work done by the force ?* (*b*) *What is*

the work done by the friction ? (c) What is the work done against the force and (d) against the friction ?

Sol. (a) $W=Fx=10\times20=+200$ joule.

(b) $f_{friction}=10$ newton.

$\therefore$ $W=10\times(-20)$, minus because displacement is opposite to friction $=-200$ joule.

(c) Work done by $=-$work done against; $\therefore$ $W=-200$ joule.
(d) $W=+200$ joule.

2. *A block of mass 20 kg is to be raised from the bottom to the top of an incline 3 m long and 1 m off the ground at the top. Assuming the incline to be frictionless, how much work must be done by a force parallel to the incline pushing the block up ?* $(g=9{\cdot}8\ m\ s^{-2})$

Sol. Let F be the force pushing the block up the plane. Because the motion is not accelerated along and perpendicular to the plane,

we have, $F=20g\sin\theta$
and $R=20g\cos\theta$

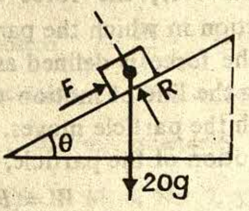


Fig. 5.1

or $F=20g.\dfrac{1}{3}=\dfrac{20g}{3}$ newton.

$(\because \sin\theta=\frac{1}{3})$

Work done $=Fx=\dfrac{20g}{3}.3=20g=20\times9{\cdot}8=196$ joule. **Ans.**

3. *Calculate the work done in compressing a spring of force constant k through x.*

Sol. The compression or elongation of a spring by an agent provides a good example of work done by an agent against a variable force.

If an agent stretches a spring so that its end point moves to a position x from the initial position, the spring will exert a force on the agent in stretching it, in proportion to the increase in length i.e.,

$$F=kx$$

where k is a constant called the 'force constant' of the spring. This relation is the 'force law' for springs. The direction of the force is always opposite to the displacement of the end point

from the initial position. Thus force is a function of x. Let us further stretch it by dx.

Then the elementary work done $=Fdx=kx\,dx$.

$\therefore$ The total work done from 0 to $x=\int_0^x kx\,dx=\frac{1}{2}kx^2$. **Ans.**

Thus the work done in stretching a spring through x

$$=\tfrac{1}{2}kx^2. \qquad \text{.. (5.5)}$$

Power : The power of an agent is defined as the rate at which work is done by it.

The average power delivered by an agent is the total work done by the agent divided by the total time interval

or
$$P=\frac{W}{t}. \qquad \text{.. (5.6)}$$

The instantaneous power of an agent is

$$P=\frac{dW}{dt}$$

and
$$W\text{ (total work done)}=\int_{t_1}^{t_2} Pdt. \qquad \text{.. (5.7)}$$

We can easily find an alternative useful formula for power in terms of velocity. We have

$$P=\frac{W}{t}=\frac{\vec{F}.\vec{x}}{t}=\vec{F}.\left(\frac{\vec{x}}{t}\right)=\vec{F}.\vec{v}. \qquad \text{.. (5.8)}$$

In SI the unit of power is joule per second (J s^{-1}), which is called watt (W). The layman's unit of power is horse power (H. P.). One horse power is equal to 746 watt.

Example

1. *A Fiat car 25 H. P. moves at a uniform speed of 40 kph. What is the forward thrust generated by the engine of the car ?*

Sol. $P=Fv$

or
$$F=\frac{P}{v}=\frac{25\times 746}{\frac{40\times 10^3}{3600}}=\frac{25\times 746\times 3{\cdot}6}{40}$$

$$=1678{\cdot}5 \text{ newton}=171{\cdot}3 \text{ kgf}.$$

Energy : Energy of a body is its capacity of doing work. In mechanics a body is capable of doing work under two circumstances:

(i) when it has motion and (ii) when it is situated in a field or when it is strained. When a bullet strikes a target it penetrates through the target and does work against the resisting force offered by the target. This energy is called kinetic energy of the body and is measured by the work it can do in being brought to rest. When a body is placed in a field, it can do work due to its position in the field. Such energy is called potential energy. Due to position in earth's gravitational field a body possesses energy called gravitational potential energy. Similarly, we have magnetic potential energy, electric potential energy. A compressed spring is capable of doing work due to strain produced in it. This is called elastic potential energy. The unit of energy is obviously the same as work.

5.2. Kinetic Energy and the Work-Energy Theorem

Let a body of mass m move with velocity v. To calculate its kinetic energy we have to see what work it can do in being brought to rest by an external opposing force. Let F be a constant opposing force applied on the body and let it be brought to rest within s. Then $v^2=2as$ where $a=$retardation produced by F.

$$\text{Work done}=\text{force}\times\text{distance}=Fs$$

$$=ma\times\frac{v^2}{2a} \quad (\because \quad F=ma, \text{ by Newton's laws of motion})$$

$$=\tfrac{1}{2}\, m\, v^2.$$

$\therefore$ Kinetic energy (K) of a body of velocity $v=\frac{1}{2}mv^2$. .. (5.9)

Now we will show that the change in the kinetic energy of a particle is always equal to the work done on the particle by the force whether the force is constant or variable.

We have in general

$$W=\int_{x_1}^{x_2} F dx = \int_{x_1}^{x_2} m\frac{dv}{dt}\, dx \quad (\because \; F=m\frac{dv}{dt} \text{ by Newton's law})$$

or $$W=\int_{x_1}^{x_2} m\frac{dv}{dx}.\frac{dx}{dt}\, dx$$

$$\text{or} \quad W = m\int_{x_1}^{x_2} \frac{dv}{dx} \cdot \; vdx \;.. \; (\because \; v = \frac{dx}{dt})$$

$$= \int_{v_1}^{v_2} m \; v \, dv = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2$$

$$\text{or} \qquad W = K_2 - K_1 = \triangle K. \qquad .. \; (5.10)$$

Thus the work done on a particle by a force (constant or varible) is always equal to the change in the kinetic energy of the particle. This is known as the work-energy theorem.

Example

1. *A block weighing 5 kg slides on a horizontal frictionless table with a speed of 1·2 ms^{-1}. It is brought to rest in compressing a spring in its path. By how much is the spring compressed if its force constant is 3·5 newton per metre ?*

Sol. By the 'work-energy' theorem

work done = change in kinetic energy.

Here work done $= \frac{1}{2}kx^2$

$\therefore \; \frac{1}{2} kx^2 = \frac{1}{2} mv^2, \qquad \text{or} \quad x = \sqrt{\frac{m}{k}} \cdot v = \sqrt{\frac{5}{3 \cdot 5}} \cdot 1 \cdot 2$

or $\qquad x = 1 \cdot 43$ m. **Ans.**

5.3. Frictional Forces : Static and Kinetic

It is a matter of common experience that when a block of mass m is set in motion with velocity v_0 on the floor, it eventually comes to rest. This means that while moving, it experiences an opposing force. This force which is neither gravitational nor elastic in nature is called frictional force and is due to microroughness of the apparently plane surfaces in contact. The frictional forces like all other forces occur in pair. Actually, whenever a body slides over that of another, each body exerts a frictional force on the other along the surfaces in contact. The frictional force on each body is in a direction opposite to its motion.

Friction is a nuisance as well as a bare necessity in our daily lives. In automobiles and machines a good percentage of useful energy is used to counteract only frictional forces. Moreover, friction causes

wear and tear of the moving parts of machines. On the other hand, without friction it would not be possible for us to walk; we could not hold a pencil in our hand; the wheel of automobiles would not rotate; a ladder would not rest against a wall and we could not climb up a ladder.

Even when there is no relative motion, frictional forces may exist between surfaces. Such forces are called forces of static friction. Consider a block resting on a horizontal table. Let us apply a small force to set the block in motion. We find that there is no visible motion of the body. We interpret that our applied force is balanced by an opposite frictional force exerted by the table on the block tangentially to the surface of contact. This force is called static frictional force. As we increase the applied force we find that a motion of the block takes place only when the force is increased to a definite value called the limiting frictional force. Once motion has started, the same force produces accelerated motion of the block. Now by a reduced force it is possible to keep the block in uniform motion. This force is smaller but it is never zero. This force is called the kinetic friction.

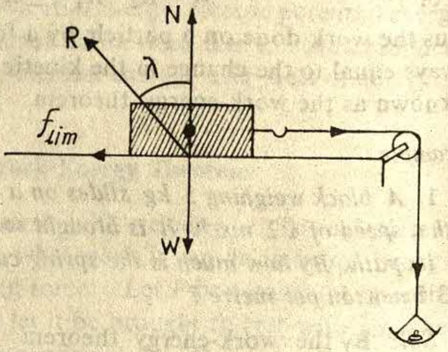


Fig. 5.2

The angle made by the reaction R which is the resultant of the normal reaction N and the limiting frictional force f_{lim} (alternatively called the limiting tangential reaction) with the normal to the surface is called angle of friction (λ).

Laws of limiting friction : (1) Friction opposes motion.

(2) The limiting frictional force is proportional to the normal reaction and is independent of the area of contact.

Thus we have $f_{lim} \propto N$ where f_{lim} is the limiting frictional force and N is the normal reaction.

$$\therefore \quad f_{lim} = \mu_s N \qquad .. \ (5.11)$$

where μ_s is a constant called the coefficient of static friction.

The force of kinetic friction f_k between two surfaces follows the same two laws as those of static friction.

Therefore, $f_k=\mu_k N$...... .. (5.12)

where μ_k is the coefficient of kinetic friction.

Usually, for a given pair of surfaces, $\mu_s > \mu_k$.

Origin of frictional forces :

Frictional forces arise from the adhesion of molecules in the surfaces of contact with one another. At the points of contact the forces of attraction are so strong that the bodies are momentarily welded together. The frictional force is the force required to break these 'cold welds' and maintain the motion. Obviously the frictional force will depend on the actual microscopic contact area which is negligible to the actual macroscopic area of contact. The cause of frictional force being proportional to the normal reaction is that the actual microscopic area of contact increases proportionately with the increase of the normal reaction.

Experimental Verification :

Fig. 5.2 may be used to verify the laws of limiting friction. In laboratories a wooden plank provided with a frictionless pulley at one end and a slider in the form of a block (rectangular) are used. This is called a Friction table. First make the table horizontal and put some weight on the pan. Gently tap the table to notice whether the block has tendency to slide over the table. If not, increase the weight on the pan bit by bit till such situation is attained. Repeat the observation by putting extra weights on the block.

Let w be the mass on the pan plus the mass of the pan weighed already before starting the experiment and W is the mass of the block plus mass on the block. Considering limiting equilibrium of the block we have

$T-f_{lim}=0$ where T is the tension of the string and f_{lim} is the limiting frictional force between the surfaces in contact.

or $$T=f_{lim}$$

and $N-Wg=0$, or $N=Wg$ where N is the normal reaction of the table.

Again considering the equilibrium of the hanging mass we have

$$T-wg=0 \text{ or } T=wg;$$

$$\therefore \quad f_{lim}=wg \text{ and } N=Wg.$$

$$\therefore \quad \mu_s=\frac{f_{lim}}{N}=\left(\frac{w}{W}\right).$$

By putting more weights on the block the experiment may be repeated to observe that with the increase of the normal reaction the limiting frictional force also increases proportionately. To check up that the limiting frictional force does not depend on the areas of the surfaces in contact, put the block vertical. Now less area will be in contact between the surfaces. It is seen by repeating the above experiment that the same weight is required on the pan to bring the block in 'just motion' for the same weight on the block. This proves the fact that frictional force does not depend on the areas in contact. In the same way, by changing the material of the sliding block, it may be shown that the frictional force depends on the material of the surfaces in contact.

Rolling friction : Lubrication

When a body rolls over another, the frictional force developed is called the force of rolling friction and the corresponding coefficient of friction is called the coefficient of rolling friction (μ_r). The rolling frictional force is much less than the sliding frictional force. This is the reason why vehicles are mounted on wheels. The advantage of this fact is also taken in the design of the moving parts of machineries by making the surfaces hard and by keeping hard rollers or balls between them. Fig. 5.2b illustrates a ball type of bearing. The balls are placed in a groove called the 'race' and the axle rests on the balls. It can be seen that the axle rotates on the balls without sliding.

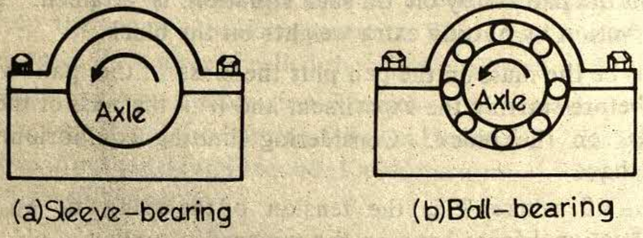


Fig. 5.2

To reduce friction between two rubbing surfaces, a suitable oil or grease, called a lubricant is generally introduced between them. This ensures smooth functioning of the different parts of a machine and prevents them from getting unduly heated. The lubricant intervenes between the surfaces and thus prevents them from coming in direct contact.

The use of air as a lubricant is very much advantageous. Purified air is compressed into the small clearance between the rubbing

surfaces. It forms an elastic cushion between the surfaces and thus solves the problem of friction and heating. It simultaneously serves another purpose. The outflow of air from the machine prevents the entry of dust into the machine. The air as lubricant is used to reduce ground friction in Hoover craft.

Determination of Coefficient of Friction :

(i) *Horizontal plane method*: See experimental verification above.

(ii) *Inclined plane method*: Place a rectangular slab D on an inclined plane AB and gradually increase the inclination to α till D just begins to slide down the plane. Ascertain this by gentle *tapping of the table.*

As there is no acceleration of the block along and perpendicular to the plane we have

$$W \cos\alpha = R$$

and $\quad W \sin\alpha = f_{lim}.$

$$\therefore \quad \frac{W \sin\alpha}{W \cos\alpha} = \frac{f_{lim}}{R} = \mu_s$$

or $\quad \tan\alpha = \mu_s. \qquad \ldots (5.13)$

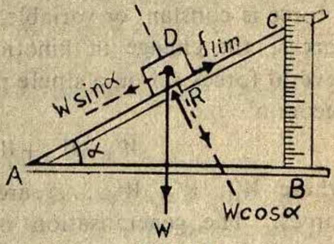


Fig. 5.3

Thus the coefficient of static friction is equal to the tangent of the inclination of the plane with the horizontal.

This particular angle is called *angle of repose.* Since $\tan\alpha = \frac{\text{height}}{\text{base}}$, the coefficient of friction is obtained by taking the height with the help of the attached scale and dividing it by the base measured by a metre scale. Repeat the experiment several times and take the mean value of μ.

Fxample

1. *An automobile is moving along a straight horizontal road with a speed* 50 *kph. If the coefficient of friction between the tyres and the road is* ·25, *what is the shortest distance in which the automobile can be stopped* ?

Sol. As there is no 'horizontal' and 'vertical' acceleration' we have $\qquad Mg = R$ and $P = f$

where P is the forward force exerted by the engine and f is the frictional force.

By the law of friction $f=\mu R$;

$$\therefore \quad P = \cdot 25\ R = \cdot 25\ Mg.$$

$$\therefore \quad a \text{ (retardation produced)} = \frac{\cdot 25Mg}{M} = \cdot 25g = \cdot 25 \times 9\ 8 \text{ m s}^{-2};$$

$$\therefore \quad \left(\frac{50 \times 10^3}{3600}\right)^2 = 2 \times \cdot 25 \times 9 \cdot 8 \times s \quad (\text{formula : } v^2 - v_0^2 = 2as)$$

$$\text{or} \quad s = \frac{25 \times 10^8}{36^2 \times 10^4 \times \cdot 5 \times 9 \cdot 8}\ m = \frac{25 \times 10^4}{36 \times 36 \times 4 \cdot 9} = 39 \cdot 4 \text{ m. } \textbf{Ans.}$$

5.4. The Conservation of Energy : Conservative and Non-conservative Forces

We have seen in the 'work-energy theorem' that in general whether a force is constant or variable, single or the resultant of a number of forces, the change in kinetic energy is equal to the work done. If several forces act on a single particle we have by the 'work-energy theorem'

$$W_1 + W_2 + W_3 + \ldots\ldots = \Delta K \qquad \ldots (5.14)$$

where W_1, W_2, W_3...... are the works done by the individual forces. The generalisation of this principle will culminate in the formulation of one of the great principles of science, the conservation of energy.

Before we proceed to generalisation of the work-energy theorem, let us first see what are conservative and non-conservative forces.

Imagine a block placed on a frictionless table having a light spring fastened at one end to a rigid wall (Fig. 5.4 a). Let us direct the block towards the spring with velocity v. After the block touches the spring, the speed and hence kinetic energy of the block decreases until finally it is brought to rest by the opposing force of the spring. The block now reverses its motion as the spring expands. It gains kinetic energy and, when it comes to its initial point of contact with the spring, we find

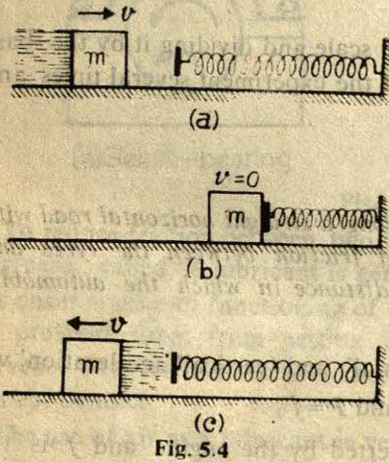


Fig. 5.4

that it has the same speed (not velocity) and kinetic energy. The block loses kinetic energy during the forward motion and gains it all back during the backward motion. In the complete trip (forward + backward motion) its kinetic energy, that is, ability to do work remains the same. Thus under the action of elastic force of the spring the ability of the block to do work has been conserved. Such force, and other forces that act in the same way, are called conservative forces. The other examples of conservative forces are gravitational force, electric force, magnetic force etc.

If, however, a particle is acted upon by such force or forces that its ability to do work in a round trip is not conserved, the force or forces is termed 'non-conservative'. Frictional force is the most familiar non-conservative force. In the above example if friction is there, the kinetic energy of the block i.e., its ability to do work will be decreased on the completion of the round trip (forward + backward motion).

A second way of presenting conservative and non-conservative forces is made available when we consider the total work done by a force on the body in a round trip. The work done by the elastic force on the block is equal and opposite in sense in the forward and backward motion of the block and hence in the complete trip net work done is zero. The work done on the block by frictional force is negative in each half of the round trip because a frictional force always opposes motion. Thus a second (equivalent to the first) definition of conservative force and non-conservative force is :

A force is conservative if the work done by the force on a particle in a round trip is zero. A force is non-conservative if work done by the force on a particle in a round trip is not zero.

Still a third way is open to define conservative and non-conservative forces. Suppose a particle passes from the position a to b along path 1 and back from b to a along path 2.

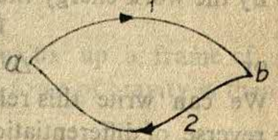


Fig. 5.5

Let us denote work done on the particle by the force from a to b along 1 by ${}_1W_{ab}$ and that from b to a along 2 by ${}_2W_{ba}$. For the round trip work done by the force on the particle is ${}_1W_{ab}+{}_2W_{ba}$. If the force is conservative,

$${}_1W_{ab}+{}_2W_{ba}=0$$

or

$${}_1W_{ab}=-{}_2W_{ba}={}_2W_{ab}.$$

Thus, if force is conservative, the work done from a position to another does not depend on the path. Hence, we have another equivalent definition of conservative and non-conservative forces :

A force is conservative if the work done by it on a particle that moves between two points depends only on these two points and not on the path followed. A force is non-conservative if the work done by that force on a particle that moves between two points depends on the path taken between those points.

We have defined potential energy of a system as the capacity of doing work by virtue of position or configuration. Here we want to add one more characteristic of potential energy of a system. The potential energy of a system is a form of stored energy which can be fully recovered and converted into kinetic energy. Complete conversion of potential energy into kinetic energy is possible only when force is conservative. Hence we see that every conservative force can be associated with potential energy. We cannot associate a potential energy with a non-conservative force such as the force of friction because the kinetic energy of a system in which such forces act does not return to its initial value when the system returns to its initial position.

Let us now focus our attention on the system (block + spring). The system possesses both potential energy and kinetic energy i.e., its ability to do work is $(U+K)$ where U is its (of the system) potential energy and K is its kinetic energy. Since the system is acted on by conservative forces, its ability to do work must remain conserved i.e.,

$$U+K=\text{a constant}$$

or

$$\triangle U+\triangle K=0$$

or

$$\triangle U=-\triangle K.$$

By the work-energy theorem

$$W=\triangle K=-\triangle U$$

or

$$\triangle U=-W=\int -F\,dx. \qquad .. \ (5.15)$$

We can write this relation (remembering that integration is just the reverse of differentiation) as

$$F=-\frac{dU}{dx}. \qquad .. \ (5.16)$$

Eq. 5·16 fixes up the mathematical criterion for a force to be conservative.

Eq. 5.15 shows how to calculate the change in potential energy of a conservative system (system under the action of conservative forces).

Let us now generalise the work-energy theorem by considering not only conservative forces but also non-conservative forces. We can rewrite the work energy theorem

$$\Sigma W_{conservative} + \Sigma W_{nonconservative} = \triangle K.$$

We have seen that each conservative force can be associated with a potential energy given by, $W = -\triangle U$.

$$\therefore \quad \Sigma W_{conservative} = -\Sigma \triangle U.$$

We cannot associate a potential energy with a non-conservative force but our experience (human experience) tells us that we can always find new forms of energy which correspond to this work ($W_{nonconservative}$). For example, by experience we learn that heat is a form of energy associated with frictional forces (an example of non-conservative forces). So we can write

$$\Sigma W_{nonconservative} = -\Sigma \triangle Q.$$

Hence the work-energy theorem can be written as

$\triangle K + \Sigma \triangle U + \Sigma \triangle Q = 0$, $\triangle K + U + Q =$ a constant

or $\Sigma \triangle K + \Sigma U + \Sigma Q =$ a constant,

or $\quad K + \Sigma U + \Sigma Q =$ a constant .. (5.17)

i.e., Kinetic energy + potential energy

+ energy in other forms = a constant.

This Eq. 5.17 is called the principle of conservation of energy which may be stated as : *Energy may be transferred from one kind to another but it cannot be created or destroyed; the total energy is a constant.*

5.5. Calculation of the Potential Energy Associated with two familiar Examples of Conservative Forces, the Gravitational Force of the Earth and the Elastic Restoring Force

(a) *Gravitational potential energy* : Let us fix up a frame of reference with the y-axis to be upward; the force of gravity is then in the negative y-direction. We have

$$F = -mg.$$

$\therefore$ The change in potential energy at a height h is found from Eq. 5.15

$$\triangle U = -\int_{y=0}^{y=h} F\,dy = -\int_{0}^{h} -mg\,dy = mgh.$$

But $\triangle U = U - U_0$ where U_0 = potential energy at the ground-level;

$$\therefore \quad U = U_0 + mgh. \quad .. \quad (5.18)$$

The potential energy at the ground level can be taken as zero, so that $U_0 = 0$ and

$$U = mgh. \quad .. \quad (5.18a)$$

Remember Eq. 5.18a does not give the absolute value of the potential energy of a body; rather it gives potential energy of the body relative to the ground level value. Eq. 5.18 gives the absolute value of the potential energy.

(*b*) *Potential energy of a compressed spring*: When a spring is compressed by x, the force exerted on the mass by the spring is

$$F = -kx.$$

The change in potential energy of the system (mass + spring) is

$$\triangle U = -\int_0^x F\,dx = -\int_0^x -kx\,dx = \tfrac{1}{2}kx^2.$$

In our ideal system kinetic energy is localised in the mass and potential energy is localised in the spring and hence the above change in the potential energy of the system is the change in potential energy of the spring. Thus the potential energy of the spring is

$$U = U_0 + \tfrac{1}{2}kx^2.$$

When $x = 0$, $U = 0$ and $\therefore \quad U_0 = 0$;

$$\therefore \quad U = \tfrac{1}{2}kx^2. \quad .. \quad (5.19)$$

Examples

1. *A 1 kg block collides with a horizontal weightless spring of $k = 2\ Nm^{-1}$. The block compresses the spring 4 m from the initial position. The coefficient of friction is ·25. What was the speed of the block at the instant of c llision ?*

Sol. Work done by the frictional force $= -\mu mgx$

(minus because displacement takes place opposite to the direction of force)

$$= -{\cdot}25 \times 1 \times g \times 4 = -9{\cdot}8\ \text{J}.$$

Work done by the elastic force $= -\tfrac{1}{2}kx^2$

$$= -\tfrac{1}{2}.2.4^2 = -16\ \text{J}.$$

Change in kinetic energy $= \tfrac{1}{2}m\,(v^2 - v_0^2)$

$$= \tfrac{1}{2}.\ 1.\ (0^2 - v_0^2) = -\frac{v_0^2}{2}.$$

By the 'work-energy' theorem

$$-\frac{v_0^2}{2} = -16 - 9{\cdot}8, \text{ or } v_0^2 = 2 \times 25{\cdot}8$$

or $$v_0 = \sqrt{51{\cdot}6} = 7{\cdot}2 \text{ m s}^{-1}. \quad \textbf{Ans.}$$

2. *A 18 kg body is pushed up a frictionless 30° incline plane 3 m long by a horizontal force F. (a) If the speed at the bottom is ·6 ms⁻¹ and at the top is 3 ms⁻¹, how much work is done by F ? (b) Suppose the plane is not frictionless and $\mu_k = {\cdot}15$. What work will the same force do ? How far above the plane does it go ?*

Sol. (a) Work done by the gravitational force

$$= -mgs \sin\theta = -18 \times 9{\cdot}8 \times 3. \sin 30°$$
$$= -18 \times 9{\cdot}8 \times 3 \times \tfrac{1}{2} = -264{\cdot}6 \text{ J}.$$

By the 'work energy' theorem

$$\tfrac{1}{2}.18(3^2 - {\cdot}6^2) = W + (-264{\cdot}6)$$

or $$W = 264{\cdot}6 + 9 \times (9 - {\cdot}36) = 342{\cdot}4 \text{ J}. \quad \textbf{Ans.}$$

Work done by $F = \vec{F}.\ \vec{s}$;

$$\therefore \quad 342{\cdot}4 = F.3 \cos 30°, \text{ or } F = \frac{342{\cdot}4 \times 2}{3\sqrt{3}}. \quad \textbf{Ans.}$$

(b) As there is no acceleration perpendicular to the plane,

$R = mg \cos\theta + F \sin\theta$

$$= 18 \times 9{\cdot}8. \cos 30° + \frac{342{\cdot}4 \times 2}{3\sqrt{3}}. \sin 30°$$

$$= 18 \times 9{\cdot}8. \frac{\sqrt{3}}{2} + \frac{342{\cdot}4 \times 2}{3\sqrt{3}}.\tfrac{1}{2}$$

$$= 152{\cdot}76 + 66 = 218{\cdot}76.$$

$\therefore$ The frictional force

$$= 218{\cdot}76 \times {\cdot}15 = 33 \text{ newton}.$$

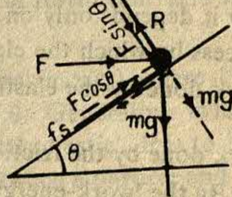


Fig. 5.6

Let the block go up by s.

$$\text{Work done by the force} = Fs \cos 30° = \frac{342{\cdot}4 \times 2}{3\sqrt{3}}.s.\ \frac{\sqrt{3}}{2}.$$

$$= 114\, s.$$

Work done by the frictional force $= -33s$.

Work done by the gravitational force $= -mg\ s \sin\theta$

$$= -18\times 9{\cdot}8.s.\tfrac{1}{2}$$
$$= -88\ s.$$

By the 'work-energy' theorem

$$\tfrac{1}{2}.18(0^2 - {\cdot}6^2) = 114s - 33s - 88s$$

or $\quad -3{\cdot}24 = -7s$, or $s = {\cdot}46$ m. **Ans.**

$\therefore$ Work done by the force $= 114\times{\cdot}46 = 52{\cdot}44$ J. **Ans.**

3. *The cable of a 4000 kg elevator snaps when the elevator is at rest at the first floor so that the bottom is at a distance of 12 m above a cushioning spring whose spring constant is $k = 10^4$kgm^{-1}. A safety device clamps the guide rails so that a constant frictional force of 1000 kg opposes the motion of the elevator. (a) Find the speed of the elevator before it hits the spring. (b) Find the total distance that the elevator will move before coming to rest.*

Sol. To solve for (a), consider the elevator as the system.

Work done by the gravitional force $= mgh$.

Work done by the frictional force $= -1000g.h$.

By the 'work-energy' theorem

$$\tfrac{1}{2}mv^2 = mgh - 1000gh \quad (\because \quad \Sigma W = \triangle K)$$

or $\quad v^2 = 2gh - \dfrac{1000\times g\times h}{2000} = 2\times 9{\cdot}8\times 12 - 6\times 9{\cdot}8$

or $\quad v = 13{\cdot}3$ m s^{-1}. **Ans.**

(b) The change in kinetic energy of the system $= 0$.

Work done by the gravitational force $= 4000g\ (12 + x)$.

(The gravitational force is a conservative force, hence the work done by it depends only on the initial and final position and not on the process by which the elevator comes to the final position.)

Work done by the elastic force $= -\tfrac{1}{2}kx^2$

$$= -\tfrac{1}{2}\times 10^4 gx^2 = -5000gx^2.$$

Work done by the frictional force $= -1000gs$.

$\therefore$ By the 'work-energy' theorem

$$0 = 4000g\ (12 + x) - \tfrac{1}{2}kx^2 - 1000gs$$

or $\quad 4000g\ (12 + x) - 5000gx^2 - 1000gs = 0$

or $\quad 4000g\ (12 + x) = 5000gx^2 + 1000gs$

or $\quad 4(12 + x) = 5x^2 + s.$

Considering the equilibrium of the elevator in the final position

$$4000g = kx + 1000g$$

or $$4000g = 10^4 gx + 1000g, \text{ or } 4 = 10x + 1$$

or $$x = \cdot 3 \text{ m.}$$

$$\therefore\ 4\,(12 + \cdot 3) = 5 \times \cdot 3^2 + s, \text{ or } s = 49\cdot 2 - \cdot 45 = 48\cdot 75 \text{ m.}$$ **Ans.**

5.6. Collisions : Elastic : Inelastic

Material particles and bodies are constantly colliding around us. The atoms or molecules of a gas are constantly colliding with each other and also on the walls of the container. We learn much about atomic, nuclear, and elementary particles by observing collisions between them. Collision is the phenomenon of exertion of mutual action and reaction of great magnitude between particles for a very short interval of time when they are in actual touch. The forces that arise between particles are elastic in nature. No body is perfectly rigid i.e., springless. When a bat strikes a baseball, the ball is deformed like a spring and exerts an elastic force on the bat.

Elastic forces are conservative and hence in elastic collisions, the kinetic energy of the system (the bodies participating in collision) is conserved. *Thus collision is 'elastic' when the kinetic energy of the system is conserved. Otherwise, the collision is said to be 'inelastic'. When two bodies stick together after collision, the collision is said to be 'completely inelastic'.*

According to Newton's experimental law on collisions of two bodies the relative velocity of any one of the colliding bodies relative to the other body after impact is proportional to the relative velocity of the same body before impact but opposite in direction. The constant of proportionality is called the 'coefficient of restitution' and it is generally denoted by 'e'. Thus,

V_{AB} (after impact) $= -e \times V_{AB}$ (before impact) where V_{AB} stands for the relative velocity of A relative to B. Generally e is less than 1. When $e = 1$, collision is said to be 'perfectly elastic'. It is to be remembered that when collision takes place between a particle and a smooth heavy wall, only the perpendicular components of velocities before and after impact are subject to Newton's law and not the ones parallel to the wall. The components of velocities before and after impact parallel to the wall remain unaltered.

Elastic collision in one dimension :

Let two particles of masses m_1 and m_2 move with velocities u_1 and u_2 along a line to the right before collision and with velocities v_1 and v_2 along the same line to the right after collision. Then by conservation of momentum

$$m_1u_1+m_2u_2=m_1v_1+m_2v_2 \quad \ldots \quad \text{(i).}$$

Since collision is elastic, kinetic energy remains conserved.

$$\therefore \quad \tfrac{1}{2}m_1u_1^2+\tfrac{1}{2}m_2u_2^2=\tfrac{1}{2}m_1v_1^2+\tfrac{1}{2}m_2v_2^2 \quad \ldots \quad \text{(ii).}$$

From (i) and (ii),

$$m_1(u_1-v_1)=m_2(v_2-u_2) \text{ and } m_1(u_1^2-v_1^2)=m_2(v_2^2-u_2^2).$$

Dividing these two we have,

$$u_1+v_1=u_2+v_2 \quad \text{or} \quad (u_1-u_2)=-(v_1-v_2).$$

Thus it is seen that in elastic collision the relative velocity remains unchanged in magnitude but reversed in direction.

Solving for v_1 and v_2 in terms of initial velocities we have

$$v_1=\left(\frac{m_1-m_2}{m_1+m_2}\right)u_1+\frac{2m_2}{m_1+m_2}u_2$$

$$v_2=\frac{m_2-m_1}{m_2+m_1}u_1+\frac{2m_1}{m_2+m_1}u_2.$$

Special cases : (i) When the colliding particles have the same masses, that is, $m_1=m_2$, it follows from the above equations $v_1=u_2$ and $v_2=u_1$. Thus in the elastic collision of two particles of equal masses, there is simply *exchange* of velocities between particles.

(ii) When the colliding particles are of the same mass and the second particle is at rest, that is, $m_1=m_2$ and $u_2=0$, it follows from above equations that $v_2=u_1$ and $v_1=0$. Thus the first particle is stopped and the second one *takes off* with the velocity of the first particle.

(iii) When a particle collides with a slow moving massive body, that is, when $m_2=\infty$, it follows from the above equations that

$$v_1=-u_1+2u_2=-(u_1-2u_2) \text{ and } v_2=u_2.$$

Thus the velocity of the particle is reversed and its magnitude is reduced by twice the velocity of the massive body.

*When the line of approch is along the centre to centre line, the collision is said be head-on.

Examples

1. *Show that in an elastic collision the speed of the centre of mass of two particles, m_1 moving with initial speed u_1 and m_2 moving with initial speed u_2 is a constant and is*

$$v_{cm}=\left(\frac{m_1}{m_1+m_2}\right)u_1+\left(\frac{m_2}{m_1+m_2}\right)u_2.$$

Sol. Let x be the initial position of the centre of mass from any origin on the line joining the particles and x_1 and x_2 are the positions of the particles respectively.

Then $$(m_1+m_2)x=m_1x_1+m_2x_2$$

or $$x=\frac{m_1}{m_1+m_2}\,x_1+\frac{m_2}{m_1+m_2}\,x_2;$$

$$\therefore\quad \frac{dx}{dt}=\frac{m_1}{m_1+m_2}\cdot\frac{dx_1}{dt}+\frac{m_2}{m_1+m_2}\cdot\frac{dx_2}{dt}$$

or $$v_{cm}=\frac{m_1}{m_1+m_2}\,u_1+\frac{m_2}{m_1+m_2}\,u_2.$$

Since the collision is elastic and the system is free from external forces, the centre of mass of the system will have no acceleration. Hence the velocity of the centre of mass will be the same as before.

$$\therefore\quad v_{cm}=u_{cm}$$

or $$u_{cm}=\frac{m_1}{m_1+m_2}\,u_1+\frac{m_2}{m_1+m_2}\,u_2.$$ Proved.

2. *Show that in an elastic collision between two particles of mass m_1 with a particle of mass m_2 initially at rest the maximum angle θ_m through which m_1 can be deflected by the collision is given by*

$$\cos^2\theta_m=1-\frac{m_2^2}{m_1^2}.$$

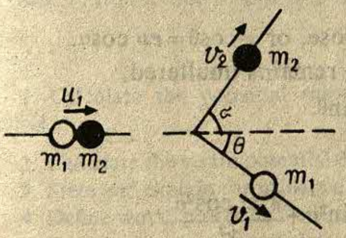


Fig. 5.7

Sol. Let m_1 hit m_2 with velocity u_1. After collision let m_1 move with velocity v_1 in direction θ with the original direction and m_2 with v_2 at an angle α on the other side of the line of approach.

Considering the conservation of momentum along and perpendicular to the line of approach we have

$$m_1u_1=m_1v_1\cos\theta+m_2v_2\cos\alpha$$

and $\qquad o = -m_1v_1 \sin\theta + m_2v_2 \sin\alpha$

or $\qquad m_1u_1 - m_1v_1 \cos\theta = m_2v_2 \cos\alpha$

and $\qquad m_1v_1 \sin\theta = m_2v_2 \sin\alpha.$

Squaring and adding to eliminate α we have

$$m_1^2u_1^2 - 2m_1^2u_1v_1 \cos\theta + m_1^2v_1^2 \cos^2\theta + m_1^2v_1^2 \sin^2\theta = m_2^2v_2^2$$

or $$m_1^2u_1^2 - 2m_1^2u_1v_1 \cos\theta + m_1^2v_1^2 = m_2^2v_2^2.$$

Since the collision is elastic

$$\tfrac{1}{2}m_1u_1^2 = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2$$

or $$m_1u_1^2 = m_1v_1^2 + m_2v_2^2$$

or $$m_1m_2u_1^2 = m_1m_2v_1^2 + m_2^2v_2^2;$$

$\therefore \quad m_1^2u_1^2 - 2m_1^2u_1v_1 \cos\theta + m_1^2v_1^2 = m_1m_2u_1^2 - m_1m_2v_1^2$

or $\quad m_1u_1^2 - 2m_1u_1v_1 \cos\theta + m_1v_1^2 = m_2u_1^2 - m_2v_1^2$

or $\quad (m_1 - m_2)\, u_1^2 - 2m_1v_1 \cos\theta\, u_1 + (m_1 + m_2)\, v_1^2 = 0.$

Since u_1 is real, we must have

$$4m_1^2v_1^2 \cos^2\theta \geqslant 4\,(m_1 - m_2)\,(m_1 + m_2)\,v_1^2$$

or $$\cos^2\theta \geqslant \frac{m_1^2 - m_2^2}{m_1^2}, \text{ or } \cos^2\theta \geqslant 1 - \frac{m_2^2}{m_1^2}.$$

$\therefore$ When $\theta = \theta_m$,

$$\cos^2\theta_m = 1 - \frac{m_2^2}{m_1^2}. \quad \textbf{Proved.}$$

3. *A smooth particle strikes a fixed plane. Its volocity before impact is of magnitude u, at an angle α with the normal to the plane. Find the velocity of the particle just after the impact if e is the coefficient of restitution.*

Sol. The relative velocity of the particle $= u \cos\alpha$ relative to the plane and towards the plane.

The relative velocity of the particle $= -v \cos\theta$ relative to the plane and towards the plane.

By Newton's law, $-v \cos\theta = -eu \cos\alpha$, or $v \cos\theta = eu \cos\alpha$.

Since velocity parallel to the plane remains unaltered,

$$u \sin\alpha = v \sin\theta.$$

Squaring and adding $\sin\theta$ and $\cos\theta$,

$$\sin^2\theta + \cos^2\theta = 1 = \frac{u^2}{v^2} \sin^2\alpha + \frac{e^2u^2 \cos^2\alpha}{v^2}$$

or $$v = u \sqrt{\sin^2\alpha + e^2 \cos^2\alpha}. \quad \textbf{Ans.}$$

QUESTIONS

(A)

1. Is (a) $\mu_s > \mu_k$, (b) $\mu_s = \mu_k$, (c) $\mu_s < \mu_k$, (d) none of these ?

2. Power of a system is (a) $\vec{F}.\vec{v}$, (b) $\vec{F}.\vec{s}$, (c) $\vec{F}\times\vec{v}$, (d) $\vec{F}\times\vec{s}$.

3. The dimensions of work are
(a) ML^2T^{-3}, (b) $ML^{-2}T^{-2}$, (c) ML^2T^{-2}, (d) MLT^{-1}.

4. A collision is elastic when (a) the potential and kinetic energy are conserved, (b) the potential, kinetic and other forms of energy are conserved, (c) only the kinetic energy is conserved, (d) only the potential energy is conserved.

5. A block is about to slide down an inclined plane of inclination 30° with the horizontal. A 5 kg weight is then put on the block. Now the system (block+weight) will just slide down the plane when its inclination is (a) greater than 30°, (b) less than 30°, (c) equal to 30°, (d) will never slide down the plane.

6. If the coefficient of static friction between a body and an inclined plane be less than 1, the 'angle of repose' (a) cannot be greater than 30°, (b) cannot be greater than 45°, (c) cannot be less than 30°, (d) cannot be less than 45°.

7. A bullet of mass m_A and velocity v_A is fired into a block of mass m_B and sticks to it. The final velocity of the system equals (a) $\frac{v_A}{m_A+m_B}\times m_B$, (b) $\frac{v_A}{m_A+m_B}\times m_A$, (c) $\frac{m_A+m_B}{m_A}\times v_A$, (d) $\frac{m_A+m_B}{m_B}\times v_A$.

(I. I. T. 1975)

8. A man pulls a body along a rough horizontal surface. If W_m is the work done by him and W_f is the work done by the frictional force, then (a) W_m is +ve, W_f is −ve, (b) W_m is −ve, W_f is +ve, (c) W_m is −ve and W_f is negative, (d) none of these.

9. If, in the example 8, W_m is the work done against man and W_f is the work done against friction, then (a) W_m is +ve and W_f is −ve, (b) W_m is −ve, W_f is +ve, (c) W_m is −ve and W_f is negative, (d) none of these.

10. If a small body moving with velocity u strikes elastically a massive body moving with velocity v ($u>v$) in the same direction, the velocity of the small body after collision is (a) $-(u-2v)$, (b) $u-2v$, (c) $-(u+v)$, (d) $u+2v$.

Ans. 1. (a), 2. (a), 3. (c), 4. (c), 5. (c), 6. (b), 7. (b). 8. (a). 9. (b). 10. (a).

(B)

1. Calculate the potential energy of a body placed at a height 'h' above the ground.

2. Calculate the strain potential energy of a compressed spring.

3. State and explain the principle of conservation of energy.

4. Define work, power and energy. State their units in SI.

5. Explain the terms 'coefficient of friction' and 'angle of friction'.

6. Show that the coefficient of friction is equal to the tangent of the angle of friction.

(C)

1. Define work, energy and power. Deduce their dimensions.

(Ran. 1973; Mag. '77; Bhag. '78)

2. State and explain the laws of limiting friction and describe experiments to verify them. (Bih. 1977; Mag. '70 S; Pat. '72; Ran. '70)

3. State and explain the 'work-energy' theorem. Explain what you mean by conservative and non-conservative forces. Is the kinetic energy completely recoverable when a non-conservative force acts on a system ?

4. What do you mean by elastic, inelastic and completely inelastic collisions ? Show that in elastic head-on collisions of two bodies the relative velocity remains unchanged in magnitude but reversed in direction.

(D)

1. Find the horizontal force required to prevent the slipping of a body of mass 20 kg on a rough plane inclined at 60° to the horizontal. (coefficient of static friction$=\cdot 1$)

(Ans. : 272·6 newton)

2. A body slides down on inclined plane through a distance of 10 m in 3 s. If the plane makes an angle of 30° with the horizontal, find the coefficient of kinetic friction between the body and the inclined plane.

(Ans. : ·3)

3. A body of mass 40 gm starts from rest and slides down a plane inclined at 30° to the horizontal. After it travelled 1 m down the plane its velocity is 8 ms^{-1}. Calculate the work done against friction. What is the work done by friction ?

(Ans. : $+\cdot 183$ J, $-\cdot 183$ J)

4. Prove that the change in kinetic energy of a body equals the work done by an external force on the body.

5. A man supports a mass of 2 kg in his hand. What is the work done by him when (a) he is stationary, (b) he moves a distance 20 m up an incline of 1 in 20, (c) he moves down the plane, (d) in the round trip, (e) he moves over a distance of 20 m on a level horizontal distance.

(Ans. : (a) Zero, (b) $+19\cdot 6$ J, (c) $-19\cdot 6$ J, (d) Zero, (e) Zero)

6. Two balls of masses 3 kg and 4 kg moving in the same direction with velocities 8 ms^{-1} and 16 ms^{-1} respectively, collide and then move on together. Find the energy lost in the collision.

(Ans. : 55 joule)

7. What is the H. P. of a pump which empties the water of a well 5 m deep and 1·5 m cross section in one hour ? Assume that the level of water was 7m below in the beginning.

(Ans. : ·256 H. P.)

8. Water is pumped up from a well through a vertical height of 5 m by means of a motor. The motor discharges 4 cubic metre of water per minute, working at an efficiency of 85%. Calculate the horse power of the motor, assume the level of water to remain stationary. (Ans. : 4·38 H. P.)

9. A cord is used to lower vertically a block of mass M — through a distance d at a constant downward acceleration $g/4$. Find the work done by the cord on the block. (Ans. : $\frac{3}{4}Mgd$)

10. A running man has half the kinetic energy of a boy whose mass is half the mass of the man. The man speeds up by 1 ms^{-1} and then has the same kinetic energy as the boy. What were the original speeds of man and boy ?

(Ans. : 2·41 ms^{-1}, 4·828 ms^{-1})

11. From what height would an automobile have to fall to gain kinetic energy equivalent to what it would have when going 100 km per hour ?

(Ans. : 39·37 m)

12. A block of mass 3·57 kg is drawn at a constant speed a distance 4·06 m along a horizontal floor by a rope exerting a constant force of 7·68 newton making 15° with the horizontal. Calculate (a) the total work done on the block, (b) the work done by the rope on the block, (c) the work done by friction on the block and (d) the coefficient of kinetic friction.

[Ans. (a) 0, since there is no net force on the block, (b) 30·1 J, (c) —30·1 J, (d) ·225]

13. A railroad flat car is loaded with crates having coefficient of friction ·25 with the car. If the train is moving at 30 kph, in how short a distance can the train be stopped ? (Ans. 1·3 m)

(Hint : Crates will not slide so long the two (car+crate) have a common retardation. The limiting frictional force between the car and the crate will produce the maximum retardation at which the crates will not slide)

14. A piece of ice slides down a 45° incline in twice the time it takes to slide down a frictionless 45° incline. What is the coefficient of kinetic friction between the ice and the incline ?

(Ans. : ·75)

(E)

1. The earth moving around the sun in a circular orbit is acted upon by a force and hence work must be done on the earth by this force. Do you agree with this statement ? (I.I.T. 1973)

2. Two springs have their force constants as k_1 and k_2 ($k_1 > k_2$). On which spring is more work done (i) when their lengths are increased by the same amount, (ii) when they are stretched by the same force ? (I.I.T. 1976)

3. A particle of mass m is moving in a horizontal circle of radius r under a centripetal force equal to $-k/r^2$, where k is a constant. What is the total energy of the particle ? (I. I. T. 1977)

4. A bullet is fired from a rifle. If the rifle recoils freely, determine whether the kinetic energy of the rifle is greater than, equal to or less than that of the bullet. (I. I. T. 1978)

5. Mountain roads rarely go straight up the slope but wind up gradually. Why ?

6. Two discs are connected by a stiff spring. Can one press the upper disc down enough so that when it is released it will spring back and raise the lower disc off the table ? (I. I. T. 1976)

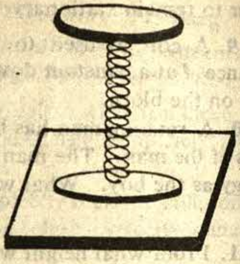


Fig. 5.8

7. A spring is kept compressed by tying its ends together tightly. It is then placed in acid and dissolves. What happens to its strain potential energy ?

8. A man rowing a boat up stream is at rest with respect to the shore. (a) Is he doing any work ? (b) If he stops rowing and moves down with the stream, is any work being done on him ?

9. Does the work done in raising a box onto a platform depend on how fast it is raised ?

10. Is it easier to push than to pull a lawn-roller ?

11. Can the coefficient of friction be greater than 1 ?

12. Increasing the area of contact should increase the actual microscopic area of contact. Why then does not the coefficient of friction depend on the total area in contact ?

[**Ans. : 1.** No, because the force is at right angles to the velocity. 2. (i) $W_1 > W_2$ (ii) $W_2 > W_1$. 3. $-\frac{1}{2}\frac{k}{r}$ (Hint : K.E$=\frac{1}{2}mv^2$; P.E. $=-\int F\,dr$ ($\because$ $F=-\left(\frac{dU}{dr}\right)$, for a conservative force). 4. less. 5. The gravitational force being conservative, the work done is the same whether the road goes straight to the top or it curls up to the top. But the time in which the work will be done will be different for the two paths. In the case of a curly path the time will be greater and hence 'sweat' of doing work will not be felt. From dynamic considerations we may say that the less the slope, the less is the force down the plane against which work is to be done. 6. Yes, if $kl > mg$, where l is the compression. 7. Its energy is converted into chemical energy. 8. (a) No. (b) No, because he is being carried horizontally. 9. No. 10. to pull, the force needed to pull$=\dfrac{\mu mg}{\cos\theta+\mu\sin\theta}$ and the force to push $=\dfrac{\mu mg}{\cos\theta-\mu\sin\theta}$ where θ is the angle made by the applied force with the horizontal and μ is the coefficient of friction. 11. Yes. 12. Though on decreasing the area of contact the number of contact points decreases, due to increase of pressure $\left(\text{pressure}=\dfrac{force}{area}\right)$ the microscopic area of 'cold welds' increases proportionately and so the same force is needed to move out the bodies. This is why limiting frictional force is independent of macroscopic area of contact.]

CHAPTER 6

CIRCULAR MOTION : CENTRIPETAL AND CENTRIFUGAL FORCES

(PSEUDO FORCE)

6.1. Uniform Circular Motion :

When a particle is constrained to move in such a way that its distance from a fixed point is always the same, its motion is said to be circular and if it describes equal lengths of the arcs of the circle in equal intervals of time, however small intervals may be, it is specially called uniform circular motion. A uniform circular motion is a periodic motion. The time in which particle returns to the same position in the path after making one complete trip round the circle is called time of revolution or time period (T) of the uniform circular motion and the number of trips made round the circle in one second is called the frequency (ν) of revolution. The line joining the centre to the particle is called *the radius vector of the particle*. As the particle moves, the radius vector describes angles at the centre of the circle. The rate of which the radius vector describes angles is called *angular velocity*. The angular velocity is an axial vector.

If θ be the angle made by the radius vector at any instant t with any reference line through the centre of the circle then

$$\omega \text{ (angular velocity)} = \frac{d\theta}{dt} \qquad .. \quad (6.1)$$

The unit of angular velocity is 'radian per second' (rad s^{-1}). In one complete revolution the radius vector sweeps 2π radians at the centre in T second.

$$\therefore \qquad \omega = \frac{2\pi}{T} \qquad .. \quad (6.2)$$

or

$$\omega = 2\pi\nu \quad \left(\because \quad \nu = \frac{1}{T}\right) \qquad .. \quad (6.3)$$

As the direction of motion is changing continuously in a uniform

circular motion, the velocity vector changes in direction but not in magnitude. Hence in case of a particle moving in a circle, its speed is uniform but not its velocity. In one complete revolution a particle describes the circumference of the circle which is $2\pi r$.

$$\therefore \quad v \text{ (speed of the particle)} = \frac{2\pi r}{T} = \omega r \qquad (\because \quad \omega = \frac{2\pi}{T})$$

or

$$v = \omega r \qquad .. \quad (6.4)$$

6.2. Velocity and Acceleration in a Uniform Circular Motion

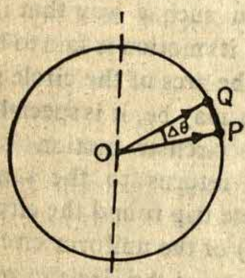


Fig. 6.1

Let r be the radius of the circular path of a particle. Fix up a frame of reference at the centre of the circle. Suppose at any instant t the particle is at P so that its position vector at time t is $\vec{OP}$ and after a short interval $\triangle t$ it is at Q so that its position vector is $\vec{OQ}$. The small displacement in time $\triangle t$ is $\vec{PQ} = \triangle \vec{s}$. Therefore average velocity during the interval $\triangle t$ is

$$\vec{\bar{v}} = \frac{\triangle \vec{s}}{\triangle t} = \frac{\triangle s}{\triangle t} . \triangle \hat{s} = r . \frac{\triangle \theta}{\triangle t} . \triangle \hat{s} \quad (\because \quad \triangle s = r \triangle \theta).$$

The instantaneous velocity of the particle as per definition is

$$\lim_{\Delta t \to 0} \frac{\triangle \vec{s}}{\triangle t}.$$

$$\therefore \quad \vec{v} = \lim_{\Delta t \to 0} \frac{\triangle s}{\triangle t} \triangle \hat{s}$$

$$= \lim_{\Delta t \to 0} r \frac{\triangle \theta}{\triangle t} \triangle \hat{s} = r . \lim_{\Delta t \to 0} \frac{\triangle \theta}{\triangle t} . \lim_{\Delta t \to 0} \triangle \hat{s}.$$

As Q approaches P, $\triangle \hat{s}$ becomes a unit vector along the tangent at P. Hence the limiting value of $\triangle \hat{s}$ is $\hat{u}_t$ (the unit vector along the tangent).

$$\therefore \qquad \vec{v} = r . \frac{d\theta}{dt} . \hat{u}_t = r\omega \hat{u}_t \qquad .. \quad (6.5)$$

Thus in a circular motion velocity is $r\omega$ in magnitude and along the tangent in direction.

We now seek to calculate the acceleration in the uniform circular motion. Let P be position of the particle at the time t and Q its position at the time $t+\triangle t$. The velocity at P is $\vec{v}$, a vector along the tangent to the circle at P. The velocity at Q is $\vec{v}'$, a vector along the tangent to the circle at Q.

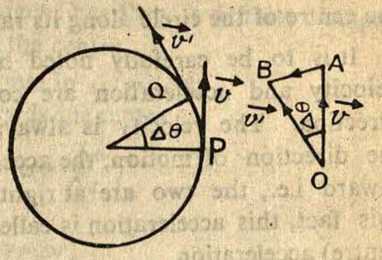


Fig. 6.2 (a)

Vectors $\vec{v}$ and $\vec{v}'$ are equal in magnitude but their directions are different. Represent $\vec{v}$ by $\vec{OA}$ and $\vec{v}'$ by $\vec{OB}$.

The change in velocity in time $\triangle t$ is

$$\triangle \vec{v} = \vec{v}' - \vec{v} = \vec{OB} - \vec{OA} = \vec{AB} = AB\hat{u}_{AB}$$

where $\hat{u}_{AB}$ is the unit vector along $\vec{AB}$

$$(\because \quad OA = OB = v = \omega r)$$

or
$$\triangle \vec{v} = \omega r \triangle\theta \, \hat{u}_{AB}$$

By definition of acceleration

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\triangle \vec{v}}{\triangle t}$$

$$= \lim_{\Delta t \to 0} \frac{\omega r \triangle\theta}{\triangle t} \hat{u}_{AB}.$$

$$= \omega r \lim_{\Delta t \to 0} \frac{\triangle\theta}{\triangle t} . \lim_{\Delta t \to 0} \hat{u}_{AB}$$

As $\triangle t_{\to 0}$, $\hat{u}_{AB}$ becomes a unit vector along the radius at P. Hence the limiting value of $\hat{u}_{AB}$ is $-\hat{u}_r$ ($\hat{u}_r$, the unit vector along the radius away from the centre).

$$\therefore \quad \vec{a} = \omega r . \frac{d\theta}{dt} . (-\hat{u}_r) = \omega r . \quad \omega(-\hat{u}_r)$$

$$= -\omega^2 r \hat{u}_r$$

$$\vec{a} = -\omega^2 r \hat{u}_r \qquad .. \quad (6.6)$$

or $$\vec{a} = -\frac{v^2}{r}\,\hat{u}_r \quad (\because\ v = \omega r) \qquad ..6.\ (6a)$$

Thus in a circular motion acceleration is $\omega^2 r$ or v^2/r directed towards the centre of the circle along its radius.

It is to be carefully noted here that in a circular motion both velocity and acceleration are constant in magnitude but not in direction. The velocity is always along the tangent to the circle in the direction of motion; the acceleration is always directed radially inward i.e., the two are at right angles to each other. Because of this fact, this acceleration is called *a radial* or *centripetal* (seeking a centre) acceleration.

ALTERNATIVE METHOD

Let a particle of mass m be constrained to move in a circle of radius r with uniform speed v. Let P be the position of the particle at any instant. The velocity at P is v along the tangent PT. After an infinite-simally small time Δt, let the position of the particle be at P'. The velocity of the particle at P' is v along the tangent $P'T'$ at P'. During this short interval the radius vector of the particle describes a small angle $\Delta\theta$ at the centre of the circle. Let us resolve the velocity at P and P' along PT and PO.

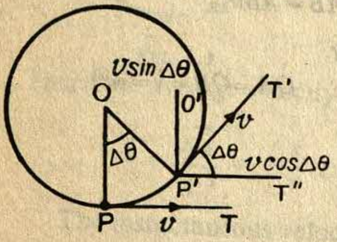


Fig. 6.2 (b)

At the point P the resolved parts are

$$v \cos 0^\circ = v \text{ along } PT$$

and

$$v \sin 0^\circ = 0 \text{ along } PO.$$

At the point P' the two resolved parts are

$$v \cos \Delta\theta \text{ along } P'T'' \parallel PT$$

and

$$v \sin \Delta\theta \text{ along } P'O' \parallel PO.$$

Since $\Delta\theta$ is very small, $\cos \Delta\theta = 1$ and $\sin \Delta\theta = \Delta\theta$.

Hence the two resolved parts at P' are

$$v \text{ along } P'T'' \parallel PT$$

and

$$v\Delta\theta \text{ along } P'O' \parallel PO.$$

Comparing the resolved parts at P and P' we find that along PT there is no change in velocity but there is change in velocity parallel to PO. Now imagine P' to be closer and closer to P. As P' approaches P, $P'O'$ tends to merge with PO and $P'T''$ with PT. So as

the particle passes from one point to the point next to it, there is no change in velocity along the tangent but there is definite change in velocity along the radius towards the centre.

Change in velocity along the normal in $\Delta t = v\Delta\theta$;

$\therefore$ the normal acceleration towards the centre

$$= \frac{v\Delta\theta}{\Delta t} = v\omega \quad (\omega = \frac{\Delta\theta}{\Delta t})$$

$$= v.\frac{v}{r} \quad (\because v = \omega r)$$

$$= \frac{v^2}{r}$$

$\therefore$ the radial (normal) acceleration $= \frac{v^2}{r}$ or $\omega^2 r$ ($\because v = \omega r$).

6.3. Centripetal and Centrifugal Forces : Pseudo Forces

According to Newton's second law ($F=ma$) of motion every accelerated body must have a force F acting on it.

If we observe a body in circular motion from a fixed frame of reference (inertial frame), the body will appear accelerated towards the centre and hence the body is **not** in equilibrium. Here we will interpret that a force is acting on the body which is directed inward along the radius of the path. Such a force is called *Centripetal force.* A force is essentially needed to keep a body move in a circle. Unless a force directed inward acts on the body, it will move in a straight line in confirmity with Newton's first law of motion. To keep the body on the circular path a force must act to deflect it continuously through small angles in small intervals of time. The force so needed is the above mentioned centripetal force. This force can always be accounted by pointing to a particular object in the environment. Therefore it is a 'real force'.* A centripetal force is not a new kind of force but it is simply a way of describing the behaviour of a force which is responsible for maintenance of circular motion. Thus a force can be centripetal and elastic as in whirling of a stone by a string, centripetal and gravitational as in the case of a 'Satellite', centripetal and electrostatic as in the circular motion of electron in an atom or centripetal and magnetic as in the circular motion of a charged particle in a magnetic field.

*Forces which can be attributed to some bodies in the environment are called '*real forces*'.

Centrifugal force : a Pseudo force. The force that we encounter can be attributed to some bodies in the environment. These forces are called *real forces.* The force on the satellite is a real force because this is due to attraction of the earth which is the only body in the environment of the satellite. The tension in a rope is a real force because this force is attributable to the bodies in the environment. But there are certain forces observed in non-inertial frames (accelerated frames) which cannot be attributed to any body in the environment. Such forces are called *pseudo forces* (or *inertial forces*). To fix up our ideas more appropriately let us imagine a large platform having a raised glass rim and rotating at a constant speed. Place a marble against the rim and let two observers watch the marble-one (O) fixed on the ground and the other (O') on the platform. Here the frame of reference of the observer O is an inertial frame and that of O' is non-inertial frame (rotating and hence accelerating). The observer O would declare the marble to be in uniform circular motion, accelerated radially inward with acceleration $a=v^2/r$. The necessary inward force called the centripetal force is the force exerted by the rim (environment of the marble) on the marble. The marble is definitely **not** in equilibrium

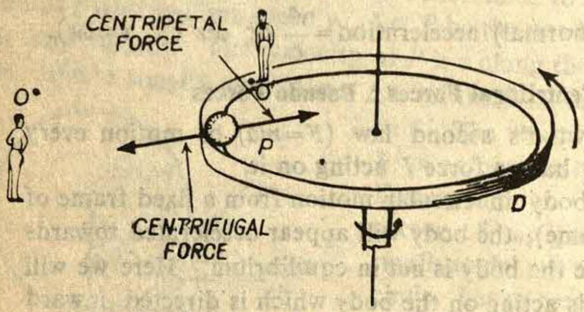


Fig. 6.3.

($\because \quad \vec{a} \neq 0$) from the point of view of this observer. There is an unbalanced force on the marble towards the centre of rotation.

The observer on the platform would declare the marble to be in equilibrium because he would see the marble always in the same position. The physical laws like law of inertia (no force no acceleration) are supreme. They are to be honoured in all frames of reference. Therefore, there cannot be any unbalanced force towards the centre in the non-inertial frame. But the marble is already acted on by a real force i.e., the force exerted by the rim on the marble. How is it then possible that there is no force on the marble

in the non-inertial frame? This is possible only when a force of the same magnitude but directed outward acts on the marble in this frame (non-inertial). What is this force due to? We cannot say. Such forces whose existence is necessitated so that laws of mechanics (classical) may be applicable in non-inertial frames as well are called *Pseudo forces* or *Inertial forces.* They are so named, because like real forces, we cannot attribute them to any body in the environment. Such forces exist only in non-inertial frames. This pseudo force or inertial force directed radially outward as observed from non-inertial frame is called *Centrifugal Force.* It is to be mentioned with emphasis that centrifugal force also really exist but only in non-inertial frames. If the observer O' displaces the marble a bit from the rim toward the centre of rotation, he observes that it moves back to the previous position. This shows that centrifugal forces really exist in non-inertial frames.

6.4. Magnitude of Centripetal and Centrifugal Forces

Please see the calulation of centripetal acceleration in Art. 6.2.

$\because$ Force = mass × acceleration $(\vec{F} = m\,\vec{a})$

$\therefore$ $\vec{F}_{centripetal} = m\,\vec{a}_{centripetal}$

or $\vec{F}_{centripetal} = -m\omega^2 r\,\hat{u}_r$ .. (6.7).

$\therefore$ The absolute value (i.e., only magnitude) of the centripetal force is

$$F_{centripetal} = m\omega^2 r \quad \text{or} \quad \frac{mv^2}{r} \qquad (\because\ v=\omega r)$$

Centrifugal acceleration $= \omega^2 r\,\hat{u}_r$

$\therefore$ $\vec{F}_{centrifugal} = m\omega^2 r\,\hat{u}_r$ .. (6·8.)

$\therefore$ The absolute value of the centrifugal force is

$$F_{centrifugal} = m\omega^2 r \quad \text{or} \quad m\frac{v^2}{r} \qquad (\because\ v=\omega r).$$

6.5. Some Illustrations

(a) *Motion of a cyclist in a curved path.* A cyclist turning a

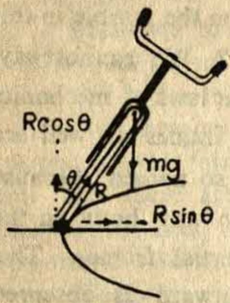


Fig. 6.4.

corner has to incline his body inwards because unless he does so he would not receive the necessary centripetal force. If he remains erect and wants to go round the curved path, it is physically impossible for him. We know a force directed radially inward is needed to keep a body move in a curved path. When he is erect there is no force towards the centre of the curved path. If he leans inward by a certain angle from the vertical, the reaction of the ground will be inclined by the same angle from the vertical. Now the reaction of the ground will have two resolved parts—one vertically upward (normal reaction) balancing the weight of the system (cycle+man) and the other along the horizontal (horizontal reaction or frictional force) towards the centre providing the necessary centripetal force.

If R is the reaction of the ground and θ is its inclination from the vertical we have

$$R\ \sin\theta = \text{centripetal force} = \frac{mv^2}{r}$$

and $R \cos\theta = mg$ (because there is no acceleration along the vertical).

Dividing we have,

$$\frac{R\ \sin\theta}{R\ \cos\theta} = \frac{\frac{mv^2}{r}}{mg}$$

or

$$\tan\theta = \frac{v^2}{rg} \quad .. \quad (6.9.)$$

(b) *Banking of highways and train road-beds on curves.* On a level road-bed around a curve having radius r, the forces acting on the train or cars are the four normal reactions of the ground at the four wheels vertically upwards, the weight mg vertically downwards at its centre of gravity and a horizontal centripetal force F. In the case of cars, this centripetal force is supplied by a '*sidewise frictional force*' exerted on the tyres by the ground; in the case of the railway car it is supplied by the outer rails exerting sidewise force on the inner rims of the car's wheels. Neither of these sidewise forces can be safely relied upon to be large enough to provide centripetal forces

for safe driving. Moreover, both cause unnecessary wear and tear of tyres or wheels. Hence the road-bed is banked on curves. When banked, the normal reactions at the wheels have not only vertical components to balance the weight of the car but also horizontal components which supply the centripetal force necessary for uniform circular motion; no additional sidewise forces are needed.

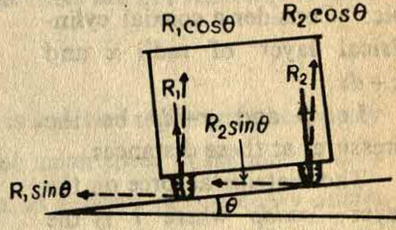


Fig. 6.5

Let θ be the correct angle of banking for speed v. Let R_1, R_2, R_3 and R_4 be the reactions at the wheels. Then centripetal force $=(R_1+R_2+R_3+R_4)\sin\theta$
$=m\times v^2/r$

and $(R_1+R_2+R_3+R_4)\cos\theta=mg$ (because there is no acceleration along the normal to the road-bed).

Dividing, we have,

$$\frac{(R_1+R_2+R_3+R_4)\sin\theta}{(R_1+R_2+R_3+R_4)\cos\theta}=\frac{\frac{mv^2}{r}}{mg}$$

or $$\tan\theta=\frac{v^2}{rg}.$$

We can now easily calculate the distance by which outer rail is to be raised above the inner rail in case of rail road-beds. Let L be the standard distance between rails and x be the height of the outer rail above the inner rail on a banked road-bed. Then,

$$\sin\theta=\frac{x}{L}.$$

When θ is small, $\tan\theta=\theta=\sin\theta$.

$$\therefore\quad \frac{x}{L}=\frac{v^2}{gr}$$

or $$x=\frac{v^2L}{gr}. \qquad (6.10)$$

6.6. Practical Applications

1. *Cream Separator*. In a cream separator a vessel containing milk is rotated fast. The cream, being lighter collects in a cylindrical

layer round the axis, whence it is drawn off and the skimmed milk is drained through an outlet fitted on the wall of the vessel.

In a rotating liquid there is essentially a 'horizontal pressure gradation' because without such gradation the rotational equilibrium of the liquid mass is not possible. Consider a coaxial cylindrical layer of radii x and $x+dx$.

Let p and $p+dp$ be the pressures at these distances.

The centripetal force on the layer $=2\pi x l dp$ where l is the height of the layer

Also, $=(2\pi x l dx \rho)\omega^2 x$

where ρ is the density of the liquid and ω is the angular speed of rotation.

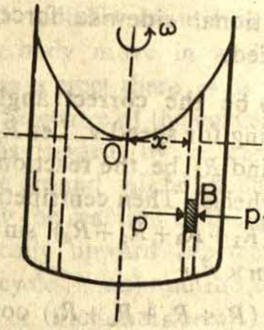


Fig. 6.6

$$\therefore \quad dp=\omega^2\rho\, x dx.$$

Now consider an elementary body B suspended in the liquid. Let its face area be $\triangle s$.

Then, $\triangle F_{inward}$ = inward force on the particle due to 'horizontal pressure gradation'

$$=\triangle s.dp$$
$$=\triangle s\rho\omega^2 x dx.$$

The 'inertia force', i.e., centrifugal force needed for rotational equilibrium of the particle $=(\triangle s.\ dx\sigma)\omega^2 x$ where σ is the density of the suspended particle.

$$\therefore \qquad \triangle F_{centrifugal}=\triangle s dx\sigma\omega^2 x.$$

$$\therefore \qquad \frac{\triangle F_{inward}}{\triangle F_{centrifugal}}=\frac{\rho}{\sigma}.$$

$\therefore$ If $\rho>\sigma$, $\triangle F_{inward}>\triangle F_{centrifugal}$

and if $\rho<\sigma$, $\triangle F_{inward}<\triangle F_{centrifugal}$.

Thus, particles whose density is less than that of the liquid are driven towards the axis of rotation and those whose density is greater than that of the liquid are driven away from the axis. Cream is lighter than milk and so it is separated from the milk and collects at the axis.

2. *The Centrifugal drier.* In laundries wet clothes are dried by packing them in a cylindrical vessel with perforated walls and rotated at a very high speed. Water particles stick to the clothings with a certain force called adhesive force. The water particles move off when the adhesive forces on the particles are not sufficient to keep them moving uniformly in a circle.

3. *Watt's Speed Governor.* This is a device to control the supply of steam to an engine such that the mean speed of the engine may remain constant at the desired value. It consists of two heavy metal balls A and B at the end of two rods hinged at the top of a spindle V which is driven by the engine itself. The two rods carrying the balls are connected by link rods to the sleeve S. When the spindle is rotated at a high speed the two balls are thrown away from the spindle and the sleeve goes up the spindle. With the decrease of the speed of rotation the sleeve falls down. This to-and-fro motion of the sleeve is used to regulate the supply of steam to the engine.

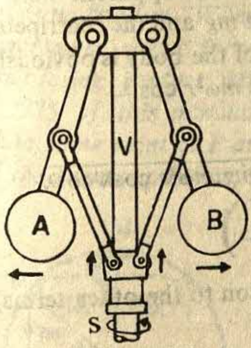


Fig. 6.7

6.7. Some Natural Consequences

1. *Flattening of the earth.* It is believed by the geologists that initially the earth consisted of molten matter. Because of the rotation of the earth, about its axis, this molten matter spread out at the equator where the centrifugal forces were greatest. As a result of this process the earth bulged at the equator and got flattened at the poles.

2. *Loss in weight due to axial rotation of the earth.* The earth

rotates about its axis once in 24 hours. Any body on its surface also rotates in a circle with the same angular speed as that of the earth. The force needed to keep it moving in a circle is provided by a part of the weight of the body. So a body on the surface of the earth loses a part of its weight.

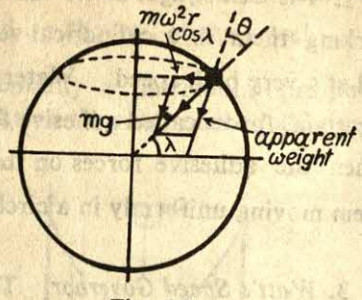


Fig. 6.8

Let us consider a body of mass '*m*' at a place of latitude λ on the surface of the earth. This body moves in a circle of radius $r\cos\lambda$ with angular velocity ω. The centripetal force on the body is $m\omega^2 r\cos\lambda$, where r is the radius of the earth. The apparent weight is obviously the vector difference of the weight mg and the centripetal force. Or in other words, the apparent weight of the body is obviously the vector sum of mg and the centrifugal force $m\omega^2 r\cos\lambda$.

$\therefore$ Apparent weight

$$=\sqrt{m^2g^2+m^2\omega^4r^2\cos^2\lambda-2mg.m\omega^2r\cos\lambda\cos\lambda}$$

$$=\sqrt{m^2g^2\left(1-\frac{2\,\omega^2r\cos^2\lambda}{g}\right)}$$

(neglecting $m^2\omega^4r^2\cos^2\lambda$ in comparison to the other terms)

$$=mg\left(1-\frac{2\,\omega^2r\cos^2\lambda}{g}\right)^{\frac{1}{2}}$$

$$=mg\left(1-\tfrac{1}{2}.\frac{2\,\omega^2r\cos^2\lambda}{g}\right)$$

(again expanding and neglecting terms of higher powers of ω)

or Apparent weight $=(mg-m\omega^2r\cos^2\lambda)$ newton

or Apparent weight $=\left(m-\dfrac{m\omega^2r\cos^2\lambda}{g}\right)$ kg.

3. *Reactions at the top and bottom on an aeroplane moving in a loop.* When the pilot of an aeroplane loops the loop (fig. 6.9), the reaction R of the seat at the lowest position must overcome his weight and also provide the necessary centripetal force.

$$\therefore\quad R-mg=\frac{mv^2}{r}$$

and at the highest point $R' + mg = \frac{mv^2}{r}$.

In ultrasonic planes the centripetal acceleration at times becomes 8 g. At such high speeds the walls of the arteries fail to circulate blood through the brain and consequently the pilot sometimes faints. To avoid such danger he lays straight horizontally.

Examples :

1. *A ball at the end of a string is whirled at a constant speed in a horizontal plane. If the radius of the circle is 1·2 m and the speed of the ball is 3 ms⁻¹, calculate the magnitnde of the radial acceleration.*

Sol. Radial accelaration $= \frac{v^2}{r} = \frac{3^2}{1 \cdot 2} = 7 \cdot 5$ ms^{-2}. **Ans.**

2. *A mass 'm' is moving inside a circular track of radius R. There is no friction. When m is at its lowest point its speed is v. Find the minimum speed for which 'm' will go completely round the circle without losing contact with the track. Suppose the particle moves with ·775 of this minimum velocity. The particle will move up the track to some point at P at which it will lose contact with the track. Find the angular position θ of the point.*

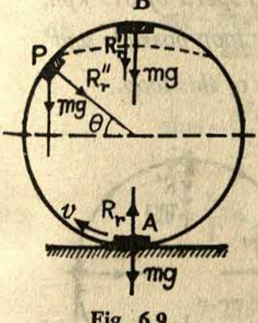


Fig. 6.9

Sol. At the lowest point A :

$R_r - mg = \frac{mv^2}{R}$ where R_r is the reaction of the track at A.

At the highest point B :

$R_r' + mg = \frac{mv'^2}{R}$ where R_r' is the reaction of the track and v' is the velocity at the highest point. Considering the conservation of energy at the highest and the lowest point,

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}mv'^2 + mg.\, 2R$$

or $$mv^2 = mv'^2 + 4\, mgR; \quad \therefore \quad R_r' = \frac{mv^2}{R} - 5\, mg.$$

It will not lose contact even at the highest if $R_r' \geqslant 0$

or $$\frac{mv^2}{R} \geqslant 5mg$$

or $$v \geqslant \sqrt{5gR}$$

$\therefore$ $$v_{minimum} = \sqrt{5gR}.$$

At the point P, $R_r'' + mg \sin\theta = \dfrac{mv''^2}{R}$.

Considering the energy conservation at A and P,

$$\tfrac{1}{2} mv^2 = \tfrac{1}{2} mv''^2 + mg\,(R + R\sin\theta)$$

or $$mv^2 = mv''^2 + 2mgR\,(1 + \sin\theta).$$

The block just loses contact at P when $R_r'' = 0$.

$\therefore$ $$mg \sin\theta = \frac{mv''^2}{R}$$

or $$mg\,R\sin\theta = mv''^2 = mv^2 - 2mgR\,(1 + \sin\theta).$$

It is given that $v = {\cdot}775\sqrt{5gR}$, or $v^2 = {\cdot}6 \times 5gR$;

$\therefore$ $$mgR \sin\theta = m \times {\cdot}6 \times 5gR - 2mgR\,(1 + \sin\theta)$$

or $$\sin\theta = 3 - 2(1 + \sin\theta)$$

or $$\sin\theta = \tfrac{1}{3}, \quad \text{or} \quad \theta = 19^\circ\, 30'. \ \textbf{Ans.}$$

3. *If a 2250 kg air plane loops the loop at a speed of 315 kph,*

(*a*) *find the radius of the largest circular loop possible and*

(*b*) *the force on the plane at the bottom of this loop.*

Sol. $$R - mg = \frac{mv^2}{r}$$

and $$R' + mg = \frac{mv^2}{r}.$$

For the largest loop $R' = 0$,

$$mg = \frac{mv^2}{r}$$

or $$r = \frac{v^2}{g}.$$

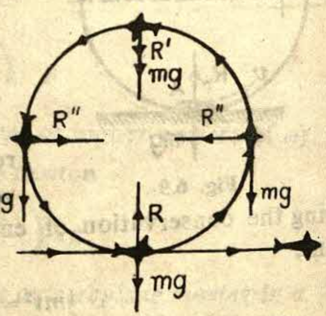


Fig. 6.10

Here $$v = \frac{315 \times 10^3}{60 \times 60} = \frac{315}{36} \times 10 = 87{\cdot}5 \text{ ms}^{-1}$$

$$\therefore \quad r = \frac{87{\cdot}5^2}{9{\cdot}8} = 781 \text{ m.}$$

At the bottom we have,

$$R = \left(mg + \frac{mv^2}{r}\right) \text{newton} = 2250\left(g + \frac{v^2}{r}\right)$$

or $$R = 2250\left(9{\cdot}8 + \frac{87{\cdot}5^2}{781}\right)$$
$$= 2250(9{\cdot}8 + 9{\cdot}8)$$
$$= 2 \times 2250 \times 9{\cdot}8 \;\; = 4500 \times 9{\cdot}8 \text{ N}$$
$$= 4500 \text{ kgf. } \textbf{Ans.}$$

4. *Due to rotation of the earth a plumb line may not hang along the direction of the earth's gravitational pull but slightly deviates from this direction. Calculate the deviation* (*a*) *at 45° latitude,* (*b*) *at the poles and* (*c*) *at the equator.* (*the radius of the earth = 6400 km.*)

Sol. Refer to the Fig 6.7. In this figure θ is the deviation. From the figure we have,

$$\frac{\text{apparent weight}}{\sin\lambda} = \frac{m\omega^2 r \cos\lambda}{\sin\theta}$$

or $$\frac{mg - m\omega^2 r \cos^2\lambda}{\sin\lambda} = \frac{m\omega^2 r \cos\lambda}{\sin\theta}$$

or $$\sin\theta = \frac{\omega^2 r \sin\lambda \cos\lambda}{g - \omega^2 r \cos^2\lambda}$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{24 \times 3600} = 7{\cdot}273 \times 10^{-5} \text{ rad/s.}$$

(a) $$\sin\theta = \frac{7{\cdot}273^2 \times 10^{-10} \times 6400 \times 10^3 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}}{9{\cdot}8 - 7{\cdot}273^2 \times 10^{-10} \times 6400 \times \frac{1}{2}}$$

$$= \frac{7{\cdot}273^2 \times 32 \times 10^{-4}}{9{\cdot}8} = 1{\cdot}73 \times 10^{-2}$$

$$= 0{\cdot}0173$$

or $$\theta = 1^\circ. \textbf{ Ans.}$$

(b) At the poles $\lambda=90°$ or $270°$,

$$\therefore \quad \theta=0. \textbf{ Ans.}$$

(c) At the equator, $\lambda=0$

$$\therefore \quad \theta=0. \textbf{ Ans.}$$

5. *In many amusement parks there is a lot of fun in a 'rotor' in which you are pinned to the wall of the rotor and you hang in space without falling down. Explain this and calculate the speed of rotation if the radius of the rotor is 2 m and the coefficient of friction between the textile material of clothing and a typical rotor wall (canvas) is ·4.*

Sol. The forces acting on the passenger are (i) mg, weight and (ii) f_s, the force of static friction between the passenger and the rotor wall and R, the reaction of the wall. Since the passenger does not move vertically,

$$f_s=mg$$

and since there is radial acceleration by $\frac{v^2}{r}$,

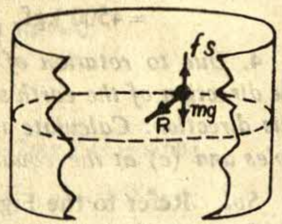


Fig. 6.11

$$\therefore \quad R=\frac{mv^2}{r}$$

But $\quad f_s=\mu R \quad$ or $\quad mg=\mu R$

$$\therefore \quad \frac{mg}{\mu}=\frac{mv^2}{r} \quad \text{or} \quad v=\sqrt{\frac{gr}{\mu}}$$

or $\quad v=\sqrt{\dfrac{9{\cdot}8\times2}{{\cdot}4}}=\sqrt{49}=7 \text{ ms}^{-1}=25{\cdot}2$ kph. **Ans.**

QUESTIONS

(A)

1. The centripetal force is (a) a real force, (b) a pseudo force, (c) not a force at all, (d) none of these.

2. A particle in a uniform circular motion has (a) constant velocity and constant acceleration, (b) varying velocity and constant acceleration, (c) constant velocity and varying acceleration, (d) varying velocity and varying acceleration.

3. A car rounds a curved path at 25 kph. If it rounds it at 50 kph, its tendency to overturn is (a) doubled, (b) tripled, (c) quadrupled, (d) halved.

4. A body of mass 2 kg tied to a string of length 2 m is moving with a uniform speed of 2 ms^{-1} in a horizontal circle. The tension in the string is (a) 2 N, (b) 3 N, (c) 8 N, (d) 4N.

5. A spirit level is tied at one end to a string and rotated rapidly in a horizontal plane. The bubble stands stationary (a) at the centre, (b) inside, (c) outside, (d) may stand anywhere.

6. When a stone is set in rotation in vertical plane, it has (a) only radial acceleration, (b) both radial and tangential acceleration, (c) only tangential acceleration, (d) none of these.

Ans : **1.** (a), **2.** (d), **3.** (c), **4.** (d), **5.** (b), **6.** (b).

(B)

1. Why are the train road-beds and highways banked on curves ?

2. How does the earth's axial rotation affect the apparent weight of a body at the equator ?

3. Can you check up whether a table top is horizontal in a moving train by a spirit level when it is moving along a curve ?

(*Hints*—Think of centrifugal force on the bubble.)

4. Distinguish between 'real force' and 'pseudo force' with examples.

(C)

1. Explain why a force is needed to keep a body moving uniformly in a circle. Calculate this force in terms of the mass of the body, its uniform speed and the radius of the circle.

2. What are Centripetal and Centrifugal forces ? Find the magnitude of these forces.

3. What are 'real' and 'pseudo' forces ? Explain centrifugal force as a pseudo force.

4. Why must a cyclist lean inwards to keep his balance when he is going round a circular course at high speed ? Deduce a relation between the speed, inclination and radius of the course.

(D)

1. The bridge over a canal is in the form of an arc of a circle of radius 15 m. What should be the maximum velocity of a car so that it may not leave the ground even at the highest point ? (Ans : 12·12 ms^{-1})

2. A centripetal force of 1 newton is acting on a body which moves with uniform speed on a circular path of radius 10 m. Calculate the kinetic energy of the body. (Ans : 5 joules)

3. Calculate the limiting velocity required by an artificial satellite for orbiting round the earth, given the radius of the earth as $6{\cdot}4\times10^6$ m and $g=9{\cdot}8$ ms^{-2}. (Ans : 7919·6 ms^{-1})

4. The distance between the two rails of a railway line is 1·67 m. A train is

moving on these rails with a speed of 50 kph in circular path of 167 m radius. By how much has the outer rail to be raised from the inner one to maintain dynamic equilibrium ? (Ans. : ·197 m)

5. Find the apparent weight of a solid of one ton at the equator, if the radius of the earth is $6{\cdot}4 \times 10^6$ m and $g = 9{\cdot}8$ ms^{-2} at the eqator. (Ans. : 996·55 kg.)

6. A body is suspended by a string of length 1 m. It is then projected horizontally with velocity 4 ms^{-1}. Calculate its tangential and radial acceleration when it rises by 60°. (Ans. : 8·5 ms^{-2}; 6·2 ms^{-2})

7. A block of mass m at the end of a string is whirled around in a vertical circle of radius R. Find the critical speed at which the string would become slack at the highest point.

(Ans : $\sqrt{5gR}$)

8. The length of a simple pendulum is one metre long. The bob of the pendulum of mass 10 gm is released when the string is horizontal. When it is at the lowest point of its path,

(i) what is its kinetic energy ?

(ii) what is its velocity ?

(iii) what is the tension of the string ?

(I. I. T. 1973) (Ans. : $9{\cdot}8 \times 10^{-2}$J; 4·43 ms^{-1}, ·2 94 N)

9. A circular curve of a highway is designed for traffic moving at 60 kph. If the radius of the curve is 120 m, what is the correct angle of banking of the road ? (Ans : 15°)

10. What is the smallest radius of a circle at which a bicyclist can travel if his speed is 28·5 kph and the coefficient of static friction between the tyres and the road is ·32 ? What is the largest angle of inclination to the vertical at which the bicyclist can ride without falling ? (Ans : 20·73m; 18°)

(E)

1. The driver of a truck travelling with a velocity v suddenly notices a broad wall in front of him at a distance r. Is it better for him to apply brakes or to make a circular turn without applying brakes in order to just avoid crashing into the wall ? (I. I. T. 1977)

2. Centrifugal force is a............force.

3. Is a body in uniform circular motion in equilibrium ?

4. Is work done by a centripetal force on a body ?

5. Is the acceleration of a particle in circular motion constant in magnitude ? direction ?

6. Is the velocity of a particle in circular motion constant in magnitude ? in direction ?

Ans. : 1. Brake, force required with brake$=\frac{1}{2}mv^2/r$; force required for circular turn$=mv^2/r$. 2. Pseudo. 3. No. 4. No. 5. Yes, No. 6. Yes, No.

CHAPTER 7

SIMPLE HARMONIC MOTION AND COMPOSITION OF TWO SIMPLE HARMONIC MOTIONS : LISSAJOUS FIGURES

7.1. Periodic Motion

Any motion that repeats itself in equal intervals of t me is called a *periodic motion*. The motion of the earth about its axis is a periodic motion. The motion of the piston of an engine, the motion of a pendulum, the motion of electron in an atom are few common examples of periodic motion. The interval of time after which motion is repeated is called the time period or periodicity of the motion.

7.2. Oscillatory Motion

If a particle in periodic motion moves back and forth over the same path, it is called an oscillatory or vibratory motion. This universe is full of oscillatory motions. Some examples are the oscillations of a simple pendulum, sonometer wire, atoms at the lattice sites of a soild, a mass attached to a spring etc.

7.3. Simple Harmonic Motion (S. H. M.)

The oscillatory motion that interests us most is the simple harmonic motion because most of the oscillatory motions in the universe are of this type. The vibratory motions of air molecules are simple harmonic as a sound wave passes by. The vibrations of atoms at the lattice sites are approximately simple harmonic. The oscillations of the simple and the compound pendulums are simple harmonic when their amplitudes are small.

Definition of Simple Harmonic Motion : The Simple Harmonic Motion (S. H. M.) is defined as *the motion in which the acceleration is always directed towards a fixed point in the path of motion and is proportional to the displacement from that point.*

7.4. Equation of Simple Harmonic Motion (S. H. M.)

Suppose that a particle is moving in a straight line in such a way that its acceleration (f) is always directed towards a fixed point O and is proportional to its displacement (x) from O.

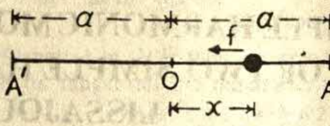


Fig. 7.1

That is

$$f \propto -x$$

or $$f = -\mu x$$

where μ is a constant.

Thus the equation of simple harmonic motion is

$$f = -\mu x$$

or $$\frac{dv}{dt} = -\mu x \qquad \left(\because\ f = \frac{dv}{dt} \text{ by Calculus}\right)$$

or $$\frac{d}{dt}(dx/dt) = -\mu x \quad (\because\ v = dx/dt \text{ by Calculus})$$

$$\frac{d}{dt}\left(\frac{dx}{dt}\right) \text{ is written as } \frac{d^2x}{dt^2};$$

$$\therefore \quad \frac{d^2x}{dt^2} = -\mu x$$

or $$\frac{d^2x}{dt^2} + \mu x = 0$$ is the equation of simple harmonic motion in the differential form.

Solution of the equation : We have

$$\frac{d}{dt}\left(\frac{dx}{dt}\right) = -\mu x \quad \text{or} \quad \frac{dv}{dt} = -\mu x \quad \left(\because\ v = \frac{dx}{dt}\right)$$

or $$\frac{dv}{dx}\cdot\frac{dx}{dt} = -\mu x$$

or $$v\,dv = -\mu x\,dx \qquad \left(\because\ v = \frac{dx}{dt}\right)$$

$\therefore$ $$\int v\,dv = -\mu \int x\,dx + c'$$ (c' is a constant to be determined)

or $$\frac{v^2}{2} = -\frac{\mu x^2}{2} + c'$$

or $\qquad v^2 = -\mu x^2 + c \qquad$ (where c is another constant)

or $\qquad v^2 = c - \mu x^2.$

From this equation it is clearly seen that as x increases v decreases and hence there will be certainly a maximum value of x when v will be zero. Let this maximum value be 'a'. This is called the amplitude of the simple harmonic motion.

Thus when $\qquad x = a, \qquad v = 0$

$\therefore \qquad 0 = c - \mu a^2 \quad$ or $\quad c = \mu a^2$

$\therefore \qquad v^2 = \mu a^2 - \mu x^2$ or $\quad v^2 = \mu(a^2 - x^2)$

or $\qquad v = \pm\sqrt{\mu}\ \sqrt{a^2 - x^2}$

or $\qquad \dfrac{dx}{dt} = \pm\sqrt{\mu}\ \sqrt{a^2 - x^2} \qquad (\because \ v = dx/dt).$

Proceeding with the positive sign we have

$$\frac{dx}{\sqrt{a^2 - x^2}} = \sqrt{\mu}\ dt$$

$$\therefore \qquad \int \frac{dx}{\sqrt{a^2 - x^2}} = \sqrt{\mu} \int dt + \text{(a constant)}$$

or $\qquad \sin^{-1}\dfrac{x}{a} = \sqrt{\mu}\ \ t + \alpha$

or $\qquad x = a\ \sin\ (\sqrt{\mu}\ \ t + \alpha).$

If we proceed with the negative sign the solution will be found to be of the same form.

We know that 'sine' repeats its value at the interval of 2π. Hence x will also repeat its value at regular intervals. This means that simple harmonic motion is an oscillatory motion. The interval after which it repeats its motion is called the time period of the motion and is usually denoted by T.

Clearly motion is repeated when

$$\sqrt{\mu}\ t + \alpha = 2\pi,\ 4\pi,\ 6\pi,\ 8\pi$$

or $\qquad t = \dfrac{2\pi - \alpha}{\sqrt{\mu}},\ \dfrac{4\pi - \alpha}{\sqrt{\mu}},\ \dfrac{6\pi - \alpha}{\sqrt{\mu}},$

$\therefore \qquad T = \dfrac{2\pi}{\sqrt{\mu}}$ or $\sqrt{\mu} = \dfrac{2\pi}{T}$

$$\therefore \quad x = a \sin\left(\frac{2\pi}{T} t + \alpha\right) \qquad \text{.. (7.1)}$$

or $x = a \sin(\omega t + \alpha)$ where $\omega = \frac{2\pi}{T}$ is called the angular frequency or the cyclic frequency.

The constant of integration (α) determines the 'initial state of motion of the particle. The '*state of motion*' of a paticle is called the '*phase*' of the particle. Thus the constant of integration determines the initial phase of the particle and is called the *epoch*. The value of epoch depends on the instant from which we reckon time. If we reckon time when particle passes through its mean position to the right then $x = 0$ at $t = 0$.

$\therefore \quad 0 = a \sin(\omega . 0 + \alpha)$ or $\alpha = 0$ or π

$\because \quad v$ (velocity) $= \frac{dx}{dt} = a\omega \cos(\omega t + \alpha)$.

$\therefore$ At $t = 0$, v is positive if $\alpha = 0$ and negative if $\alpha = \pi$.

Since we reckon time when the particle passes through its mean position to the right, out of the two possible values 0 and π, the former is acceptable. Thus epoch is zero if we reckon time from the instant when the particle passes through its mean position. In this case displacement at any instant is given by

$$x = a \sin \omega t. \qquad \text{.. (7.1. a)}$$

7.5. Terms Associated with Simple Harmoic Motion

(*a*) *Amplitude* : In simple harmonic motion the limits of oscillation are equally spaced about the mean position. *The maximum displacement of the particle executing simple harmonic motion from the mean position* is called the *amplitude of the motion*. It is denoted by a

(*b*) *Period* : *The time taken by the particle executing simple harmonic motion to repeat its motion is called its time period*. It is usually denoted by T.

(*c*) *Frequency* : *The number of oscillations* (i.e. journey from particular state of motion and back to the same state) *executed by the particle in one second is called* its *frequency*. It is usually denoted by n or v.

If T represents the period and n frequeucy, then,

$$n = \frac{1}{T}. \qquad \text{.. (7.2)}$$

(d) *Phase* : The '*state of motion*' of the particle executing simple harmonic motion at a time is called the phase of the particle. The phrase '*state of motion*' means '*where is the particle*' and '*what is its direction of motion*'. The phase of a particle is pictorially shown by a dot beaming an arrow over its top in the direction of motion. For example, the phase of a particle passing through its mean position to the right will be shown as , the phase of the particle at its extreme right will be shown as , the phase of the particle $+a/2$ and moving to the left will be shown as and so on.

To measure the phase we shall have to choose such a physical quantity which will simultaneously fix up the positon and direction of motion of the particle. We have for simple harmonic motion

$$x = a \sin(\omega t + \alpha)$$

and $$v = \frac{dx}{dt} = a\,\omega \cos(\omega t + \alpha).$$

By inspecting these two expressions for x and v we find that both are determined by the associated angle $(\omega t + \alpha)$. The phase of the particle is fully determined by x and v because the value of x fixes up the position of the particle and the sign (positive or negative) of v fixes its direction of motion. This is why $(\omega t + \alpha)$ is called the phase angle of the particle and this is taken as measure of '*phase*'.

(*e*) *Epoch* : Epoch is *the phase of the particle executing simple harmonic motion at the commencement of time.*

In general we have

$x = a\sin(\omega t + \alpha)$ for simple harmonic motion. The initial phase is given by

$$x_{initial} = a\sin\alpha$$

and $$v_{initial} = a\omega\cos\alpha.$$

Thus initial phase is determined by α. This is why α is called the 'EPOCH' of the particle executing simple harmonic motion. The

value of the epoch depends on the phase of the particle at the commencement of time.

7.6. Velocity of the Particle Executing S. H. M.

We have,

$$x = a\sin(\omega t + \alpha)$$

$$\therefore \quad v = \frac{dx}{dt} = a\omega \cos(\omega t + \alpha)$$

or $\quad v = v_{max} \cos(\omega t + \alpha)$ where $v_{max.} = a\omega$, maximum velocity of the particle in the simple harmonic motion.

When the particle is timed with the mean position, $\alpha = 0$,

$$v = a\omega\cos\omega t \qquad .. \quad (7.3)$$

or

$$v = a\omega\sqrt{1 - \frac{x^2}{a^2}}$$

or

$$v = \omega\sqrt{a^2 - x^2} \qquad .. \quad (7.4).$$

7.7. Acceleration of S. H. M.

We have $\quad f = \frac{dv}{dt}$ by Calculus

and $\quad v = a\omega\cos(\omega t + \alpha)$

$$\therefore \quad f = a\omega[-\omega\sin(\omega t + \alpha)] = -a\omega^2\sin(\omega t + \alpha).$$

When the particle is timed with the mean position, $\alpha = 0$,

$$f = -a\omega^2\sin\omega t \qquad .. \quad (7.5).$$

But, $\quad x = a\sin(\omega t + \alpha)$ in general

$$\therefore \quad f = -\omega^2 x \qquad .. \quad (7.6).$$

This is the 'displacement-acceleration' formula of simple harmonic motion. This is a very important formula which will be referred to on many occasions, specially in numerical problems.

7.8. Energy of S. H. M.

(i) The Kinetic energy K at any instant is $\frac{1}{2}mv^2$ where m is the mass of the particle. Using the relations

$$x = a\sin(\omega t + \alpha) \text{ and } v = \frac{dx}{dt} = a\omega\cos(\omega t + \alpha)$$

we obtain $$K=\tfrac{1}{2}ma^2\omega^2\cos^2(\omega t+\alpha) \qquad .. \quad (7.7)$$

or $$K=\tfrac{1}{2}ma^2\omega^2\left(1-\frac{x^2}{a^2}\right)$$

or $$K=\tfrac{1}{2}m\omega^2(a^2-x^2) \qquad .. \quad (7.8)$$

(ii) The potential energy U at any instant = the work done by the force acting on the particle through x.

Now the force acting on the particle when the displacement is

$$z = \text{mass} \times \text{acceleration}$$
$$= m(-\omega^2 z)$$
$$= -m\omega^2 z.$$

The elementary work done by the force in displacing the particle through $dz = -m\omega^2 z \times (-dz)$ (minus dz because displacement by dz in the positive direction of z is actually displacement by $-dz$ in the direction of the force)

$$= m\omega^2 z\,dz.$$

$\therefore$ The potential energy (U) at time t

= the work done in displacing through x

$$= \int_0^x m\omega^2 z\,dz$$
$$= m\omega^2\left[\frac{z^2}{2}\right]_0^x$$
$$= \tfrac{1}{2}m\omega^2 x^2.$$

$\therefore$ The potential energy of the particle at any instant

$$U=\tfrac{1}{2}m\omega^2 x^2$$

or $$U=\tfrac{1}{2}m\omega^2 a^2\sin^2(\omega t+\alpha) \qquad .. \quad (7.9).$$

$\therefore$ The total energy E at time $t = U + K$,

$$E=\tfrac{1}{2}m\omega^2 a^2\sin^2(\omega t+\alpha) + \tfrac{1}{2}m\omega^2 a^2\cos^2(\omega t+\alpha)$$

$$E=\tfrac{1}{2}ma^2\omega^2. \qquad .. \quad (7.10)$$

Thus energy remains constant in simple harmonic motion and hence the force acting on the particle executing simple harmonic is '*Conversative*'.

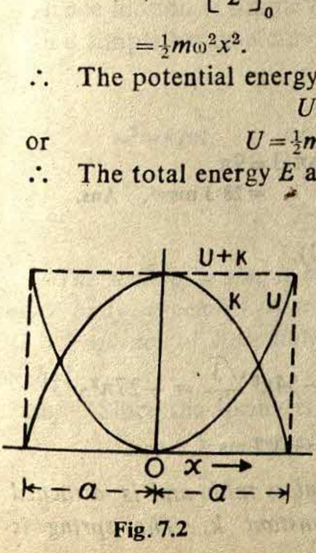


Fig. 7.2

Examples :

1. A body oscillates with simple harmonic motion according to the equation :

$$x=6\ sin\left(3\pi t+\frac{\pi}{3}\right)\ metres.$$

What is (a) the displacement, (b) the velocity, and (c) the acceleration at the time $t=2$ s. Find also the phase constant, the frequency and the period of the motion.

Sol. Comparing it with the standard form

$$x=a\ \sin\ (\omega\ t+\alpha),$$

we have, $a=6$ m, $\omega=3\pi$; or $\frac{2\pi}{T}=3\pi$

or $$T=2/3\ \text{s}.$$

α, phase constant $=\pi/3$.

Putting $t=2$,

$$x=6\ \sin\left(3\pi.2+\frac{\pi}{3}\right)=6\ \sin\ \pi/3=6.\frac{\sqrt{3}}{3}=3\sqrt{3}$$

$$=5{\cdot}2\ \text{m}.\quad \textbf{Ans.}$$

$$v=\frac{dx}{dt}=6\times 3\pi\ \cos\left(3\pi\ t+\frac{\pi}{3}\right)$$

$$v=18\pi\ \cos\left(3\pi\ t+\frac{\pi}{3}\right).$$

Putting $t=2$ s,

$$v=18\pi\ \cos\ (6\pi+\pi/3)=18\pi\ \cos\pi/3=9\pi$$

$$=28{\cdot}3\ \text{ms}^{-1}.\quad \textbf{Ans.}$$

$$f=\frac{dv}{dt}=-54\pi^2\ \sin\ (3\pi\ t+\pi/3).$$

Putting $t=2$ s,

$$f=-54\pi^2\ \sin\ (6\pi+\pi/3)=-54\pi^2\frac{\sqrt{3}}{2}=-27\pi^2\sqrt{3}$$

$$=-461{\cdot}7\ \text{ms}^{-2}.\quad \textbf{Ans.}$$

2. A body of mass 'm' rests on a frictionless table and is attached to a horizontal massless spring of force constant k. The spring is

compressed a little and then released. Show that the motion of the body is simple harmonic and the frequency of oscillation is given by,

$$f=\frac{1}{2\pi}\sqrt{\frac{k}{m}}.$$

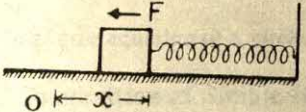


Fig. 7.3

Sol. Suppose '*O*' is the position of rest of the body. Let us displace it and then release. Let x be the displacement at any instant to the right (positive direction of x). The spring will exert a force on the body in the negative direction in proportion to x. That is,

$$F\propto -x$$

or $$F=-kx$$

where k is a constant called force constant. It depends on the length (inversely proportional to the length) and stiffness of the spring (directly proportional to the stiffness).

$$\text{Force}=\text{mass}\times\text{acceleration}$$

$\therefore$ $$-kx=mf$$

or, $$f=-\frac{k}{m}.x$$

Thus $$f\propto -x.$$

Hence motion is simple harmonic.

In a simple harmonic motion

$$f=-\omega^2 x$$

$\therefore$ $\omega^2=k/m$ or $\omega=\sqrt{\dfrac{k}{m}}$ or $2\pi f=\sqrt{\dfrac{k}{m}}$

or $$f=\frac{1}{2\pi}\sqrt{\frac{k}{m}}.\ \textbf{Proved.}$$

3. *The scale of a simple balance reading from 0 to 15 kg is 10 cm long. A body suspended from the balance is found to oscillate vertically with a frequency of 2 oscillations per second. How much does the body weigh ?*

Sol. Since the spring is elongated through 10 cm by 15 kg load,

$\therefore$ $$-15g=-k.\ 10\times 10^{-2}$$

or $$k=\frac{15\times g}{10^{-1}}=150g \text{ newton per metre.}$$

We have, $$f=\frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

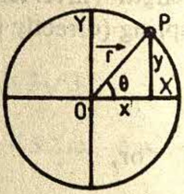
$\therefore$ $$2=\frac{1}{2\pi}\sqrt{\frac{150g}{m}} \quad \text{or} \quad 16\pi^2=\frac{150g}{m}$$

or $$m=\frac{150g}{16\pi^2}$$

$$=9{\cdot}3 \text{ kg. } \textbf{Ans.}$$

4. *Show that a uniform circular motion in a circle of radius a and angular speed ω may be resolved into two mutually perpendicular S. H. M's of amplitude a and cyclic frequency ω but differing in phase by π/2 and vice versa.*

Sol. Let us fix a two dimensional co-ordinate frame at the centre of the circle. Suppose θ is the angle (counted anticlockwise from the x-axis) made by the position vector of the particle on the circle at any instant t and (x, y) are its co-ordinates. Then



$$x=a\cos\theta;\quad y=a\sin\theta;\quad \theta=\omega t.$$

The position vector of the particle is given by

$$\vec{r}=x\hat{i}+y\hat{j}.$$

$$=a\cos\theta\,\hat{i}+a\sin\theta\,\hat{j}$$

Thus the motion of the particle on the circle is equal to the sum of the vectors $a\cos\theta\,\hat{i}$ and $a\sin\theta\,\hat{j}$ along x and y-axes.

Now $$x=a\cos\theta=a\cos\omega t=a\sin(\omega t+\pi/2).$$

This is a S. H. M. of amplitude a and cyclic frequency ω and epoch $\pi/2$.

$$y=a\sin\theta=a\sin\omega t.$$

This is a S. H. M. of amplitude a, cyclic frequency ω and epoch 0 (zero). Thus it is proved that a circular motion may be resolved into two mutually perpendicular S. H. M's of the same amplitude and frequency but differing in phase by $\pi/2$. P is called the generating point of S. H. M.'s and the circle is called the generating circle.

Converse : Consider a S. H. M. of amplitude a, cyclic frequency ω and zero epoch along x-axis. Then

$$x=a\sin\omega t \quad \text{.. (i)}$$

Consider another S. H. M. of the same amplitude and frequency but having phase difference $\pi/2$ with the first along the y-axis. Then

$$y = a\sin(\omega t + \pi/2).$$

But $$\sin(\omega t + \pi/2) = \cos\omega t$$

$\therefore$ $$y = a\cos\omega t \quad \text{.. (ii).}$$

Squaring and adding (i) and (ii)

$$x^2 + y^2 = a^2\sin^2 \omega t + a^2\cos^2 \omega t$$

or $$x^2 + y^2 = a^2.$$

This is the equation of a circle.

7.9. Composition of Simple Harmonic Motions

(*i*) *Composition of two collinear simple harmonic motions of the same frequency but of different phase and amplitude.*

Often two simple harmonic motions along the same line combine to form stationary waves, beats, interference in sound and light. Consider a simple harmonic motion of amplitude 'a' and frequency n along the x-axis. The equation of this simple harmonic motion may be written as

$$x_1 = a\sin\omega t \quad \text{where } \omega = 2\pi n.$$

The equation of simple harmonic motion of the same frequency but of different phase and amplitude must be written as

$$x_2 = b\sin(\omega t + \delta)$$

where δ is the 'phase difference' between the two simple harmonic motions and b is the amplitude of the second motion.

Since the displacements are along the same line the resultant displacement is given by

$$x = x_1 + x_2$$

or, $$x = a\sin\omega t + b\sin(\omega t + \delta)$$
$$= a\sin\omega t + b\sin\omega t.\cos\delta + b\cos\omega t.\sin\delta$$
$$= (a + b\cos\delta)\sin\omega t + b\sin\delta \cos\omega t$$

Put $$a + b\cos\delta = c\cos\varphi$$
$$b\sin\delta = c\sin\varphi.$$

This type of substitution is called '*sin-cos*' substitution. It is always adopted whenever two simple harmonic terms are to be added.

To evaluate c, the two are squared and added

$$c^2(\sin^2\varphi + \cos^2\varphi) = b^2\sin^2\delta + (a + b\cos\delta)^2$$

or $$c^2=b^2\sin^2\delta+a^2+2ab\cos\delta+b^2\cos^2\delta$$
$$=a^2+b^2+2ab\cos\delta$$

or $$c=\sqrt{a^2+b^2+2ab\cos\delta}.$$

To evaluate φ, one is divided by the other.

$$\tan\varphi=\frac{b\sin\delta}{a+b\cos\delta}.$$

$$\therefore \quad x=c\sin\omega t.\cos\varphi+c\cos\omega t.\sin\varphi$$
$$x=c\sin(\omega t+\varphi) \qquad \text{.. (7.10).}$$

This is also a simple harmonic motion of the same frequency but of different amplitude 'c' and phase difference φ with the first. Thus *always remember that the resultant of two collinear simple harmonic motions of the same frequency but of different amplitude and phase is also a simple harmonic motion of the same frequency but of different amplitude and phase.*

Special cases. The following special cases are of interest.

(a) If $\delta=0$, $c=a+b$ and $\varphi=0$

$\therefore \quad x=(a+b)\sin\omega t.$

That is, the resultant motion is simple harmonic of amplitude equal to the sum of the component amplitudes.

(b) If $\delta=\pi$, $c=a\sim b$ and $\varphi=0$

$\therefore \quad x=(a\sim b)\sin\omega t.$

That is, the resultant motion is simple harmonic of amplitude equal to the difference of the component amplitudes.

(c) If $\delta=\frac{\pi}{2}$, $c=\sqrt{a^2+b^2}$ and $\varphi=\tan^{-1}\frac{b}{a}$.

$\therefore \quad x=(\sqrt{a^2+b^2})\sin(\omega t+\varphi).$

(ii) *Composition of two rectangular S. H. M's of the same frequency but of different amplitude and phase.*

Often two simple harmonic motions *at right angles* are combined so as to produce elliptically polarised light.

Consider a simple harmonic motion of amplitude a and frequency n along the x-axis. The equation of this simple harmonic motion may be written as.

$$x=a\sin\omega t \qquad \text{.. (1).}$$

where $\omega=2\pi n$.

The equation of a simple harmonic motion at right angles to it i e., along y-axis must be written as

$$y = b \sin(\omega t + \delta) \quad \text{.. (ii).}$$

To compose the resultant vibration we take the value of $\sin\omega t$ from (i) and substitute in (ii).

$$\sin\omega t = x/a$$

and
$$y/b = \sin\omega t \cos\delta + \cos\omega t \sin\delta$$

or
$$\frac{y}{b} = \frac{x}{a}\cos\delta + \sqrt{1 - \frac{x^2}{a^2}}\sin\delta \quad (\because \cos\delta = \sqrt{1-\sin^2\delta})$$

or
$$\left(\frac{y}{b} - \frac{x}{a}\cos\delta\right)^2 = \left(1 - \frac{x^2}{a^2}\right)\sin^2\delta$$

or
$$\frac{y^2}{b^2} - \frac{2xy}{ab}\cos\delta + \frac{x^2}{a^2}\cos^2\delta = \sin^2\delta - \frac{x^2}{a^2}\sin^2\delta$$

or
$$\frac{y^2}{b^2} - \frac{2xy}{ab}\cos\delta + \frac{x^2}{a^2}(\cos^2\delta + \sin^2\delta) = \sin^2\delta$$

or
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\delta = \sin^2\delta \quad \text{.. (7,11).}$$

This is the general equation to an ellipse. The resulting motion is thus elliptic in general. The ellipse is of semi-major axis a and semi-minor axis b.

Special cases. The following cases are of interest.

(a) When $\delta = 0$, $\sin\delta = 0$ and $\cos\delta = 1$.

$$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} = 0$$

or
$$y = \frac{b}{a}x.$$

This represents a straight line passing through the origin and inclined at an angle $\tan^{-1}\frac{b}{a}$ to the x-axis (Fig. 7.4 a).

(b) When $\delta = \frac{\pi}{2}$, $\sin\delta = 1$ and $\cos\delta = 0$.

$$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This represents an ellipse of which the axes of symmetry called

major and minor axes are coincident with the co-ordinate axes (fig. 7.4 b)

(c) $\delta=\pi$, $\sin\delta=0$ and $\cos\delta=-1$.

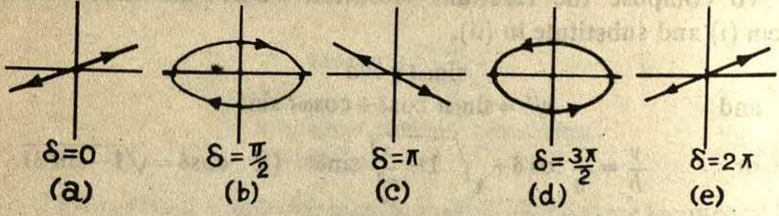


Fig. 7.4

$$\therefore \qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{2xy}{ab}=0 \quad \text{or} \quad y=-\frac{b}{a}x.$$

This is also a straight line passing through the origin but inclined at an angle $\tan^{-1}\left(-\frac{b}{a}\right)$. This is shown in Fig. 7.4 c.

(d) $\delta=\frac{3\pi}{2}$, $\sin\frac{3\pi}{2}=-1$, $\cos\frac{3\pi}{2}=0$.

$\therefore \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. That is same as Fig. 7.4 b.

(e) $\delta=2\pi$, $\sin 2\pi=0$, $\cos 2\pi=1$.
Hence it is same as (a).

(f) $\delta=\frac{\pi}{2}$ and $a=b$.

Equation 7.11 reduces to

$$x^2+y^2=a^2.$$

This is the equation of a circle of radius a. Thus *rectangular simple harmonic motions of the same frequency and amplitude but differing in phase by $\pi/2$ combine into a circular motion and conversely a uniform circular motion can be resolved into two rectangular S. H. M's of the same amplitude and frequency but differing in phase by $\pi/2$.*

7.10. Lissajous Figures

The composition of two rectangular simple harmonic motions when $f_x=f_y$ is in general an elliptical trace which reduces to a line or circle depending on their phase difference. Other simple number frequency ratios such as $f_x:f_y=1:2$, $f_x:f_y=3:2$, $f_x:f_y=1:3$

and so on give more complex figures know as the *Lissajous Figures*, some of which are shown in Fig. 7.5 (a), (b), (c) and (d).

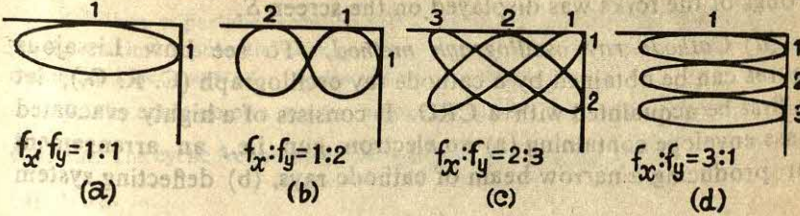


Fig. 7.5

These Lissajous figures provide an accurate method of comparing two frequencies. In any case the frequency ratio can be found from the inspection by a horizontal and a vertical line touching the figure tangentially. The ratio of the frequencies is the inverse ratio of the number of points at which the two lines touch the trace.

$$\frac{f_y}{f_x}=\frac{\text{number of points at which } x\text{-line touches the trace } (n_x)}{\text{number of points at which } y\text{-line touches the trace } (n_y)}$$

or
$$\frac{f_y}{f_x}=\frac{n_x}{n_y} \quad .. \ (7.12).$$

7.11. Methods to obtain Lissajous Figures

(*i*) *Optical Method*. Lissajous himself obtained the curves by compounding the vibrations of two tuning forks in two perpendicular directions. One prong of each fork was provided with a light mirror. A narrow beam of light passed through a lens L was

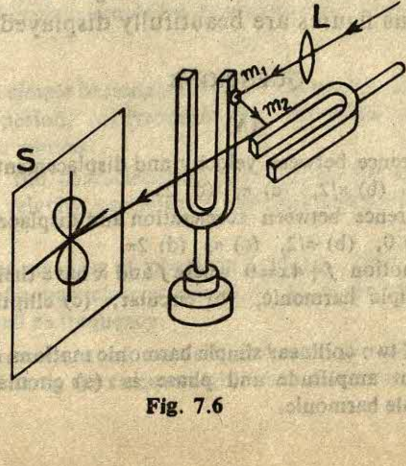


Fig. 7.6

successively reflected at the mirrors. The beam was finally focussed on a screen. The resultant of the two S. H. M's executed by the prongs of the forks was displayed on the screen S.

(*ii*) *Cathode ray oscillograph method*. To see how Lissajous figures can be obtained by a cathode ray oscillograph (C. R. O.), let us first be acquainted with a CRO. It consists of a highly evacuated glass envelope containing (a) an electron gun i.e., an arrangement for producing a narrow beam of cathode rays, (b) deflecting system

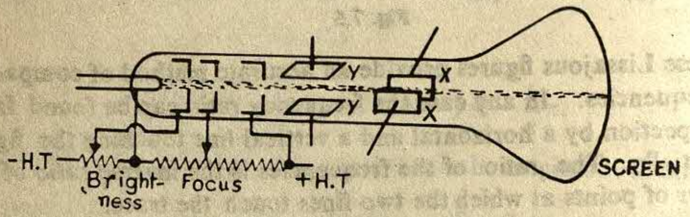


Fig 7.7

to deflect the beam horizontally and vertically, (c) a fluorescent screen on which the electron beam produces a spot of light. Apart from these three essential parts there are a few more provisions such as provision for focussing, vertical and horizontal gain, time base selection and horizontal and vertical centering of the spot of light.

To obtain Lissajous figures two alternating voltages are required. One of these voltages may be a small fraction of the 50 Hz supply voltage. This is applied to the X-plates. The other voltage is obtained from an A. F. (audio frequency) signal generator. This is applied to the Y-plates. When the two voltages are simultaneously applied, the Lissajous figures are beautifully displayed on the screen.

QUESTIONS

(A)

1. The phase difference between velocity and displacement of a simple harmonic motion is (a) 0, (b) $\pi/2$, (c) π, (d) 2π.

2. The phase difference between acceleration and displacement of a simple harmonic motion is (a) 0, (b) $\pi/2$, (c) π, (d) 2π.

3. In a certain motion $f+4x=0$ where f and x have their usual meaning. The motion is (a) simple harmonic, (b) circular, (c) elliptical, (d) none of these.

4. The resultant of two collinear simple harmonic motions of the same frequency but of different amplitude and phase is (a) circular, (b) elliptical, (c) parabolic, (d) simple harmonic.

5. The resultant of two rectangular simple harmonic motions of the same frequency and amplitude but differing in phase by $\pi/2$ is (a) circular, (b) elliptical, (c) linear, (d) simple harmonic.

6. When a particle executing SHM passes through the mean position it has (a) the maximum potential energy, (b) the minimum potential energy, (c) the maximum kinetic energy, (d) the minimum kinetic energy.

7. The equation of a simple harmonic motion is, $f+\mu x=0$ where μ is a constant. The cyclic frequency of the motion is given by (a) μ, (b) $\sqrt{\mu}$, (c) μ^2, (d) $1/\sqrt{\mu}$.

8. Two simple harmonic motions of the same frequency and amplitude but having a phase difference of π and acting at right angles will combine to form a resultant motion which is (a) linear, (b) circular, (c) elliptical, (d) parabolic.

9. The equations of two SHM's are (A) $x_1=a\sin(\omega t+\pi)$ and (B) $x_2=a\sin\left(\omega t+\frac{4\pi}{3}\right)$. (a) A will acquire maximum velocity $T/6$ prior to B, (b) A will acquire max. velocity $T/6$ after B, (c) A will acquire max. velocity $T/3$ prior to B, (d) A will acquire max. velocity after B.

10. The kinetic energy of a particle in simple harmonic motion of frequency n fluctuates at the frequency (a) n, (b) $n/2$, (c) $2n$, (d) $3n$.

Ans. : 1. (b), 2. (c), 3. (a), 4. (d), 5. (a), 6. (b+c), 7. (b), 8. (a), 9. (b), 10. (c).

(B)

1. Find an expression for the energy of a particle executing simple harmonic motion.

2. Show graphically how the kinetic energy, the potential energy and the total energy of a particle executing SHM vary in one complete cycle.

3. Show that the force acting on a particle executing SHM is conservative. (Hint : See arts. 5-4 and 7.8)

4. State and explain simple harmonic motion. Give examples.

(C)

1. What is a simple harmonic motion ? Define the following terms—(a) amplitude, (b) time-period, (c) frequency and (d) phase. Deduce the equation of a simple harmonic motion.

2. What are the charactéristics of the simple harmonic motion ? Derive expression for the velocity, acceleration and time period of a particle in simple harmonic motion.

3. Explain the meaning of 'phase' in S. H. M. How is it measured ? Prove that the energy of a particle executing S. H. M. is proportional to the square of its amplitude as well as frequency.

4. Find the resultant of two simple harmomic motions having the same period and acting along the same straight line. Describe the special cases.

5. A particle has two simultaneous simple harmonic motions at right angles to each other with the same period but of different amplitude and phase. What will be the resultant motion ? Deduce its equation and show when the motion will be (i) linear, (ii) elliptical, (iii) circular. (Ran. 1973 and '76)

6. What do you understand by 'Lissajous' Figures ? What is their importance ? Describe the cathode ray oscilloscope method of obtaining Lissajous Figures.

(D)

1. A particle describes S. H. M. in a line 4 cm long. Its velocity when passing through the centre of the line is 12 cm per second. Find the period.

(Ans : 1·047 s)

2. The maximum velocity of a point executing S. H. M. is 4 ms^{-1} and its amplitude is ·2 m. Calculate the time period of its motion.

(Ans. ·314 s)

3. A body executing simple harmonic motion has an amplitude of ·05 m and its time period is 1 s. Find the time taken by the body in moving a distance of 2·5 cm from the mean position.

(Ans. 1/12 s)

4. A particle executing S. H. M. has a displacement amplitude of 4 cm and its acceleration at a distance of 1 cm from the mean position is 3 cm per $second^2$. Calculate its velocity at a distance of 2 cm from the mean position.

(Ans. 6 cm per second)

5. The velocities of a particle executing S. H. M. are 8 cm per second and 6 cm per second when its displacements are 3 cm and 4 cm respectively. Calculate the amplitude and the angular frequency.

(Ans. 5 cm., 2 rad s^{-1})

6. A test tube of mass 6 gm and extermal diameter 2 cm is floated in water vertically by placing 10 gm of mercury at its bottom. The tube is now depressed a bit and then released. Calculate the period of oscillation.

(Ans. ·45 s)

7. When the displacement is one half of the amplitude, what fraction of the total energy is kinetic and what fraction is potential ? At what displacement is the energy half kinetic and half potential ?

(Ans. 3/4; $\frac{1}{4}$; $\frac{a}{\sqrt{2}}$)

8. A block is on a piston which is moving vertically with S. H. M. of period 1 s. (a) At what amplitude of motion will the block and piston separate ? (b) If the piston has an amplitude of 5×10^{-2} m, what is the maximum frequency for which the block and piston will be in contact continuously ? (*Hints* : They will

separate when acceleration of the piston = acceleration due to gravity.)

(Ans. ·25m; 14Hz)

9. A 2 kg mass hangs from a spring. A 300 gm body hung from below the mass stretches the spring by 2 cm further. If the 300 gm body is removed and the mass is set into oscillations, find the period of motion. (Ans. ·73 s)

10. A particle executes S. H. M. of amplitude 4 cm and time period 2 s. Find its maximum velocity and maximum potential energy if the mass of the particle is 2 gm. **(Ans.** ·126 ms^{-1}; $1{\cdot}587 \times 10^{-5}$ J**)**

(E)

1. In simple harmonic motion acceleration is.........to the displacement and is directed.........to it.

2. The initial phase of a particle executing S. H. M. is called the...............

3. The peak value of velocity of a simple harmonic oscillation of amplitude a and cyclic frequency ω is.........

4. The peak value of the acceleration of an oscillator as in 3 is.........

Ans. 1. proportional; opposite. **2.** epoch. **3.** $a\omega$. **4.** $a\omega^2$.

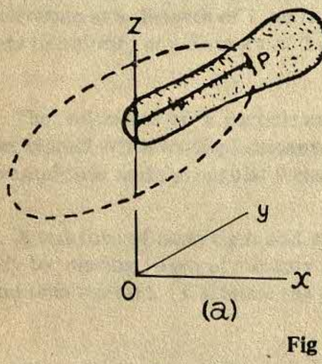
CHAPTER 8

ROTATIONAL MOTION: MOMENT OF INERTIA

8.1. Rotational Kinematics

A rigid body is said to move in pure translation if each particle of the body undergoes the same displacement as every other particle in any given time interval.

A rigid body is said to be in pure rotation if every particle of it moves in a circle the centres of which are on a straight line called the axis of rotation. The rotational motion of a rigid body is described by the angular position of any particle of it with respect to the two dimensional reference frame in the plane of rotation of the particle. In Fig. 8.1 (a) a rigid body is rotating about a fixed axis taken as the



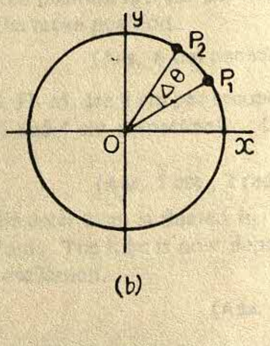


Fig 8.1

z axis of the reference frame. P is a point of the body. Figure 8.1 (b) shows the plane of rotation of the particle. The angle $P_1OX = \theta_1$ in Fig. 8.1 (b) is the angular position of the particle with respect to the two dimensional reference frame in the plane of rotation of the particle. We arbitrarily take the positive sense of rotation to be counter clockwise. Let θ_1 be the angular position of the particle at time t_1 and at a later time t_2 let it be θ_2. The angular displacement of P will be $\theta_2 - \theta_1 = \triangle\theta$ during the interval $t_2 - t_1 = \triangle t$. The *average angular speed* $\bar{\omega}$ of the particle and hence of the body in this interval

of time is defined as

$$\overline{\omega}=\frac{\theta_2-\theta_1}{t_2-t_1}=\frac{\triangle\theta}{\triangle t}.$$

The instantaneous angular speed ω is defined as the limit approached by this ratio as $\triangle t$ approaches zero :

$$\omega=\lim_{\Delta t\to 0}\frac{\triangle\theta}{\triangle t}=\frac{d\theta}{dt} \qquad .. \quad (8.1).$$

The unit of angular velocity is radian per second (rad s^{-1}) and its dimension is T^{-1}.

If the angular speed of P is not constant, then the particle has an angular acceleration denoted by α. Let ω_1 and ω_2 be the instantaneous angular speed at the times t_1 and t_2 respectively; then the average angular acceleration α of the particle and therefore of the body is defined as

$$\overline{\alpha}=\frac{\omega_2-\omega_1}{t_2-t_1}=\frac{\triangle\omega}{\triangle t}.$$

The instantaneous angular acceleration is the limit of this ratio as $\triangle t$ approachess zero:

$$\alpha=\lim_{\Delta t\to 0}\frac{\triangle\omega}{\triangle t}=\frac{d\omega}{dt}. \qquad .. \quad (8.2)$$

The unit of angular acceleration is radian per second per second (rad s^{-2}) and its dimension is T^{-2}.

Kinematic equation for a uniformly accelerated rotating body. Since definitions of angular speed and acceleration are taken exactly in the same way as linear velocity and acceleration we have exactly the same set of kinematic equations for rotational motions as well :

(i) $\omega=\omega_0+\alpha t$ cf. $v=v_0+at$

(ii) $\theta=\omega_0 t+\frac{1}{2}\alpha t^2$ cf. $s=v_0 t+\frac{1}{2}at^2$

(iii) $\omega^2-\omega_0^2=2\alpha\theta$ cf. $v^2-v_0^2=2\,as$.

These equations can be obtained by integrating (8.1) and (8.2).

Examples :

1. *The angular speed of an electric motor is increased from* 1200 *rpm to* 3000 *rpm in* 12 *s. What is its angular acceleration, assuming it to be uniform? How many revolutions does the engine make during this time ?*

Sol. We have, $\omega=\omega_0+\alpha t$.

Here $\omega_0=1200$ rpm $=2\pi\times\frac{1200}{60}=40\pi$ radian/second

$\omega=3000$ rpm $=2\pi\times\frac{3000}{60}=100\pi$ radian/second

$\therefore\quad 100\pi=40\pi+\alpha\times 12$ or $\alpha=\frac{60\pi}{12}=5\pi=15{\cdot}7$ radian sec^{-2}.

$\because\quad \theta=\omega_0 t+\frac{1}{2}\alpha t^2 \quad \therefore\quad \theta=40\pi\times 12+\frac{1}{2}\times 5\pi\times 12^2$

or $\theta=840\pi$ radian $=\frac{840\pi}{2\pi}=420$ rev. **Ans.**

($\because$ 2π radian $=1$ revolution)

2. *The earth rotates about its axis in* 24 *hours. What is its angular velocity of rotation* ?

Sol. $\omega=\frac{2\pi}{24\times 3600}=7{\cdot}273\times 10^{-5}$ radian per second. **Ans.**

3. *A wheel has a constant acceleration of* 3 *radian second*$^{-2}$. *In a* 4s *interval it turns through an angle of* 120 *radians. Assuming the wheel started from rest how long had it been in motion at the start of this* 4s *interval* ?

Sol. We have $\theta=\omega_0 t+\frac{1}{2}\alpha t^2$

$120=\omega_0 4+\frac{1}{2}.\ 3.4^2$ or $\omega_0=24$

$\omega=\omega_0+\alpha t$

$\therefore\quad 24=0+3.t$ or $t=8$ s. **Ans**

8.2. Vector Representation of Rotational Quantities

The linear displacement (s), velocity (v) and acceleration (a) are vectors. The corresponding angular quantities—angular displacement (θ), angular velocity (ω) and angular acceleration (α) respectively may be treated as vectors in the direction of the axis of rotation provided the θ's are small. *Large angular displacements are not vectors* but *infinitesimal angular displacements are vectors* becasue if they are vectors they must add like vectors i.e., commutative law for vector addition ($\vec{a}+\vec{b}=\vec{b}+\vec{a}$) must hold. This law tells us that the order in which we add them does not affect their sum. Let us see whether this is true for large angular displacements. The most handy

example is perhaps a book lying flat on a table. Give two successive rotations θ_1 and θ_2 to the book. Let θ_1 be 90° clockwise about a vertical axis passing through its centre from its initial north-south position as it is viewed from above. Let θ_2 be 90° clockwise turn about the north-south axis through the centre of the book as it is

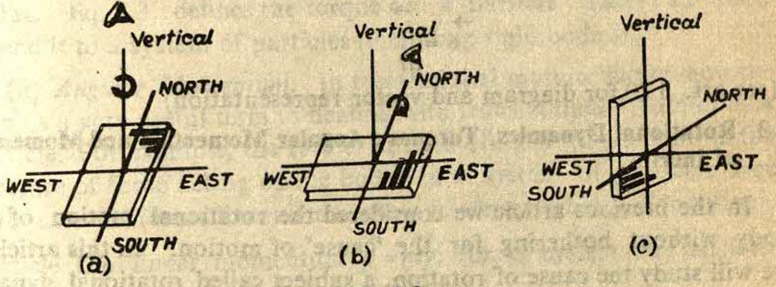


Fig. 8.2

viewed from the north. Now reverse the process. Turn the book through 90° clockwise about the north south axis viewing it from the

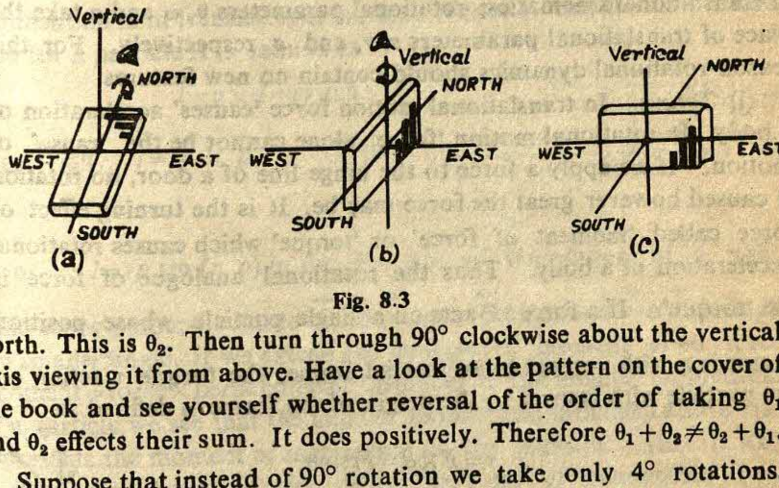


Fig. 8.3

north. This is θ_2. Then turn through 90° clockwise about the vertical axis viewing it from above. Have a look at the pattern on the cover of the book and see yourself whether reversal of the order of taking θ_1 and θ_2 effects their sum. It does positively. Therefore $\theta_1+\theta_2 \neq \theta_2+\theta_1$.

Suppose that instead of 90° rotation we take only 4° rotations. The result $\theta_1+\theta_2$ would still differ from the result $\theta_2+\theta_1$, but the difference would be much smaller. Hence if the angular displacements are infinitesimally small, the order of addition no longer effects their sum. Hence infinitesimal angular displacements are vectors.

Obviously angular velocity which is defined in terms of infinitesimal angular displacements is a vector. It is represented by an arrow drawn along the axis of rotation. The length of the arrow is taken proportional to the magnitude of the angular velocity. The

senses of the rotation determine the direction of the angular velocity, that is, the direction of the axis of rotation. By convention the direction of the axis is the direction of translation of a right hand screw placed along the axis of rotation.

Angular acceleration is also a vector quantity.

$$\vec{\alpha}=\frac{d\vec{\omega}}{dt}$$

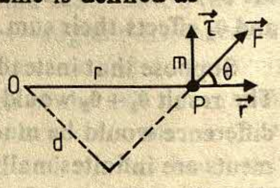
(see art. 3.20 for diagram and vector representation)

8.3. Rotational Dynamics, Torques : Angular Momentum and Moment of Inertia

In the previous article we considered the rotational motion of a body without bothering for the 'cause' of motion. In this article we will study the cause of rotation, a subject called rotational dynamics. The study of the 'cause' of rotational motion proceeds on the pattern of our study of translational motion. We have seen that rotational kinematics contained no new feature, it is simply 'recast' of translational kinematics; rotational parameters θ, ω and α take the place of translational parameters s, v, and a respectively. For this reason rotational dynamics should contain no new features.

(i) *Torque*. In translational motion force 'causes' acceleration of a body. In rotational motion 'force' alone cannot be the 'cause' of motion. If we apply a force to the hinge line of a door, no rotation is caused however great the force may be. It is the turning effect of force called 'moment of force' or 'torque' which causes rotational acceleration of a body. Thus the rotational analogue of force is the 'torque'. If a force $\vec{F}$ acts on a single particle whose position vector is $\vec{r}$ with respect to an inertial frame, the torque $\vec{\tau}$ acting on the particle with respect to the origin of the frame is defined as

$$\tau=F\times d=Fr\sin\theta.$$



Vectorially, $\vec{\tau}=\vec{r}\times\vec{F}$ .. (8.3).

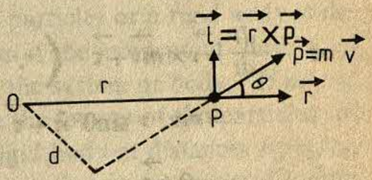
or $\tau=r\,F\sin\,(r,\,F)$ in the direction of translation of a right hand screw rotating from $\vec{r}$ to $\vec{F}$ through the smaller angle.

It has the dimensions ML^2T^{-2} in terms of M, L and T as fundamental units or FL in terms of force, length and time as fundamental quantities. Hence the unit of torque is newton-metre (Nm) which is the same as that of work or energy. However, torque and work are different physical quantities. *Torque* is a vector and *work* is a scalar. Eq. 8.3 defines the torque on a particle. Later we shall extend it to a system of particles (including rigid bodies).

(ii) *Angular Momentum.* In translational motion 'linear momentum' is a very useful term in dealing with translational dynamics of a particle. For example, the rate of change of linear momentum is the measure of force acting on the body. In a system of particles free from external forces linear momentum is conserved etc. For a single particle the linear momentum $\vec{p}=m\vec{v}$; for a system of particles $\vec{P}=M\vec{v}_{CM}$ where v_{CM} is the velocity of the centre of mass of the system. In rotational motion also it is useful to introduce a similar term which will be as useful a concept in rotational motion as linear momentum is in translational motion. We call it 'angular momentum' and for a particle it is defined as

$$l=p\times d=pr\sin\theta.$$

Vectorially, $\vec{l}=\vec{r}\times\vec{p}$ .. (8.4).



or $\quad l=rp\sin(r, p)$ in the direction of translation of a right hand screw perpendicular to $\vec{r}$ and $\vec{p}$ and rotating from $\vec{r}$ to $\vec{p}$ through the smaller angle. If we extend the concept of moment of a force (moment = force × perpendicular distance) to the angular momentum of a particle we see that the angular momentum as defined above also represents the moment of momentum of the particle. This is why angular momentum is also called 'moment of momentum'. Its dimension is ML^2T^{-1} or FLT. Hence its unit is newton-metre-second (Nms).

RELATION OF TORQUE AND ANGULAR MOMENTUM OF A PARTICLE

We have

$$\vec{F}=\frac{d}{dt}(m\vec{v})=\frac{d\vec{p}}{dt} \quad \text{(From Newton's second law)}$$

Let us take the vector product of $\vec{r}$ with both sides

$$\vec{r}\times\vec{F}=\vec{r}\times\frac{d\vec{p}}{dt}$$

By the definition of torque on a particle

$$\vec{\tau}=\vec{r}\times\vec{F}$$

$$\therefore \quad \vec{\tau}=\vec{r}\times\frac{d\vec{p}}{dt} \qquad \text{.. (8.5)}$$

By the definition of angular momentum of a particle we have,

$$\vec{l}=\vec{r}\times\vec{p}$$

Differentiating this

$$\frac{d\vec{l}}{dt}=\frac{d}{dt}\left(\vec{r}\times\vec{p}\right)=\frac{d\vec{r}}{dt}\times\vec{p}+\vec{r}\times\frac{d\vec{p}}{dt}$$

$$=\vec{v}\times\vec{p}+\vec{r}\times\frac{d\vec{p}}{dt}^{**}\left(\because\ \vec{v}=\frac{d\vec{r}}{dt}\right)$$

or $$\frac{d\vec{l}}{dt}=\vec{v}\times m\vec{v}+\vec{\tau}\quad\left(\because\ \vec{p}=m\vec{v};\ \vec{\tau}=\vec{r}\times\frac{d\vec{p}}{dt}\text{ by Eq. 8·5}\right)$$

$$=mv.\ v\sin 0^\circ\,\hat{n}+\vec{\tau}$$

$$=0+\vec{\tau}$$

or $$\vec{\tau}=\frac{d\vec{l}}{dt} \qquad \text{.. (8.6)}$$

which states that *the rate of change of the angular momentum is equal to the torque acting on it*—a statement similar to Newton's Second law for linear motion.

(iii) *Moment of Inertia : Radius of Gyration.*

Moment of Inertia : In translatory motion a body continues in its state of rest or of uniform motion in a straight line, unless acted upon by an external force. This inability of a body to change its state of rest or uniform motion in a straight line without the help of a

** The rule for the differentiation of a vector product is the same as the rule for the differentiation of an ordinary product, except that we must not change the order of the terms.

force is called '*inertia*' of the body. We know, by experience, that the greater the mass of a body, the greater is its inertia for rest as well as for motion in a straight line. This is why 'mass' is taken as the measure of inertia for translatory motion.

Exactly in the same manner a body in rotational motion continues in its state of rest or of uniform rotational motion, unless acted upon by an external torque. This means that just as a body has inertia for uniform linear motion, so also it has inertia for rotational motion It is the inertia for rotational motion of the body which is called its 'moment of inertia'. Here also we learn by experience that the 'moment of inertia' of a particle depends not only on its mass but also on its distance from the axis of rotation. The moment of inertia of a particle varies as the square of its distance from the axis of rotation. The moment of inertia of a particle of mass 'm' and at a distance r from the axis of rotation is defined as

$$i = mr^2 \quad .. \quad (8.7)$$

It has the dimension ML^2 and its unit is kgm^2.

Eq. 8.7 defines the moment of inertia of a particle. Having moment of inertia of a particle defined in this way, the moment of inertia of a system of particles or a rigid body is defined as the sum of the moment of inertia of the particles of the system or body. If $m_1, m_2, m_3, m_4, \ldots$ are the masses of the particles of a system or a rigid body at distances $r_1, r_2, r_3, r_4 \ldots$ respectively then the moment of inertia of the system or the rigid body is defined as

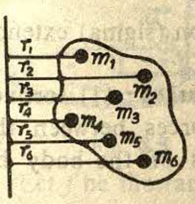


Fig. 8.4

$$I = i_1 + i_2 + i_3 + i_4 + \ldots$$
$$= m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + m_4r_4^2 + \ldots$$

or, $I = \Sigma m_i r_i^2$ where Σ extends over all the elementary particles.

$$I = \Sigma m_i r_i^2 \quad .. \quad (8.8).$$

For a rigid body that is not composed of discrete point masses but is instead a continuous distribution of matter such as a rod, disc, cylinder, sphere etc., the summation in I i.e., $\Sigma m_i r_i^2$ becomes an integration. We divide the body into infinitesimally small masses 'dm'. Then I is obtained from,

$$I = \int dmr^2 \quad .. \quad (8.8\text{ a}).$$

where $\int$ is carried over such limiting values of r that all elementary masses are included in the integration.

Redius of Gyration. The moment of inertia of a body is defined as $I = \Sigma mr^2$ where mr^2 is the moment of inertia of any particle of the body.

Let, $I = \Sigma mr^2 = MK^2$ where $M = \Sigma m$ is the mass of the body and K is a suitable constant of the dimension of length. This K is called the 'radius of gyration'.

DEFINITION

MK^2 represents the moment of inertia of a particle of mass equal to the mass of the body about the same axis. Thus the radius of gyration may be defined as *the distance of the point where the whole mass of the body may be supposed to be concentrated so far its moment of inertia is concerned.*

8.4. Relation of Torque and Angular Momentum of a Body

A rigid body is a system of particles. Hence the angular momentum $\vec{L}$ of a body about a given axis will be equal to the vector sum of the angular momenta of all the individual particles of the body about the same axis. For a body we have, then,

$\vec{L} = \vec{l_1} + \vec{l_2} + \vec{l_3} + \vec{l_4} + \ldots = \Sigma \vec{l}$ where summation (sigma) extends over all the particles.

The torque on a body may arise from two sources : (1) torques exerted on the particles of the body by internal forces between the particles and (2) torques exerted on the particles of the body by external forces.

On account of Newton's third law of motion the first source contributes nothing towards the total torque on a body. Hence the torque on a body about an axis is also equal to the vector sum of the external torques on the individual particles of the body about the same axis. Therefore

$$\vec{\Gamma} = \vec{\tau_1} + \vec{\tau_2} + \vec{\tau_3} + \ldots + \ldots = \Sigma \vec{\tau}$$

For a particle we have,

$$\vec{\tau} = \frac{d\vec{l}}{dt}$$

$$\therefore \quad \Sigma\vec{\tau} = \Sigma \frac{d\vec{l}}{dt}$$

$$\vec{\Gamma} = \frac{d}{dt}(\Sigma\vec{l})$$

$$\vec{\Gamma} = \frac{d\vec{L}}{dt} \qquad \text{.. 8.9.}$$

This equation is just the generalisation of Eq. 8.6 to a rigid body or a system of particles and it states that the rate of change of the angular momentum of a rigid body or a system of particles about the origin of an inertial frame is equal to the external torque acting on the system.

8.5. Equation of Rotational Motion of a Body

In translatory motion we have

force = mass × acceleration

$$\vec{F} = m\vec{a}$$

In rotational motion we have exactly a similar relation viz. Torque = moment of inertia × angular acceleration

$$\Gamma = I\alpha$$

Vectorially $\qquad \vec{\Gamma} = I\vec{\alpha}$

Let us investigate this relationship between torque and angular acceleration. Suppose that a rigid body (Fig. 8.5) is rotating about a fixed axis with uniform angular acceleration α.

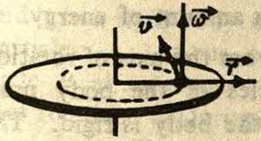


Fig. 8.5

Let f be the tangential force on the particle of mass m. Then τ = torque on it = force × perpendicular distance

$$= \left(m\frac{dv}{dt}\right) \times r = mr\,\frac{d}{dt}(\omega r) = mr^2\,\frac{d\omega}{dt} = mr^2\alpha$$

where α is the angular acceleration of the particle which is the same for all particle.

$\therefore \quad \Gamma$ (torque on the body) $= \Sigma\tau = \Sigma mr^2\alpha = I\alpha \quad (\because \quad I = \Sigma mr^2)$.

$$\therefore \qquad \vec{\Gamma} = I\vec{\alpha} \qquad \text{.. (8.10)}$$

$$\text{or} \qquad \Gamma = I\alpha \qquad \text{.. 8.10 (a).}$$

This is the equation of rotational motion and it states that, *torque = moment of inertia × angular acceleration.*

8.6. Relation between Angular Momentum and Angular Velocity

Proceeding with definitions of angular momentum $(\vec{l} = \vec{r} \times \vec{p})$ of

a particle and torque on the particle ($\vec{\tau}=\vec{r}\times\vec{F}$) we have shown in art. 8.4 and 8.3 that for a body or system of particles.

$$\vec{\Gamma}=\frac{d\vec{L}}{dt}$$

$$\text{and} \quad \vec{\Gamma}=I\vec{\alpha}=I\frac{d\vec{\omega}}{dt} \qquad (\because \ \vec{\alpha}=\frac{d\vec{\omega}}{dt})$$

$$\therefore \quad \frac{d\vec{L}}{dt}=I\frac{d\vec{\omega}}{dt}=\frac{d}{dt}(I\vec{\omega})$$ (∵ I is a constant, pushing of I inside the bracket is justified)

$$\therefore \qquad \vec{L}=I\vec{\omega} \qquad .. \quad (8.11).$$

$$\text{or} \qquad L=I\omega \qquad .. \quad (8.11\text{ a}).$$

8.7. Kinetic Energy of a Rotating Body

Imagine a rigid body rotating with angular speed ω about an axis that is fixed in a particular inertial frame of reference. Each particle has a certain amount of kinetic energy. Hence the body or system of particles as a whole has also certain amount of energy.

A particle of mass m, at a distance r from the axis of rotation moves in a circle of radius r. All the particles of the body must rotate with the same angular speed because the body is rigid. The linear speed of the particle is

$$v=\omega r$$

$$\therefore \text{ the kinetic energy of the particle}=\tfrac{1}{2}mv^2$$
$$=\tfrac{1}{2}mr^2\omega^2$$

Hence the kinetic energy K of the body or system of particles is

$$K=\Sigma\tfrac{1}{2}mr^2\omega^2=\tfrac{1}{2}(\Sigma mr^2)\omega^2$$
$$=\tfrac{1}{2}I\omega^2 \qquad (\because \ \Sigma mr^2=I)$$
$$\text{or} \qquad K=\tfrac{1}{2}I\omega^2 \qquad .. \quad (8.12).$$

8.8. Kinetic Energy of a Rolling Body without Sliding

The kinetic energy of a body which moves along a straight line while rolling arises from two sources : (1) rotational motion of the body and (2) translational motion of the body.

If I is the moment of inertia of the body about its axis of rotation and ω its angular speed, its kinetic energy of rotation $=\tfrac{1}{2}I\omega^2$

$$=\tfrac{1}{2}MK^2\omega^2$$

where M is the mass of the body and K is its radius of gyration.

If r be the radius of the body and v the linear velocity of the centre of mass of the body, then if

$v=\omega r$, the lowest point has zero velocity and the body is said to '*roll without sliding*'.

$\therefore$ the kinetic energy of rotation $=\frac{1}{2}MK^2\frac{v^2}{r^2}$

and the kinetic energy of translation $=\frac{1}{2}Mv^2$

$\therefore$ the total kinetic energy ($K.E$) of the body

$$=\frac{1}{2}MK^2\frac{v^2}{r^2}+\frac{1}{2}Mv^2$$

or,
$$K.E=\frac{1}{2}M\left(\frac{K^2}{r^2}+1\right)v^2 \quad .. \quad (8.13).$$

8.9. Work and Power in Rotational Motion

To calculate work done by the torque on a rotating body, we proceed on the usual pattern, that is, first calculate the work done by the torque on a particle and then extend it to a rigid body or system of particles. Let us consider a particle P at a distance r from the axis of rotation. During a short interval dt, it will move an infinitesimal distance ds along a circular path of radius r as the body rotates through an infinitesimal angle $d\theta$, where

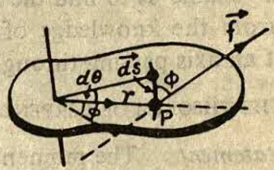


Fig. 8.6

$$ds=rd\theta$$

The work dw done by the force f acting on the particle during this small rotation is

$$dw=\vec{f}.\vec{ds}=f\cos\varphi\ ds$$ where φ is the angle between $\vec{f}$ and $\vec{ds}$.

or,
$$dw=(f\cos\varphi)(r.d\theta)$$
$$=(f\cos\varphi\ r)d\theta$$

The term ($fr\cos\varphi$), however, is the magnitude of the instantaneous torque exerted by f on the particle.

$$\therefore \quad dw=\tau d\theta \quad .. \quad (8.14).$$

Carrying the summation of this expression for all particles of the rigid body or system of particles

$$dW = \Gamma d\theta \qquad .. \quad (8.15).$$

This differential expression for the work done in rotation is equivalent to the expression $dW = Fdx$ for the work done in translational motion.

To obtain the rate of doing work in rotational motion, we have from Eq. 8.15.

$$\frac{dW}{dt} = \Gamma \frac{d\theta}{dt}$$

or,

$$P = \Gamma\omega \qquad .. \quad (8.16).$$

where ω is the angular velocity of rotation. This equation is the rotational equivalent of $P = Fv$ for translational motion.

8.10. Theorems of Axes of Moment of Inertia

There are two important theorems regarding the axes of rotation which enable us to find the moment of inertia of a body about any axis from the knowledge of moment of inertia of the same body about an axis passing through its centre of mass.

(A) THE THEOREM OF PERPENDICULAR AXES

Statement : The moment of inertia of a plane lamina about an axis perpendicular to the plane of the lamina is equal to the sum of the moments of inertia of the same lamina about two axes at right angles to each other, in its own plane, and passing through the point where the perpendicular axis meets the body.

If the perpendicular axis is considered as z-axis and the two axes in the plane of lamina as x and y axes, then this theorem mathematically states

$$I_z = I_x + I_y \qquad .. \quad (8.17).$$

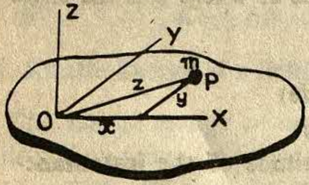


Fig. 8.7

Proof. Consider a particle of mass m at P at distances y, x and z respectiveley from axes OX, OY and OZ.

$I_y = \Sigma mx^2$; $I_x = \Sigma my^2$ and $I_z = \Sigma mz^2$

Now, $\quad x^2 + y^2 = z^2$

$\therefore \quad mx^2 + my^2 = mz^2$

or $\quad \Sigma mx^2 + \Sigma my^2 = \Sigma mz^2$

$$I_y + I_x = I_z$$

or $$I_z = I_x + I_y$$

This proves the theorem.

(B) THE THEOREM OF PARALLEL AXES

Statement : The moment of inertia of a body about any axis is equal to its moment of inertia about an axis parallel to this axis and passing through its centre of mass, plus the product of the mass of the body and the square of the distance between the axes.

If I is the moment of inertia of a body about any axis [(AB in the Fig. 8.8 (a)] and I_o is the moment of inertia of the same body about an axis (CD) parallel to the previous axis and passing through its centre of mass (CM), then this theorem states that,

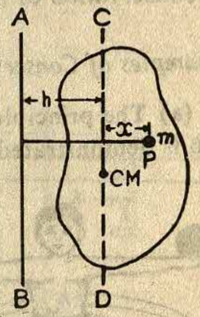


Fig. 8.8 (a)

$$I = I_o + Mh^2 \quad \text{.. (8.18).}$$

where M is the mass of the body and h is the distance between the axes.

Proof. Consider a particle of mass m at a distance x from CD and $(x+h)$ from AB. Then,

$$I = \Sigma m(x+h)^2 = \Sigma m(x^2 + 2xh + x^2)$$
$$= \Sigma mx^2 + \Sigma m2xh + \Sigma mh^2$$

or $$I = \Sigma mx^2 + 2h\Sigma mx + h^2\Sigma m$$

Now, $$\Sigma mx^2 = I_0$$
$$\Sigma m = M$$

and $\Sigma mx =$ algebraic sum of the moments of masses about $CM = 0$

$$\therefore \quad I = I_0 + Mh^2$$

This proves the theorem.

8.11. Principle of Conservation of Angular Momentum

In rotational motion of a body or system of particles

$$\vec{\Gamma} = \frac{d\vec{L}}{dt}$$

If the system or body is free from external torque

then, $$\vec{\Gamma} = 0$$

$$\text{or,} \qquad \frac{d\vec{L}}{dt}=0 \quad \text{or } L=a \text{ constant} \qquad .. \quad (8.19)$$

Thus if a body or a system of particles is free from external torques the angular momentum of the body or system of particles remains conserved. This is known as the principle of conservation of angular momentum. This principle is analogous to the principle of conservation of linear momentum.

Examples of Conservation of Angular Momentum

(a) The principle of conservation of angular momentum of a system is demonstrated by the famous turn-table experiment. A man sits on a stool that is free to rotate about a vertical axis. He holds his arms extended horizontally with two heavy weights in his hand. The turn-table is set to rotation and then left to itself. Now he pulls his hands to his sides and the system begins to rotate faster. This is due to the principle of conservation of angular momentum of the system (man+weights+turn-table). When the system is rotated with a certain speed, it acquires a certain angular momentum When the man pulls his hands to his sides the moment of inertia of the system decreases and the angular speed increases to maintain the value of the angular momentum of the system at its previous value.

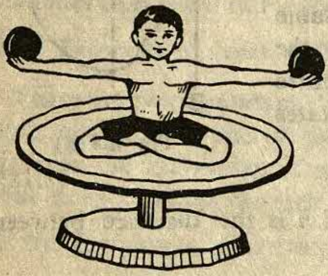


Fig. 8 8 (b)

(b) A diver makes use of this principle and takes a few somersaults before striking water. As he leaves the diving board he has a certain angular speed about a horizontal axis through his centre of mass so that he would rotate through half a turn before he strikes water. On the way he pulls his arms and legs toward the centre of the body, when he automatically makes more somersaults. After leaving the board there is no external torque on the diver and hence his initial angular momentum must be conserved. Note here that though, gravity on the man is an external force it exerts no torque about the centre of mass of the diver. When he pulls his arms and legs toward the centre of the body, his moment of inertia decreases and consequently he spins faster.

Acrobats (circus), ballet dancers, ice skaters and others often use this principle.

8.12. Calculation of the Moment of Inertia of Bodies of Known Geometrical Shapes

(*a*) *A thin rod about an axis through its centre and perpendicular to its length.*

Consider a thin rod of length l and an axis through its centre and perpendicular to its length, and consider an element of length dx at a distance x from the axis.

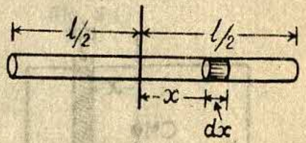


Fig. 8.9

If M be the mass of the rod, the mass of the element $=\frac{M}{l}dx$ and the moment of inertia of the element about the axis $=\left(\frac{M}{l}dx\right)x^2$

$$=\frac{M}{l}x^2.dx.$$

Integrating this expression between the limits $x=0$ to $x=\frac{l}{2}$, the moment of inertia of one half of the rod is obtained. The moment of inertia (I) of the entire rod will be just double of that integrated value.

$$\therefore \qquad I=2\int_0^{l/2}\frac{M}{l}x^2dx=\frac{2M}{l}\left[\frac{x^3}{3}\right]_0^{l/2}$$

$$=\frac{2M}{l}\left[\frac{l^3}{24}-0\right]$$

$$=\frac{2M}{l}\cdot\frac{l^3}{24}=\frac{1}{12}Ml^2$$

or, $$I=\frac{1}{12}Ml^2 \qquad \text{.. (8.20)}$$

$\because \quad I = MK^2 \qquad \therefore \quad K = \dfrac{l}{\sqrt{12}}$ .. (8.20 a)

Using this expression and the theorems of axes the moment of inertia of the rod about any other axis can be calculated.

(b) *Moment of inertia of a rectangular lamina about an axis through its centre of mass and parallel to one of its sides.*

Consider a rectangular lamina of mass M and length and breath l and b respectively. The axis is parallel to the side b and passes through the centre of mass CM of the rectangle.

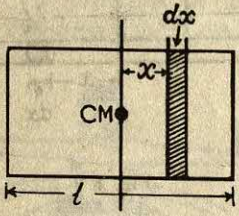


Fig. 8 10

Consider a small strip of width dx at a distance x from the axis. The mass of the strip will obviously be $\left(\frac{M}{l}dx\right)$, and, therefore, the moment of inertia of the strip about the axis will be $\left(\frac{M}{l}dx\right)x^2$.

Integrating this expression between the limits $x=0$ to $x=l/2$, the moment of inertia of one-half of the rectangle is obtained. The moment of inertia (I) of the rectangle will be just double of the integrated value.

$$\therefore \qquad I = 2\int_0^{l/2} \frac{M}{l} x^2 dx$$

$$= \frac{2M}{l}\left[\frac{x^3}{3}\right]_0^{l/2}$$

$$= \frac{2M}{l}\left[\frac{l^3}{24} - 0\right] = \frac{1}{12}Ml^2$$

Using this expression and the theorems of axes, the moment of inertia of the rectangle about any other axis can be calculated.

For example; Suppose we have to calculate the moment of inertia of the rectangle about an axis passing through its centre of mass and perpendicular to its plane. Take the axis through the centre of mass and perpendicular to b as x-axis and that perpendicular to l as y-axis.

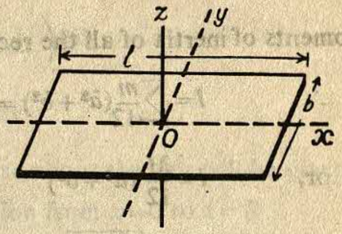


Fig. 8.11

Then, $I_x = \frac{1}{12} Mb^2$

and $I_y = \frac{1}{12} Ml^2$

By the theorem of perpendicular axes

$$I_z = I_x + I_y$$

$$\therefore \quad I_z = \frac{1}{12} Mb^2 + \frac{1}{12} Ml^2$$

$$\text{or,} \quad I_z = \frac{M}{12}(l^2 + b^2) \qquad \ldots \quad (8.21)$$

$$\text{and,} \quad K = \sqrt{\frac{l^2 + b^2}{12}} \qquad \ldots \quad (8.21\text{ a})$$

(*c*) *Moment of inertia of a rectangular bar about an axis passing through its centre of mass and perpendicular to the face containing length and breadth.*

Consider a rectangular bar of mass M, length a, breadth b, and thickness c. The axis is perpendicular to the top face containing length and breadth and passes through the centre of mass. Divide the bar into a large number of rectangular strips parallel to the top face. Let m_1, m_2, m_3..... be the masses of the strips so that $M = m_1 + m_2 + m_3 + \ldots\ldots = \Sigma m$. The moment of inertia of a strip of mass m is

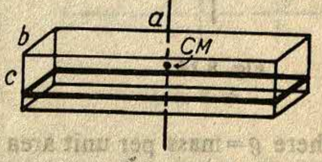


Fig. 8.12

$\frac{m}{12}(a^2+b^2)$. The moment of inertia of the bar is the sum of the moments of inertia of all the rectangles.

$$\therefore \quad I=\sum\frac{m}{12}(a^2+b^2)=\frac{a^2+b^2}{12}\ (\Sigma m)$$

$$\text{or,} \quad I=\frac{M}{12}(a^2+b^2) \quad \ldots \quad (8.22)$$

$$\text{or,} \quad K=\sqrt{\frac{a^2+b^2}{12}} \quad \ldots \quad (8.22\text{ a})$$

(*d*) *Moment of inertia of a loop or circular ring about an axis passing through its centre of mass and perpendicular to its plane.*

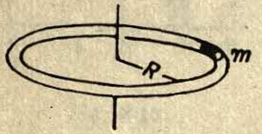


Fig. 8.13

Consider a circular ring of mass M and radius R. Consider a representative elementary mass m of the ring. The moment of inertia of m about the axis is mR^2 The moment of inertia of the loop is the sum of the moments of inertia of all the elementary masses.

$$\therefore \quad I=\Sigma mR^2=R^2(\Sigma m)$$

$$\text{or,} \quad I=MR^2 \quad \ldots \quad (8.23)$$

$$\text{and} \quad K=R \quad \ldots \quad (8.23\text{ a})$$

(*e*) *Moment of inertia of a circular disc about an axis passing through its centre of mass and perpendicular to its plane.*

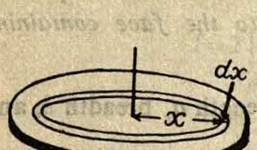


Fig. 8.14

Consider a disc of radius R and mass M, and consider a narrow concentric ring of radius x and width dx. The area of the ring

$=$ circumference $\times$ width

$=2\pi x dx.$

$\therefore$ the mass of the ring

$=2\pi x dx \varrho$

where $\varrho=$ mass per unit area of the disc $=\frac{M}{\pi R^2}$

or the mass of the ring $=2\pi x dx.\ \frac{M}{\pi R^2}=\frac{2M}{R^2}\,x dx$

The moment of inertia of the ring about the axis

$$=\left(\frac{2M}{R^2}xdx\right)x^2$$

$$=\frac{2M}{R^2}x^3dx$$

The moment of inertia of the disc about the axis will be given by the integration of the above expression from $x=0$ to $x=R$

$$\therefore \quad I=\int_0^R \frac{2M}{R^2}x^3dx=\frac{2M}{R^2}\left[\frac{x^4}{4}\right]_0^R$$

$$=\frac{2M}{R^2}\left[\frac{R^4}{4}-0\right]$$

or, $\quad I=\frac{1}{2}MR^2$ .. (8.24)

and, $\quad K=\frac{R}{\sqrt{2}}$ .. (8.24a)

If I_d is the moment of inertia of the disc about a diameter, then by the theorem of perpendicular axes,

$$I_d+I_d=\tfrac{1}{2}MR^2$$

or $$I_d=\tfrac{1}{4}MR^2.$$

If I_t is the moment of inertia of the disc about a tangent in its plane, then by the theorem of parallel axes,

$$I_t=\tfrac{1}{4}MR^2+MR^2$$

or $$I_t=\tfrac{5}{4}MR^2.$$

(*f*) *Moment of inertia of a solid cylinder about an axis passing through its centre of mass and perpendicular to its length.*

Consider a solid cylinder of mass M, length l and radius r. The axis is perpendicular to its length and it passes through its centre of mass.

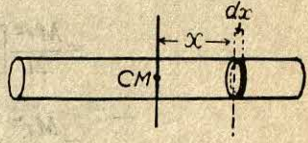


Fig. 8.15

Consider a disc of width dx at a distance x from the axis. The mass of the disc $=\frac{M}{l}dx$. The axis passing through the centre of mass of the disc and parallel to the given axis is the diameter of the disc. The moment of inertia of a disc about its diameter is $\frac{1}{4}$ mass $\times$ radius^2

(deduce it from Eq. 8.24 by applying the theorem of perpendicular axes)

$\therefore$ the moment of inertia of the disc about a diametrical axis parallel to the given axis

$$=\tfrac{1}{4}\left(\frac{M}{l}dx\right)r^2=\frac{Mr^2}{4l}dx$$

By applying the theorem of parallel axes we have,

the moment of inertia of the disc about the given axis

$$=\frac{Mr^2}{4l}dx+\left(\frac{M}{l}dx\right)x^2$$

$$=\frac{Mr^2}{4l}dx+\frac{M}{l}x^2dx$$

The moment of inertia of the cylinder will be obtained by integrating the above expression between the limits

$x=0$ to $x=\dfrac{l}{2}$ and then doubling it.

$$\therefore \qquad I=2\int_0^{l/2}\left(\frac{Mr^2}{4l}dx+\frac{M}{l}x^2dx\right)$$

$$=\frac{Mr^2}{2l}\int_0^{l/2}dx+\frac{2M}{l}\int_0^{l/2}x^2dx$$

$$=\frac{Mr^2}{2l}\Big[x\Big]_0^{l/2}+\frac{2M}{l}\left[\frac{x^3}{3}\right]_0^{l/2}$$

$$=\frac{Mr^2}{2l}\left[\frac{l}{2}-0\right]+\frac{2M}{l}\left[\frac{l^3}{24}-0\right]$$

$$=\frac{Mr^2}{4}+\frac{Ml^2}{12}$$

or, $$I=M\left(\frac{l^2}{12}+\frac{r^2}{4}\right) \qquad .. (8.25)$$

and $$K=\sqrt{\frac{l^2}{12}+\frac{r^2}{4}} \qquad .. (8.25\text{ a})$$

(g) *Moment of inertia of a solid sphere about a diameter.*

Consider a sphere of M and radius R. Let AB be the diameter about which the moment of inertia of the sphere is to be found out.

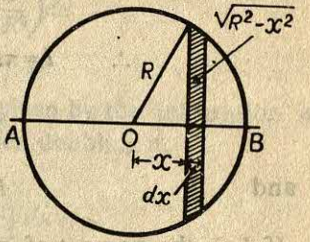


Fig. 8.16

Consider a *thin disc* perpendicular to AB at a distance x from the centre of the sphere and let its thickness be dx.

The radius of the disc $=\sqrt{R^2-x^2}$

The mass of the disc = volume × density

$=\pi(\sqrt{R^2-x^2})^2 dx\,\varrho$ where ϱ is the density of the sphere.

$=\pi\varrho(R^2-x^2)dx.$

The moment of inertia of the disc about AB

$=\frac{1}{2}$ mass × radius²

$=\frac{1}{2}\times\pi\varrho(R^2-x^2)dx\,.\,(\sqrt{R^2-x^2})^2$

$=\frac{\pi\varrho}{2}(R^2-x^2)^2 dx$

The moment of inertia of the sphere is given by the integration of the above expression from $x=0$ to $x=R$ and then doubling it.

$$\therefore \quad I=2\int_0^R \frac{\pi\varrho}{2}(R^2-x^2)^2\,dx$$

$$=\pi\varrho\int_0^R (R^4-2R^2x^2+x^4)dx$$

$$=\pi\varrho\left[R^4x-2R^2\,\frac{x^3}{3}+\frac{x^5}{5}\right]_0^R$$

$$=\pi\varrho\left[R^5-\frac{2R^5}{3}+\frac{R^5}{5}\right]=\pi\varrho.\frac{8}{15}R^5$$

Now, $\varrho = \dfrac{\text{mass of the sphere}}{\text{volume of the sphere}} = \dfrac{M}{\frac{4\pi}{3}R^3} = \dfrac{3M}{4\pi R^3}$

$$\therefore \quad I = \pi . \frac{3M}{4\pi R^3} \frac{8}{15} R^5 = \frac{2}{5} MR^2$$

$$I = \tfrac{2}{5} MR^2 \quad .. \quad (8.26)$$

and
$$K = \sqrt{\frac{2}{5}}\, R. \quad .. \quad (8.26)$$

If I_t is the moment of inertia of the sphere about a tangent to it, then by the theorem of parallel axes,

$$I_t = \tfrac{2}{5} MR^2 + MR^2 = \tfrac{7}{5} MR^2.$$

(*h*) *Moment of inertia of a spherical shell about a diameter.*

Consider a spherical shell of mass M and radius R. Let AB be the diameter about which the moment of inertia of the sphere is to be found out.

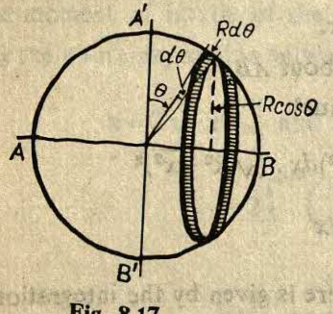


Fig. 8.17

Consider a *thin ring* perpendicular to AB Let θ be the angular position of the ring from the diameter $A'\,B'$ which is perpendicular to AB.

The radius of the ring $= R \cos\theta$

The width of the ring $= Rd\theta$

The mass of the ring = area × surface density of mass

$$= 2\pi\ R \cos\theta .\ Rd\theta\varrho$$
$$= 2\pi\ R^2\ \varrho \cos\theta\ d\theta$$

The moment of inertia of the ring about AB

$$= \text{mass} \times \text{radius}^2$$
$$= (2\pi R^2 \varrho \cos\theta\ d\theta) R^2 \cos^2\theta$$
$$= 2\pi R^4 \varrho \cos^3\theta\ d\theta$$

If x is the distance of the ring along AB from centre of the shell, then $x = R \sin\theta$

$$\therefore \qquad dx = R \cos\theta\ d\theta$$

$\therefore$ The moment of inertia of the ring about AB

$$= 2\pi\ R^4 \varrho\ \cos^3\theta \frac{dx}{R \cos\theta}$$

$$=2\pi\ R^3\varrho\ \cos^2\theta\ dx$$
$$=2\pi\ R^3\varrho\ (1-\sin^2\theta)\ dx$$
$$=2\pi\ R^3\varrho\left(1-\frac{x^2}{R^2}\right)dx$$
$$=2\pi\ R\ \varrho(R^2-x^2)\ dx$$

The moment of inertia of the shell is given by the integration of this expression from $x=0$ to $x=R$ and then doubling it.

$$\therefore \qquad I=2\int_0^R 2\pi\ R\varrho\ (R^2-x^2)\ dx$$
$$=4\pi\ R\varrho\left[R^2x-\frac{x^3}{3}\right]_0^R$$
$$=4\pi\ R\varrho\left[R^3-\frac{R^3}{3}\right]$$
$$=4\pi\ R\varrho\frac{2}{3}R^3$$

or $$I=\frac{8\pi R^4\varrho}{3}$$

Now $$\varrho=\frac{\text{mass of the shell}}{\text{area of the shell}}=\frac{M}{4\pi R^2}$$

$$\therefore \qquad I=\frac{8\pi R^4}{3}\cdot\frac{M}{4\pi R^2}=\frac{2}{3}MR^2$$

$$I=\tfrac{2}{3}MR^2 \qquad \text{.. (8.27)}$$

and $$K=\sqrt{\frac{2}{3}}R \qquad \text{.. (8.27a)}$$

If I_t is the moment of inertia of the shell about a tangent to it, then by the theorem of parallel axes,

$$I_t=\tfrac{2}{3}\ MR^2+MR^2=\tfrac{5}{3}\ MR^2.$$

8.13. Analogy between Translatory and Rotational Motion

We have shown that all the laws and formulae of translatory motion can be extended to rotational motion simply by replacing 'mass' by 'moment of inertia', 'linear displacement by 'angular displacement' and force by torque. The complete analogy is tabulated

in Table 1.

Table 1

Linear motion	Angular motion
Linear displacemens (x)	Angular displacement (θ)
Linear velocity $\left(v=\frac{dx}{dt}\right)$	Angular veloclty $\left(\omega=\frac{d\theta}{dt}\right)$
Linear acceleration $\left(a=\frac{dv}{dt}=\frac{d^2x}{dt^2}\right)$	Angular acceleration $\left(\alpha=\frac{d\omega}{dt}=\frac{d^2\theta}{dt^2}\right)$
Force = mass $\times$ acceleration $\vec{F}=m\vec{a}$	Torque = moment of inertia $\times$ angular acceleration $\vec{\tau}=I\vec{\alpha}$
Momentum = mass $\times$ velocity $\vec{p}=m\vec{v}$	Angular momentum = $M.I\times$ angular velocity $\vec{L}=I\vec{\omega}$
Work done = force $\times$ displacement $W=Fx$	Work down = torque $\times$ angular displacement $W=\tau.\theta$
Kinetic energy = $\frac{1}{2}\times$ mass $\times$ velocity^2 $K=\frac{1}{2}mv^2$	Kinetic energy = $\frac{1}{2}M\ I\times$ angular velocity^2 $K=\frac{1}{2}I\omega^2$
Power = Force $\times$ velocity $P=F.v$	Power = Torque $\times$ angular velocity $P=\tau.\omega$

8.14. Determination of Moment of Inertia : Principle of Torsional Pendulam

The moment of inertia of a body can be easily found by constructing a *torsion pendulum* with it and finding the period of oscillation :

Let a body be suspended by a wire from a rigid support. This simple arrangement is called a torsion pendulam. If the body is turned through a small angle and subsequently released, it will be set to torsional oscillations twisting the wire forward and backward. If at any instant θ be the angular displacement, the torque on the body tending to reduce θ is proportional to θ. That is the torque on the body $=-\tau\theta$ where τ is a constant depending upon the length, radius and material of the wire.. The minus

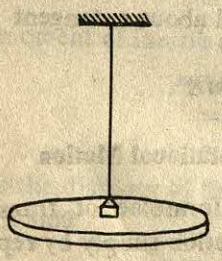


Fig. 8.18

sign is put to account for the fact that the torque is in the decreasing direction of θ. If I is the moment of inertia of the body then

$$-\tau\theta = I\alpha$$

or
$$\alpha = -\frac{\tau}{I}\theta \quad \therefore \quad \alpha \propto -\theta$$

Therefore the motion is simple harmonic.

Here $\omega^2 = \frac{\tau}{I}$ or $\omega = \sqrt{\frac{\tau}{I}}$ or $T = 2\pi\sqrt{\frac{I}{\tau}}$ .. (8.28)

To find the moment of inertia of the suspended body, first the pendulum is set to torsional oscillations and its time-period is found with the help of a stop-watch. Let it be T_1. Then

$$T_1 = 2\pi\sqrt{\frac{I}{\tau}} \quad .. \text{(i)}$$

Next a body of known moment of inertia (I_1) is placed on the body and again the pendulum is set to torsional oscillations and its new time period is determined. Let it be T_2.

$$T_2 = 2\pi\sqrt{\frac{I+I_1}{\tau}} \quad .. \text{(ii)}$$

Squaring and dividing (ii) and (i)

$$\frac{T_2^2}{T_1^2} = \frac{I+I_1}{I} \quad \text{or} \quad I_1 = \frac{T_2^2 - T_1^2}{T_1^2} . I$$

or
$$I = \frac{T_1^2 I_1}{T_2^2 - T_1^2}$$

8.15. Gyrostat

A gyrostat is a heavy circular disc free to rotate about an axle passing through its centre of mass and perpendicular to its plane. The axle is mounted in a ring free to rotate about an axis perpendicular to the axle of the disc inside a second ring which, in its turn, is free to rotate about a third axis. Such a device has got the property to preserve its axis of rotation against disturbing small torques. This fact endows it with the capacity of 'stabilising direction of motion.' It

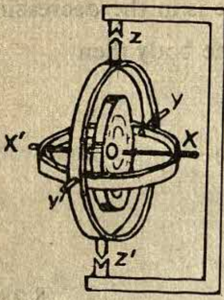


Fig. 8.19

is used in the construction of gyrocompass, turn and bank indicator of aircrafts rifling of barrels of fire arms, stabilisation of rolling of ships.

It works on the principle that if a torque is applied to a rotating disc so as to turn its plane perpendicular to its plane of rotation, its axis of rotation turns in the plane which is perpendicular to both the plane of rotation of the disc and the plane of rotation of the torque without change in the magnitude of the angular velocity of the disc. Speaking in terms of the frame of reference, if x-axis is the axis of rotatation (i. e. YOZ is the plane of rotation) of the disc, y-axis is the axis of the torque (i e. ZOX is the plane of rotation of the torque), then the axis of rotation of the disc turns about the z-axis (i.e., it rotates in the YOX plane) without change in the angular speed of the disc. This is called the 'precession' of the axis and the angular speed (ν) with which this axis rotates is called the 'precessional velocity' of the axis of the disc. Since here the direction of the angular velocity (ω) of the disc (the angular velocity as vector is always along the axis of rotation) changes with time, the disc possesses angular acceleration due to the precession of its axis. This is a case somewhat similar to circular motion where a body can have acceleration without change in the magnitude of its velocity.

To produce an angular acceleration, a torque is always needed. The torque needed to produce precession of the axis is called precessional torque.

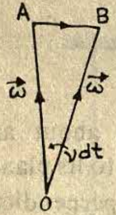


Fig. 8.20

The change in the angular velocity

$$=\omega \times \nu \ dt$$

$\therefore$ The rate of change of angular

velocity $$=\frac{\omega \times \nu dt}{dt}=\omega \times \nu$$

$\therefore$ τ (the precessional torque)

$$=I \times (\omega \times \nu)=(I\omega)\times \nu = L \times \nu$$

where L is the angular momentum of the disc.

$$\therefore \qquad \nu=\frac{\tau}{L}$$

This gives the precessional velocity of the disc. Obviously the greater the angular momentum, the less the precessional velocity of the axis of the disc for a given external torque. Thus a gyrostat has got the property of maintaining the direction of its axis of rotation against external small disturbing torques. An electron revolving around the nucleus is an example of gyrostats in real existence in nature.

Examples

1. *A uniform disc of 3 kg and radius 20 cm is mounted on an axle supported in fixed frictionless bearings. A light cord wrapped around the rim of the wheel carries a 500 gm mass. Find the angular acceleration of the wheel and the tangential acceleration of a point on the rim.*

Sol. Considering the free-body diagram of the weight we have

$$mg-T=ma$$

$$T=mg-ma$$

The torque on the wheel = moment of tension about the axis of rotation of the wheel.

$$=T\times R=\tfrac{1}{2}MR^2\alpha$$

$(\because I=\tfrac{1}{2}MR^2$ and $\tau=I\alpha)$.

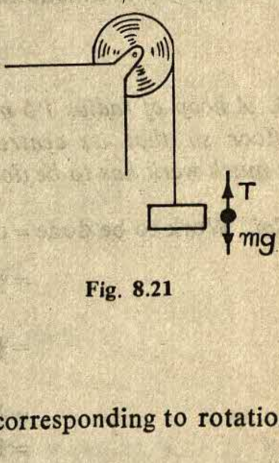


Fig. 8.21

$$\text{or}\qquad \alpha=\frac{2T}{MR}\quad\text{or}\quad T=\frac{MR\alpha}{2}$$

If x be the displacement of the mass corresponding to rotation of the wheel by θ, then

$$x=R\theta \text{ or } \frac{dx}{dt}=R\frac{d\theta}{dt}$$

$$\text{or}\qquad v=R\omega \text{ and } \frac{dv}{dt}=R\frac{d\omega}{dt}\quad\text{or}\quad a=R\alpha$$

$$\therefore\quad T=\tfrac{1}{2}MR\alpha=mg-ma=mg-mR\alpha$$

$$\text{or}\quad \alpha=\frac{mg}{mR+\frac{MR}{2}}=\frac{\cdot5\times9\cdot8}{\cdot5\times\cdot2+\frac{3\times\cdot2}{2}}=\frac{4\cdot90}{\cdot1+\cdot3}$$

or $\alpha=12\cdot25$ rad sec^{-2} **Ans.**

The tangential acceleration $= a = R\alpha = {\cdot}2 \times 12{\cdot}25$

$= 2{\cdot}45\ ms^{-2}$ **Ans.**

2. *Compute the work done in 2 s on the disc of Ex. 1 and the increase in the kinetic energy.*

Sol. $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

$\therefore\ \theta = 0 \times 2 + \frac{1}{2} \times 12{\cdot}25 \times 2^2$

$= 24{\cdot}50$

the torque on the disc $= TR = \frac{1}{2}MR^2\alpha$

$= \frac{1}{2} \times 3 \times {\cdot}2^2 \times 12{\cdot}25$

$= {\cdot}735\ Nm.$

$\therefore$ Work done $= \tau\theta = {\cdot}735 \times 24{\cdot}5 = 18J$ **Ans.**

$\omega = \omega_0 + \alpha t$

$\therefore\ \omega = 0 + 12{\cdot}25 \times 2 = 24{\cdot}5$ rad s^{-1}

$\therefore$ Increase in kinetic energy $= \frac{1}{2}I\omega^2$

$= \frac{1}{2}\ \frac{1}{2}MR^2.\omega^2$

$= \frac{1}{4} \times 3 \times {\cdot}2^2 \times 24{\cdot}5^2 = 18J$ **Ans.**

3. *A hoop of radius 1·5 m weighs 150 kg. It rolls along a horizontal floor so that its centre of mass has a speed of 15 cm per second. How much work has to be done to stop it.*

Sol. Work to be done = change in kinetic energy

$= \frac{1}{2}\ Mv^2 + \frac{1}{2}\ I\omega^2$

$= \frac{1}{2}\ Mv^2 + \frac{1}{2}MR^2\frac{v^2}{R^2}\ (\because\ \ v = \omega R)$

$= \frac{1}{2}\ Mv^2 + \frac{1}{2}\ Mv^2 = Mv^2$

$= 150 \times {\cdot}15^2$

$= 3{\cdot}375$ joule. **Ans.**

4. *You are given two spheres of the same mass and size and appearance, but one of them is hollow and the other solid. How will you find which one is hollow and which one is solid ?*

Sol. Allow them to roll without slipping down an inclined plane. The one which reaches the bottom of the plane earlier is solid and the other is hollow.

In rolling down an incline a rolling body loses potential energy of an amount Mgh, where h is the height of the plane. If S is the length of the plane and θ is its inclination with the horizontal then

$$h = S\sin\theta$$

$\therefore$ Loss in potential energy $= MgS \sin\theta$

Gain in kinetic energy $= \frac{1}{2}I\omega^2 + \frac{1}{2}Mv^2$

$$= \frac{1}{2}MK^2\omega^2 + \frac{1}{2}Mv^2$$

$$= \frac{1}{2}MK^2\frac{v^2}{r^2} + \frac{1}{2}Mv^2$$

$$= \frac{1}{2}Mv^2\left(\frac{K^2}{r^2}+1\right)$$

$\therefore$ $$MgS \sin\theta = \frac{1}{2}Mv^2\,(K^2/r^2+1)$$

or $$v^2 = \frac{2gS \sin\theta}{K^2/r^2+1}$$

If f is the effective accleration down the plane then

$$v^2 = 2fS$$

$\therefore$ $$f = \frac{g \sin\theta}{K^2/r^2+1}$$

For a solid sphere, $K^2 = 2/5\, r^2$

$\therefore$ $$f_{sphere} = \frac{g \sin\theta}{2/5+1} = \frac{5}{7}g \sin\theta = {\cdot}71\, g \sin\theta$$

For a hollow sphere (shell) $K^2 = 2/3\, r^2$

$\therefore$ $$f_{shell} = \frac{g \sin\theta}{2/3+1} = \frac{3}{5}g\sin\theta = {\cdot}6g \sin\theta.$$

$\therefore$ $f_{sphere} > f_{shell}$

5. *A solid right circular cylinder of mass m and radius r is pulled along a horizontal plane by a force P applied to the end of a string wound around the cylinder. (a) Assuming no slip, find the acceleration of the centre. (b) What coefficient of friction is required between the cylinder and the plane to prevent slipping under these conditions.*

Sol. (a) Considering the linear motion of the body

$$P - F_{friction} = mf \quad \text{.. (i)}$$

Considering rotational motion

$$Pr + F_{friction} \times r = I\alpha$$

But $$\omega r = v \quad (\because \text{ there is no slip})$$

or $$\alpha r = f$$

$$Pr + F_{friction} \times r = 1/2\, mr^2 \times f/r \quad (\therefore\ I = 1/2\, mr^2)$$

or $$P + F_{friction} = 1/2\, mf \quad \text{.. (ii).}$$

Adding (i) and (ii)

$$2\,P=mf+1/2\,mf=3/2\,mf$$

or $$f=4/3\frac{P}{m} \quad \text{Ans.}$$

(b) Slipping is prevented so long friction between the cylinder and the plane does not exceed the limiting frictional force. That is

$$\mu mg \geqslant F_{friction} \quad \text{or} \quad \mu mg \geqslant \frac{1}{3}P \ \text{or}\ \mu \geqslant \frac{1}{3}\,\frac{P}{mg} \quad \text{Ans.}$$

QUESTIONS

(A)

1. Angular velocity is a (a) polar vector, (b) pseudo vector, (c) axial vector, (d) a scalar.

2. The dimensions of angular momentum is (a) MLT, (b) MLT^{-2}, (c) ML^2T^{-1} (d) MLT^{-1}.

3. The moment of inertia of a spherical shell about a tangent is (a) MR^2, (b) $\frac{1}{2}MR^2$, (c) 5/2 MR^2, (d) 5/3 MR^2.

4. The moment of inertia of a disc about its tangent is (a) MR^2, (b) $\frac{1}{4}MR^2$, (c) $\frac{3}{4}MR^2$, (d) 5/4 MR^2.

5. When a man spinning on a turntable with his hands outstretched lowers his hands, the spinning rate (a) decreases, (b) increases, (c) remains unchanged, (d) becomes zero.

6. A solid sphere and a thin spherical shell have equal masses and equal moments of inertia about their diameters. Their radii must bear the ratio of (a) 3 : 5, (b) 5 : 3, (c) $\sqrt{3} : \sqrt{5}$, (d) $\sqrt{5} : \sqrt{3}$

7. A fly swings around at a constant speed v in a circle of radius r. After a short while it starts swinging at the same speed in a bigger circle. Which of the following remain unchanged (a) moment of inertia, (b) angular speed, (c) kinetic energy, (d) mass.

8. A solid sphere of radius R is flattened to a disc of radius 5 R. Their moment of inertia about their respective diameter must bear the ratio (a) 1 : 4, (b) 16 : 1, (c) 8 : 125, (d) 125 : 8.

9. A sphere, a disc and a ring of the same mass and radius are allowed to roll down an inclined plane from the same height without slipping. The order in which they reach the bottom of the plane earlier is (a) sphere, d sc and ring, (b) sphere, ring and disc, (c) ring, disc and sphere, (d) ring, sphere and disc.

10. The moment of inertia about any diameter of a circular disc of unit diameter and unit mass is (a) unit, (b) $\frac{1}{4}$ unit, (c) 1/8 unit, (d) 1/16 unit.

Ans. : 1. c, 2. c, 3. d. 4. d, 5. b, 6. d. 7. c+d, 8. c, 9. a, 10. d.

(B)

1. What is moment of inertia ? What is its physical significance ?

2. Show that the work done in rotational motion is the product of torque and angular displacement.

3. State Newton's three laws of motion for rotating bodies.

4. State the principle of conservation of angular momentum and deduce it.

(C)

1. Explain average and instantaneous angular velocity of a rigid body rotating about an axis. How do you represent it vectorially ? Prove rotational kinetic equation.

$$\theta=\omega_0 t+\tfrac{1}{2}\alpha t^2$$

2. Define 'torque' and 'angular momentum'. Show that for a particle

$$\vec{\tau}=\frac{d\vec{l}}{dt}$$

3. What do you mean by moment of inertia and radius of gyration of a rotating body. Show that in rotational motion $\vec{\Gamma}=I\vec{\alpha}$

where Γ, I and α have their usual meaning.

4. State and prove the theories of perpendicular and parallel axes as applied to moment of inertia.

Derive an expression for the moment of inertia of a rectangular bar about an axis passing through its centre of mass and perpendicular to its plane.

5. Calculate the moment of inertia of a thin circular disc of mass M and radius R about an axis through its centre and perpendicular to its plane.

6. Derive expressions for the moment of inertia of a solid sphere about its diameter.

7. Calculate the moment of inertia of a solid cylinder about an axis passing through its centre and perpendicular to its length.

(D)

1. A thin uniform rod of length 12 cm and mass 10 gm makes 10 rotations per second about an axis passing normally through its centre. Calculate the angular momentum of the rod about this axis.

(Ans. $7{\cdot}54\times10^{-4}$ kg m^{2}s^{-1})

2. A thin circular disc of mass 5 gm and radius 2 cm is rotated at the rate of 10 rotations per second about its diameter. Calculate its K. E. and angular momentum.

(Ans. $9{\cdot}86\times10^{-4}$ J; $3{\cdot}14\times10^{-5}$ kg m^2s^{-1}).

3. A coin of diameter 3 cm and mass 9 gm is allowed to roll on a table with a

speed of $6\times10^{-2} ms^{-1}$. Calculate the total kinetic energy of the coin.

Ans. $2{\cdot}43\times10^{-5}$ joule).

4. A circular wheel rolls without slipping on an inclined plane whose height is 20 m. Calculate its velocity on reaching the bottom of the plane. ($g=9.8$ ms^{-2}).

(Ans. $16{\cdot}2$ ms^{-1})

5. A wheel of moment of inertia 5×10^{-2} kgm^2 is making 10 rotations per minute. What amount of work will be needed to make it rotate five times faster ?

(Ans. ·658 joule).

6. Two thin discs each of mass 100 gm and radius 5 cm are placed at either end of a rod of 20 cm long and mass 120 gm. What is the moment of inertia of the system about an axis passing through the centre of the rod and perpendicular to its length ? The rod is perpendicular to the discs at their centres.

(Ans. ·002525 kgm^2)

7. An electric motor of 100 H. P. rotates the armature of a pump at 1800 rev/min. What torque does it apply ? (1 H. P.$=$746 watt). (Ans. $395{\cdot}7$ Nm).

8. A body of radius R and mass m is rolling horizontally without slipping with speed v. It then rolls up an incline to a maximum height h. If $h=\frac{3}{4}\frac{v^2}{g}$, what is the moment of inertia of the body ? (Ans. $\frac{1}{2}Mr^2$)

9. A horizontal turntable of radius r carries a gun at its edge and rotates with initial angular velocity ω_o about its geometric axis. Calculate the increment of angular velocity $\triangle\omega$ that the turntable will obtain if the gun fires a bullet of mass m with tangential muzzle velocity v. The moment of inertia of the turntable and the gun is I_o

Ans. $\dfrac{mvr}{I_o+mr^2}$

[Hints : $(I_o+mr^2)\omega_o=I_o\omega-m(v-\omega r)r$ from the principle of conservation of angular momentum]

10. A homogeneous thin rigid hoop of radius r rolls without slipping along a horizontal plane with velocity v_c and strikes an inclined plane. With what velocity will the hoop start up the inclined plane if $\alpha=45°$? Neglect any tendency to rebound or slip. (Ans. $\cdot863\ v_c$).

[Hints Consider conservation of angular momentum about the point of contact of the hoop with the plane immediately after it begins to start up the plane. Remember that moment of momentum is also angular momentum. Apply condition for no sliding].

11. A solid homogeneous right circular cylinder of mass m and diameter d is pulled up a 30° incline by a constant force $P=\frac{1}{2}\ mg$ applied to the end of a string wound around its circumference. Assuming no slip at point of contact, find the acceleration of the centre of mass up the plane. (Ans. $g/3$)

[Hint. Consider the dynamics of linear and angular motion. Apply the condition for no sliding.]

15. At what height h above the table surface should a billiard ball of radius r be struck by a horizontal impact in order to have no sliding at the point of contact. (Ans. $7/5\ r$)

[Hint. Apply the dynamics of linear and rotational motion of the ball. Apply the condition for no sliding.]

(E)

1. Many great rivers flow toward the equator. The sediment they carry to the sea.........the rotation of the earth.

2. The moment of inertia of a cube about its edge is.........Ma^2 where M is the mass of the cube and a is the length of the cube on each edge.

3. If two circular discs of the same mass and thickness are made from metals having different densities, which disc will have the larger rotational inertia about its central axis.

4. Can you distinguish between a raw egg and a hard boiled one by spinning each one ?

5. A particle of mass m moves parallel to the x-axis with velocity v at a distance d from it. Does it possess any angular momentum about the origin ? If so, by how much ?

(Ans. : **1.** decreases **2.** 2/3 **3.** The one having less density will have greater rotational inertia **4.** Yes, the hard boiled egg will have greater rotational inertia. **5.** Yes, because moment of momentum is angular momentum. By mvd.)

CHAPTER 9

GRAVITATION: ESCAPE VELOCITY: SATELLITES: WEIGHTLESSNESS

9.1. Historical Background

It was occasioned by the fall of an apple as Newton sat in a contemplative mood under an apple tree that he started thinking why the apple should fall towards the earth instead of going away from the earth. There must then exist some force of attraction between the earth and the apple. It occurred to him almost simultaneously that the force between the earth and the moon providing necessary centripetal force have the same origin. In fact, he argued everybody in this universe attracts every other body towards itself. This is called gravitation.

9.2. Newton's law of Universal Gravitation

The force of attractions between any two particles varies directly as the product of their masses and inversely proportional to the square of the distance between them.

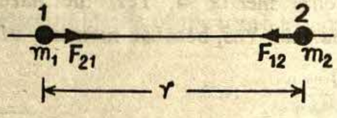


Fig. 9.1

If m_1 and m_2 be two point masses (particles) and r the distance between them, then the magnitude of the force is

$$F = G\frac{m_1 m_2}{r^2} \quad .. (9.1).$$

where G is a universal constant having the same value for all pairs of particles. In fact there are two forces: one is the force exerted by the first particle on the second particle (F_{12}) towards the first particle along the line joining the two. Likewise, the second particle exerts a force on the first (F_{21}) that is directed toward the second particle along the line joining the two. Thus proper vectorial representation of the gravitational force are

$$\vec{F}_{12} = -G\,\frac{m_1 m_2}{r^2_{12}}\,.\,\hat{r}_{12} \quad .. (9.1\text{ a}).$$

and $$\vec{F}_{21} = -G\frac{m_1 m_2}{r^2_{21}}\,\hat{r}_{21} \quad \text{.. (9.1 b).}$$

The constant G is called the universal gravitational constant. Its value is $6{\cdot}67 \times 10^{-11} Nm^2kg^{-2}$ in SI.

It has the dimensions $M^{-1}L^3T^{-2}$ when M, L and T are the base units.

It is to be noticed carefully that the law expresses the force between two point masses (particles). This point should be clearly understood and one should be careful about its application. Large bodies are to be divided into particles and then the interaction between all particles must be computed. Another important feature of the law to be noticed is that the gravitational force does not depend on the properties of the intervening space which is not found to be true with the electric forces between two point charges or magnetic force between two point charges in motion.

9.3. Experimental Determination of G

(A) CAVENDISH'S METHOD

The laboratory class of experiments to measure force of attraction between two massive balls and hence determine G was first designed by Rev. John Michell who could not use it due to his sudden death. Later, this apparatus fell into the hands of Cavendish who successfully used it to determine G.

The apparatus consisted of a long bar PQ about 1·8m long attached to a wheel W which was operated from outside. It carried two large and equal lead spheres C and D.

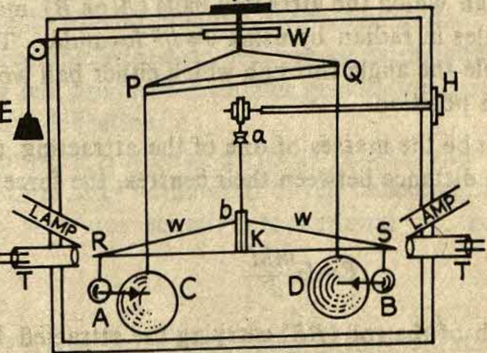


Fig. 9.2

Immediately below this bar another light rod RS was suspended by silvered copper wire from a torsion head T which could also be turned from outside. Two wires (w, w) fastened to the ends of the rods and tied to a vertical rod in the middle gave additional strength to the rods to bear the weights without bending. Two small lead balls A and B were suspended from the two ends of the rod RS in such a way that the centres of the four balls lay in the same horizontal plane. To measure the angle of turning, a vernier scale was provided at one end of the rod (RS) which moved over a scale placed on a stand without touching the scale. To guard against any change of temperature and consequent air currents, the whole apparatus was closed in a gilt glass case and observations were taken with the help of telescope T fixed into the walls of the chamber.

Procedure : In the actual experiment the big lead spheres C and D were set opposite to A and B respectively by turning the wheel W with the help of string wound round it. The reading of the vernier was noted by the method of oscillations. The attracting balls (C and D) were then moved to C' and D' position such that the distances between the centres of the attracting balls (C and D) and attracted balls (A and B) were the same as before. The reading of the vernier scale was again noted by the method of oscillations. The distance through which the attracted balls (A or B) moved was reduced to angles in radian by using $\theta = l/r$ formula. This angle was evidently double the angle through which either ball would turn from its equilibrium position.

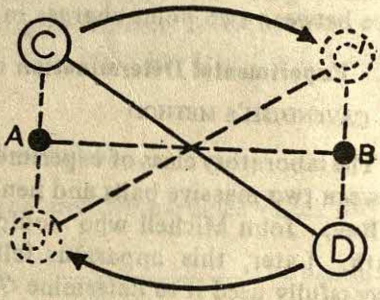


Fig. 9.3

If M and m be the masses of one of the attracting and attracted balls and d the distance between their centres, the force of attraction on A or B is

$$F = G\frac{mM}{d^2} \qquad \text{.. (i).}$$

If the length of the rod (RS) carrying the attracted balls (A and B) be l the moment of the couple formed by the two forces on A

and B, that is, the torque experienced by the rod (RS) is

$$c=\left(\frac{GmM}{d^2}\right).\, l=\frac{GmMl}{d^2}. \qquad \text{.. (ii).}$$

This torque is opposed by the torsion in the suspending fibre. If τ represents the torque per unit twist, the torque for a twist $\theta=\tau\theta$ and for eqilibrium,

$$\frac{GmMl}{d^2}=\tau\theta \qquad \text{.. (iii).}$$

In order to determine τ, the rod (RS) alone was set to torsional oscillations and its time-period was measured. By the principle of torsional pendulum

$$T=2\pi\sqrt{\frac{I}{\tau}}$$ where I is the moment of inertia of the oscillating system (rod + balls)

or $$\tau=\frac{4\pi^2 I}{T^2} \qquad \text{.. (iv).}$$

τ is calculated from (iv) and this is used in (iii) to calcutate G. The value of G obtained by Cavendish was $6{\cdot}754\times10^{-11}$ SI units.

Defects of Cavendish's method: In the Cavendish's experiment the accuracy was limited owing to the follwing defects :

(i) The suspension wire was rather thick. This made τ comparatively large and consequently the deflection small

(ii) The small angle of rotation was measured by ordinary vernier method.

(iii) Temperature gradients could not be avoided completely in the large chamber in which the apparatus was housed.

(iv) The appreciable attraction between an attracted ball and the distant attracting ball produced a counter gravitational couple which further reduced the deflection.

(v) The torsion wire, not being perfectly elastic, did not return to its normal position after withdrawal of the deflecting couple and thus the torque was not strictly proportional to the angle of rotation.

(B) BOY'S METHOD

Sir Charles Vernon Boys took into account all the defects of Cavendish's experiment and designed a new apparatus.

The apparatus consisted of two attracted small gold balls (A and B) suspended at different heights by two quartz fibres from the ends of a small mirror S which in its turn was suspended by a long quartz fibre from a torsion head T supported on the top of a long vertical glass tube. The quartz suspension was much better than ordinary silvered copper wire. It was almost perfectly elastic and sufficiently strong to bear the heavy weights of balls. Moreover the restoring torque per unit twist (called torsional rigidity) of quartz was much smaller and consequently deflection was much larger.

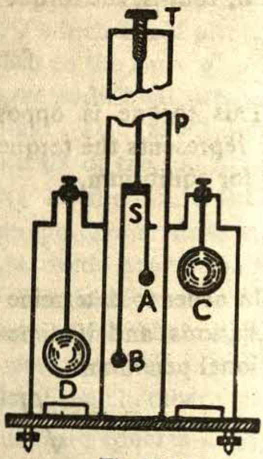


Fig. 9.4

In an outer coaxial tube, which could be rotated and its angle of rotation could be measured accurately, two attracting large lead spheres C and D were suspended at different heights in levels with the centres of A and B respectively.

The angle of rotation was measured by the 'lamp and scale' method. Light from a lamp was directed on to the mirror S and the reflected light was received on a distant graduated scale. The deflection of the spot of light on the scale bears the relation

$$\theta=\frac{d}{2D} \quad \text{.. (i).}$$

with the angle of rotation of the mirror. Here d is the deflection of the spot of light and D is the distance of the scale from the mirror. A circular scale graduated in degrees was provided to measure the angle of rotation of the tube.

•The whole apparatus rested on a platform provided with levelling screws and rubber pads are placed below the lead spheres as safe guard against breakage, in case they should fall accidentally.

Boys removed all the defects of Cavendish's method. By using fine quartz fibre he removed the first defect, by using the 'lamp and scale' arrangement he removed the second defect, by reducing the size of the apparatus he removed the fourth defect and finally by arranging the pairs of balls at different heights the counter gravitational couple was reduced to a minimum.

Procedure. The outer tube is rotated and brought in the position in which the centres of all the four balls lie in the same vertical plane. In this position the mirror experiences no torque. From this position if it is rotated through 180°, the torque on the mirror will again be zero. Obviously there will be a particular position of the outer tube where the torque on the mirror will be maximum. In the experimental procedure from the first position of 'no torque' i.e., when C is opposite to A and closest to it and D is opposite and closest to B, the outer tube is rotated slowly till the deflection of the spot of light on the scale on one side is maximum. This position of the spot of light is noted. The position of the tube on the circular scale is also noted. The tube (outer one) is then rotated in the oppsite direction to produce maximum deflection on the other side of the centre of the scale. The position of the spot in this position of the tube is again noted. The difference gives the deflectioh of the spot of light on the scale. This deflection is then converted into radian by using the formula (i). Half of this value is obviously the angle of twist of the suspension wire from the mean position. Let the angle of twist be θ in radian.

The difference of the readings in the two positions of the tube on the circular scale gives twice the angle of rotation of the tube from the mean position.

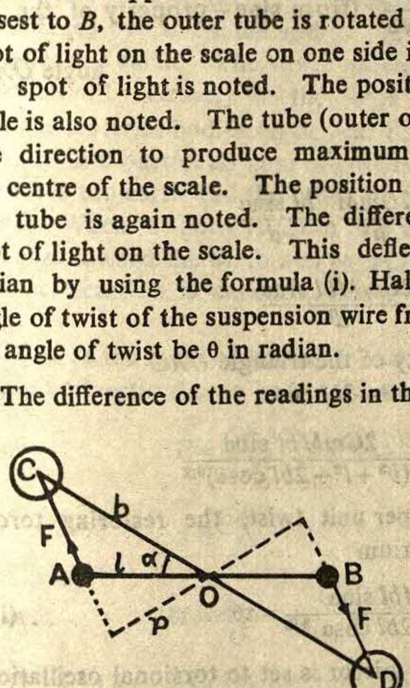


Fig. 9.5

Let m and M be the masses of one of the gold balls and lead spheres respectively and d is the distance between the centres of the neighbouring attracted and attracting masses. The force of attraction is

$$F = G\frac{mM}{d^2} \quad \text{.. (ii).}$$

Let $2l$ be the length of the mirror i.e., the distance between the centres of the gold balls and $2b$ be the distance between the centres of lead spheres. These two distances can be measured in the experiment with great accuracy. Let $2p$ be the perpendicular distance between the forces of attraction between the pairs of attracting and

attracted masses. Then the deflecting torque

$$= \text{force} \times \text{perpendicular distance}$$

$$= F \times 2p = \frac{2GmMp}{d^2} \qquad \text{.. (iii).}$$

With reference to the Fig. 9.5

$$p = b \sin \angle ACO$$

and $\dfrac{l}{\sin \angle ACO} = \dfrac{d}{\sin \alpha}$ (from sine – property of the triangle OAC)

$$\therefore \quad p = \frac{bl \sin\alpha}{d}$$

and the deflecting torque $= \dfrac{2GmM}{d^2} \cdot \dfrac{bl \sin\alpha}{d}$

$$= \frac{2GmMbl \sin\alpha}{d^3}$$

Again from cosine – property of the triangle OAC

$$d^2 = b^2 + l^2 - 2bl \cos\alpha$$

$\therefore$ the deflecting torque $= \dfrac{2GmMbl \sin\alpha}{(b^2 + l^2 - 2bl \cos\alpha)^{3/2}}$

If τ be the torsional couple per unit twist, the restoring torque for twist θ is $\tau\theta$, and for equilibrium

$$\frac{2GmMbl \sin\alpha}{(b^2 + l^2 - 2bl \cos\alpha)^{3/2}} = \tau\theta \qquad \text{.. (iv).}$$

In order to determine τ, the mirror is set to torsional oscillations and its time period is measured. By the principle of torsional pendulum

$$T = 2\pi \sqrt{\frac{I}{\tau}}$$

where I is the moment of inertia of the oscillating system (mirror + gold balls)

or
$$\tau = \frac{4\pi^2 I}{T^2} \qquad \text{.. (v).}$$

Form this formula (v), τ is calculated and this value is used in (iv) to calculate G.

The value of G obtained by Boys was

$$6{\cdot}6576 \times 10^{-11} \ SI \text{ units.}$$

9.4. Mean Density and Mass of the Earth

From a knowledge of the radius of the earth and the constants g and G, the mean density and mass of the earth can be calculated. Among these three quantities the acceleration due to gravity (g) and the radius of the earth (R) were known but not G. This was first determined by Cavendish and so Cavendish is often said to have weighed the earth. The word 'gravity' must not be confused with 'gravitation'. Gravity is the force of attraction between the earth and the material bodies on or near the surface of the earth. Thus 'gravity' is a particular case of gravitation when one of the attracting bodies is the earth and the other body is near the surface of the earth. When a body is taken to a small height and allowed to go, it moves toward the earth due to 'gravity' with uniform acceleration. This acceleration is called, acceleration due to gravity. It is denoted by g.

If M be the mass of the earth and R, its radius, the 'gravity i.e., force of attraction' on a body of mass m is

$$F = \frac{GmM}{R^2} = \text{mass} \times \text{acceleration} = mg$$

$$\therefore \quad g = \frac{GM}{R^2} \qquad .. \ (9.2),$$

or

$$M = \frac{gR^2}{G}$$

Knowing G, g and R, the mass (M) of the earth is calculated from this formula.

If the earth be a homogeneous sphere, its volume

$$V = \frac{4\pi}{3} R^3$$

$$\therefore \quad M = \frac{4\pi}{3} R^3 \varrho = \frac{gR^2}{G}$$

or

$$\varrho = \frac{3g}{4\pi RG}$$

This equation gives the mean density of the earth.

9.5. Gravitational Field and Potential

Gravitational Field. The word 'field' is a new concept to bring basic facts in science within our perception. It is a fact that two particles attract each other. How? We do not know. A charged body attracts or repels another charged body. How? Again we do not know. But it is a fact that such force does exist between two charged bodies. A moving charged body exerts a force on another charged body in motion. This is a third basic fact in science. These basic facts are beyond human perception. To bring these facts within human perception it is said that a *body modifies the space around it in some way due to various 'possessions'—mass, charge, charge + motion etc. and exerts force on other bodies. This is called the field of the body.* The field concept, therefore, plays an intermediate role in our thinking about the forces between material bodies on account of their various 'possessions'. The field set by a body in the space surrounding it due to possession of 'mass' is called gravitational field. If a particle of mass Δm placed at a point in the gravitational field experiences force $\Delta\vec{F}$, the intensity of the gravitational field at that point is

$$\vec{g}=\frac{\Delta\vec{F}}{\Delta m} \qquad \text{.. (9.3).}$$

If we put $\Delta m=1$, $\vec{g}=\Delta\vec{F}$

Thus *intensity of gravitational field at a point is the force experienced by unit mass placed at that point.* Its unit is newton per kilogramme (Nkg^{-1}).

Gravitational Potential. Consider an element of mass Δm placed in a gravitational field of intensity g. The mass will experience a gravitational force of magnitude Δmg. Due to the action of this force the element of mass will move in the direction of the field with acceleration. In order to hold the mass in the field an external agent must apply a force equal and opposite to the gravitational force. Suppose you are yourself that

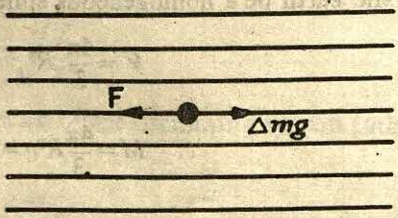


Fig. 9.6

external agent. If you move the mass through a small distance in the direction of the force applied by you (the external agent), work will be done by you against the gravitational force. In this case energy will flow from your body to the mass. You lose energy and the mass gains that energy. The work done is thus associated with transfer of energy from one body to another. This energy transferred from your body (the external agent) to the element of mass in bringing it from infinity is stored as energy of the mass and is called the gravitational potential energy of the mass. Conventionally when there is inflow of energy from the external agent to the mass, the potential energy is taken as positive. Or we may say the work done against the gravitational force which is same as the work done by the external agent is conventionally taken as positive work. This work done by the external agent against the gravitational force per unit mass from infinity up to a point in a gravitational field is defined as gravitational potential.

If $\triangle W$ is the work done by the external agent against the gravitational force in bringing $\triangle m$ mass from infinity up to a point P, then potential (V) of the point is defined as

$$V=\frac{\triangle W_{\infty\to P}(\text{external agent})}{\triangle m} \quad .. \ (9.4).$$

If we put $\triangle m=1$, then $V=\triangle W_{\infty\to P}$. Thus *potential at a point may be defined as the work done by the external agent in bringing unit mass from infinity up to that point.*

Its unit is joule per kilogramme (Jkg^{-1}).

9.6. Relation of Gravitational Field and Potential of a Point

Suppose P is a point at a distance r from an arbitrary origin where the gravitational field and potential are g and V respectively. Let Q be a point close to P at a distance $r+\triangle r$ and the field and potential at Q are $g+\triangle g$ and $V+\triangle V$ respectively. Potential difference between P and

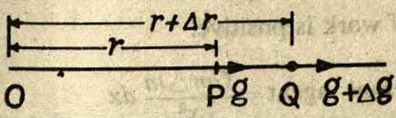


Fig. 9.7

$$\begin{aligned} Q &= V-(V+\triangle V) \\ &= -\triangle V \\ &= \text{Work done by external} \end{aligned}$$

agent from Q to $P=\triangle W_{Q\to P}$

The external agent must apply a force opposite to the gravitational force. Assuming the force at P to be uniform over PQ, we have

$$-\triangle V \simeq g \triangle r \quad \text{or} \quad g \simeq -\frac{\triangle V}{\triangle r}$$

This is approximately equal because, in fact, g is not uniform over PQ. However, it is to be admitted that this approximate relation will tend to the precise relation as $\triangle r$ tends to zero.

$$\therefore \quad g = -\lim_{\triangle r \to 0} \frac{\triangle V}{\triangle r} = -\frac{dV}{dr}$$

or $$g = -\frac{dV}{dr} \qquad \text{.. (9.5).}$$

9.7. Gravitational Potential at a Point Due to a Point Mass

Consider a particle of mass m. P is a point at a distance r from it. Consider any point X at a distance x on the line joining m with P.

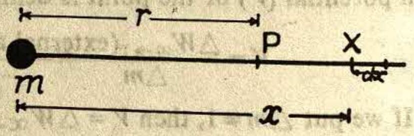


Fig. 9·8

The gravitational force on $\triangle m$ at $X = \dfrac{Gm\triangle m}{x^2}$

This force is to the left because m attracts $\triangle m$. The external agent must apply a force on $\triangle m$ to the right.

Let the mass $\triangle m$ at X be moved through dx away from m. Work will be done by the external force as the displacement dx is in its direction and hence this small bit of work is positive.

$\therefore$ The work done by the external agent $= \dfrac{Gm\triangle m}{x^2}\, dx$

$\triangle W_{P\to\infty}$(work done from P to ∞) by the external agent

$$= \int_r^{\infty} \frac{Gm\triangle m}{x^2}\, dx$$

$$= Gm\triangle m\left[-\frac{1}{x}\right]_r^\infty$$

$$= \frac{Gm\triangle m}{r}$$

$$\therefore \quad \triangle W_{\infty\to P} = -\frac{Gm\triangle m}{r}$$

$(\because \quad \triangle W_{P\to\infty}(\text{agent}) = -\triangle W_{\infty\to P}\ (\text{agent})$

By definition, the potential at P is

$$V = \frac{\triangle W_{\infty\to P}\ (\text{agent})}{\triangle m}$$

$$\therefore \quad V = -\frac{Gm\triangle m}{r\triangle m} = -\frac{Gm}{r}$$

$$\therefore \quad V = -G\frac{m}{r} \qquad \text{.. (9.6).}$$

9.8. Potential and Field at a Point Due to a Spherical Shell

Consider a spherical shell of mass M and radius a. P is a point at a distance r from the centre O of the shell.

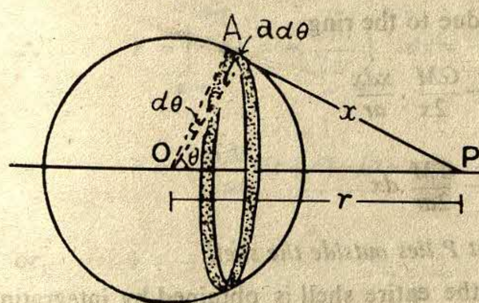


Fig. 9.9

Consider a ring at right angles to OP. Let θ be the angular position of the ring from the line OP.

The radius of the ring $= a\sin\theta$ and its width $= ad\theta$

The surface area of the ring $= (2\pi a \sin\theta)\, ad\theta$

$$= 2\pi a^2 \sin\theta\, d\theta$$

The mass of the ring $= (2\pi a^2 \sin\theta\, d\theta) \times$ mass of the shell per unit area of it

$$= 2\pi a^2 \sin\theta\ d\theta \times \frac{M}{4\pi a^2}$$

$$=\frac{M}{2}\sin\theta\, d\theta$$

If x represents the distance of P from a point of the ring and $\triangle m$ is an elementary mass of the ring, the potential at P due to the ring is

$$dV=-\sum\frac{G\triangle m}{x}=-\frac{G}{x}\sum\triangle m=-\frac{G}{x}\times\text{mass of the ring}$$

$$=-\frac{G}{x}\;\frac{M}{2}\sin\theta\, d\theta$$

$$=-\frac{GM}{2x}\sin\theta\, d\theta$$

From the 'cosine-property' of the triangle OAP

$$x^2=a^2+r^2-2ar\cos\theta$$

Differentiating,

$$2x\,dx=2ar\sin\theta\, d\theta$$

$$\sin\theta\, d\theta=\frac{xdx}{ar}$$

$\therefore$ The potential at P due to the ring

$$dV=-\frac{GM}{2x}\cdot\frac{xdx}{ar}$$

$$=-\frac{GM}{2ar}.dx$$

Case I. When the point P lies outside the shell.

The potential at P to the entire shell is obtained by integrating the potential due to the ring from

$$x=r-a \text{ to } x=r+a$$

$$\therefore\qquad V=\int_{r-a}^{r+a}dV=-\frac{GM}{2ar}\int_{r-a}^{r+a}dx=-\frac{GM}{2ar}\Big[x\Big]_{r-a}^{r+a}$$

or, $$V=-\frac{GM}{2ar}[(r+a)-(r-a)]$$

$$= -\frac{GM}{2ar}.2a = -\frac{GM}{r}$$

or $$V = -\frac{GM}{r}. \quad \text{.. (9.7)}$$

This is the potential at P due to a point mass M at O. Thus *it is proved that for points lying outside the whole mass of the shell may be supposed to be concentrated at its centre so far as its potential is concerned.*

Field outside the shell :

$$g = -\frac{dV}{dr} = -\frac{d}{dr}\left(-\frac{GM}{r}\right) = -\frac{GM}{r^2}$$

$$g = -\frac{GM}{r^2}. \quad \text{.. (9.18)}$$

Case II. When the point P lies inside :

The potential at P due to the entire shell is obtained by integrating the potential due to the ring from $x=a-r$ to $x=a+r$

$$\therefore \quad V = \int_{a-r}^{a+r} dV = -\frac{GM}{2ar}\int_{a-r}^{a+r} dx = -\frac{GM}{2ar}\Big[x\Big]_{a-r}^{a+r}$$

or $$V = -\frac{GM}{2ar}[(r+a)-(a-r)] = -\frac{GM}{2ar}, 2r$$

or, $$V = -\frac{GM}{a}$$

This expression is independent of r.

Thus *the potential at every point inside the shell is the same* and *it is equal to the potential of the surface of the shell.*

Field inside the shell.

$$g = -\frac{dV}{dr} = -\frac{d}{dr}\left(-\frac{GM}{a}\right) = -\frac{d}{dr}(\text{a constant}) = 0$$

Thus *the gravitational field inside a shell is zero everywhere.*

9.9. Gravitational Potential and Field due to a Homogeneous Solid Sphere

Consider a homogeneous solid sphere of mass M and radius a. P is a point at a distance r from the centre of the sphere.

Case I. When the point P is external to the sphere.

The solid sphere may be supposed to be divided into thin spherical shells. Consider one such shell of mass $\triangle m$.

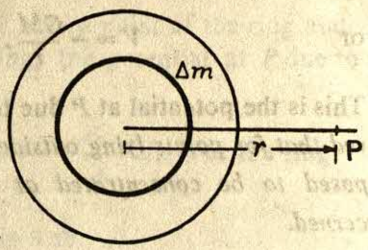


Fig. 9.10

Since the point P lies out the sphere, it must also lie outside the shell,

$\therefore$ the potential at P due to the shell $= -\dfrac{G\triangle m}{r}$

$\therefore$ the potential at P due to the entire sphere

$$V = -\sum \frac{G\triangle m}{r} = -\frac{G}{r}\sum \triangle m = -\frac{GM}{r} \quad \therefore (9.9)$$

$$(\because \; M = \sum \triangle m)$$

Thus for points external to the sphere whole mass of the sphere may be supposed to be concentrated at its centre.

The gravitational field.

$$g = -\frac{dV}{dr} = -\frac{d}{dr}\left(-\frac{GM}{r}\right) = -\frac{GM}{r^2} \quad .. (9.10)$$

Case II. When the point P is inside the material of the sphere.

Imagine a concentric spherical surface through P. The potential at P arises out of the inner sphere and the outer thick spherical shell.

Let $V = V_1 + V_2$, where V_1 = potential due to the inner sphere and V_2 = potential due to the outer thick shell.

The mass of the inner sphere $= \dfrac{4\pi r^3}{3}\varrho$ where ϱ = density of the

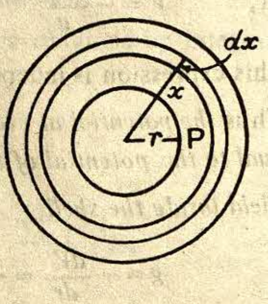


Fig. 9.11

$$\text{sphere} = \frac{M}{\frac{4\pi a^3}{3}} = \frac{3M}{4\pi a^3}$$

The potential at P due to this sphere

$$V_1 = -\frac{G\left(\frac{4\pi r^3}{3}\varrho\right)}{r} = -\frac{4\pi}{3}G\varrho r^2 \quad \text{.. (i)}$$

To find V_2, consider a thin concentric shell of radius x and thickness dx.

The volume of the shell $= 4\pi x^2 dx$

and its mass $= 4\pi x^2 dx \varrho$

The potential at P due to the shell

= self potential of the shell

($\because$ P lies inside the shell)

$$= -\frac{G(4\pi x^2 dx\varrho)}{x}$$

$$= -4\pi G\varrho x dx$$

$\therefore$ the potential at P due to the outer thick shell

$$= -\int_{x=r}^{x=a} 4\pi G\varrho x dx$$

$$= -4\pi G\varrho\left[\frac{x^2}{2}\right]_{x=r}^{x=a} = -4\pi G\varrho\left[\frac{a^2}{2} - \frac{r^2}{2}\right]$$

$$= -2\pi G\varrho(a^2 - r^2) \quad \text{.. (ii)}$$

$$\therefore \quad V = V_1 + V_2 = -\frac{4\pi}{3}G\varrho r^2 - 2\pi G\varrho(a^2 - r^2)$$

$$= -\frac{4\pi}{3}G\varrho\left(r^2 + \frac{3}{2}.a^2 - \frac{3}{2}r^2\right)$$

$$=-\frac{4\pi}{3}G\rho\left(\frac{3}{2}a^2-\frac{r^2}{2}\right)$$

$$=-\frac{4\pi}{3}G.\frac{3M}{4\pi a^3}\left(\frac{3a^2-r^2}{2}\right)$$

or $$V=-\frac{GM}{2a^3}\,(3a^2-r^2) \qquad .. \quad 9.11$$

The gravitational field inside a sphere.

$$g=-\frac{dV}{dr}=-\frac{d}{dr}\left\{-\frac{GM}{2a^3}\,(3a^2-r^2)\right\}$$

$$=\frac{GM}{2a^3}\,\frac{d}{dr}\,(3a^2-r^2)$$

$$=-\frac{GM}{2a^3}\,.\,2r=-\frac{GM}{a^3}\,.\,r.$$

$$g=-\frac{GM}{a^3}\,r. \qquad .. \quad 9.12$$

9.10. Variation in the Acceleration Due to Gravity with Distance From the Centre of the Earth.

If the earth is supposed to be a homogeneous solid sphere, the intensity of gravitational field inside the earth will change according to the Eq. 9.12 and outside the earth according to the Eq. 9.10. Graphically the variation of g with distance is shown in the Fig 9.12. The acceleration due to gravity is zero at the centre of the earth. It increases linearly with distance up to the surface of the earth and then decreases rapidly and again becomes zero at infinity. Note that the acceleration due to gravity is maximum on the surface of the earth.

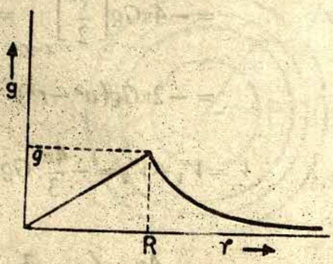


Fig. 9.12

9.11. Escape Velocity

Every body on the surface of the earth possesses gravitational potential energy which is equal to the work done in bringing it from infinity up to the surface. The potential of the surface of the earth is the same as the work done per unit mass.

If V is the potential of the surface of the earth, then the potential energy per unit mass of every body is V joule

$\therefore$ Gravitational potential energy of a body of mass $m = mV$

$$= m\frac{GM}{R} \quad (\because V = \frac{GM}{R} \quad \text{numerically})$$

If we could give a projectile energy by an amount more than this at the surface of the earth, then, neglecting the resistance of the earth's atmosphere, it would escape from the earth never to return. The velocity required to impart this minimum energy is called escape velocity. Let v be the escape velocity.

Then $$\frac{1}{2}mv^2 = m\frac{GM}{R}$$

or $$v = \sqrt{\frac{2GM}{R}} \quad .. \quad 9.13$$

$\because$ $$g = \frac{GM}{R^2}.$$

$\therefore$ $$v = \sqrt{2gR} \quad .. \quad 9.13a$$

The molecules of a gas may possess such tremendous speed to escape. So it is believed that gaseous molecules, specially the lighter ones, are slowly escaping into the outer space. Hydrogen gas, which might have been present in the earth's atmosphere a long time ago, has now disappeared from it. The escape velocity for the sun is so great that none of the gas molecules can escape from it. On the other hand the escape velocity for the moon is too small to keep any atmosphere at all.

9.12. Inertial and Gravitational Mass

The mass occuring in the law of motion $F = ma$ is called the

inertial mass. The mass occuring in the law of gravitation $F=\frac{Gmm'}{d^2}$ is called the gravitational mass of the body. Are the gravitational mass and the inertial mass of a body really the same ?

Let us consider two particles A and B of gravitational masses m'_A and m'_B and inertial masses m_A and m_B respectively. Let us first consider their gravitational interaction with a third particle of gravitational mass m'_C. Then

$$F_{AC}=G\frac{m'_A m'_C}{d^2} \text{ and } F_{BC}=\frac{Gm'_B m'_C}{d^2}$$

$$\therefore \quad \frac{F_{AC}}{F_{BC}}=\frac{m'_A}{m'_B}$$

Now suppose that the third body C is the earth. Then F_{AC} and F_{BC} are the weights of the bodies. By applying law of motion we have,

$$F_{AC}=m_A g \text{ and } F_{BC}=m_B g.$$

$$\therefore \quad \frac{F_{AC}}{F_{BC}}=\frac{m_A}{m_B}$$

$$\therefore \quad \frac{m'_A}{m'_B}=\frac{m_A}{m_B}. \text{ or } \frac{m'_A}{m_A}=\frac{m'_B}{m_B}$$

Thus inertial mass and gravitational mass are proportional to each other. When we consider the period of oscillation of a simple pendulum it is given by

$T=2\pi\sqrt{\frac{ml}{m'g}}$ where m and m' are respectively the inertial and gravitational masses of the bob of the pendulum. (See Q. 5, Gr. C, Ch. 10)

But experimentally, $T=2\pi\sqrt{\frac{l}{g}}$

$$\therefore \quad m=m'$$

i.e. inertial mass of a body = its gravitational mass.

9.13. Satellites : Orbital Velocity

Satellites are heavenly bodies revolving continuously round some planets of the solar system. The moon is a natural satellite of the earth. Saturn is surrounded by a cluster of ten satellites. Till recently, there were no artificial satellites. In 1957 Russian scientists and engineers struck the whole world with wonder and surprise by launching their first artificial satellite Sputnik *I*. This was about 83.6 kg in weight and 58 cm in diameter. It orbited the earth once in every 96·3 minutes at a height of 950 km above the earth's surface. This was made possible due to developments of high speed rockets. Now a days many countries have launched artificial satellites. India has Aryabhatta, Rohini, Apple and Insat-1 as artificial satellites to her credit.

Condition for a satellite launching. The idea of artificial satellite has been derived from the natural satellites such as the moon round the earth. The gravitational pull provides the necessary centripetal force and due to that the moon maintains her orbital motion round the earth. Exactly in the same way if a heavy body is taken by some means to a certain height above the earth and then thrown parallel to the surface of the earth with the specific speed which the gravitational pull at that height can possibly maintain, then the body must go round the earth, uninterrupted provided there is no friction to the motion by the atmosphere there. This velocity is *orbital velocity*.

If m is the mass of the satellite, M is the mass of the earth, R, its radius and v, the velocity of the satellite, we have centripetal acceleration of the satellite

$= \frac{v^2}{(R+h)}$ where h is the height of the satellite above the earth.

Gravitational pull on the satellite $= \frac{G\,Mm}{(R+h)^2}$.

$$\therefore \quad \frac{GM\,m}{(R+h)^2} = m\frac{v^2}{R+h} \quad \text{(Force = mass} \times \text{acceleration)}$$

or

$$v = \sqrt{\left(\frac{GM}{R+h}\right)}$$

If g is the acceleration due to gravity at the surface of the earth

then $$g=\frac{GM}{R^2}$$

$$\therefore \qquad v=\sqrt{\frac{gR^2}{R+h}}$$

Now suppose that we want to launch a satellite at a height of 1000 km above the surface of the earth. The velocity at which the satellite is to be launched is

$$v=\sqrt{\frac{9\cdot8\times(6\cdot4\times10^6)^2}{6\cdot4\times10^6+10^6}} \quad (\because \quad R=6\cdot4\times10^6 m)$$

$$=6\cdot4\sqrt{\frac{9\ 8}{7\cdot4}}\times10^3 ms^{-1}$$

$$=7\cdot37\times10^3 ms^{-1}=7\cdot37\ \text{kms}^{-1}$$

Now, the question is how to hurl the satellite with such a high velocity. This formidable problem is, however, solved by carrying the satellite on a multi-stage rocket, for no single rocket can possibly achieve the requisite velocity all by itself.

9.14. Weightlessness

The gravitational pull due to the earth toward its centre is the weight of a body. The pull is zero at the centre of the earth and at infinite distanee from surface of the earth. In a manned artificial satellite the cosmonaut feels weightlessness. In an artificial satellite the whole of the gravitational pull provides the necessary centripetal force and no part of the gravitational pull remains unbalanced. So a body is 'weightless' in an artificial satellite.

9.15. The Motions of Planets

Johaness Kepler studied the motions of planets under the guidance of Tycho Brahe and found certain regularities in the motion of the planets. These regularities are known as Kepler's laws of planetary motion.

Law I. All planets move in elliptical orbits having the sun as one focus.

Law II. The line joining the planet to the sun sweeps out equal areas in equal times.

Law III. The period of any planet about the sun is proportional to the ($\frac{3}{2}$) power of the semi-major axis of the planet's orbit.

These laws can be deduced from the laws of motion and the law of gravitation.

Let us consider a planet of mass m moving round the sun of mass M. Let the distance between the sun and the planet be r and for simplicity's sake, let us assume a circular orbit for the planet. A circle is also an ellipse of equal major and minor axes

$$\text{Centripetal force on the planet} = \frac{G\,m\,M}{r^2} = m\omega^2 r$$

where ω is the angular speed of the planet.

If T is the period of the planet, $\omega = \dfrac{2\pi}{T}$

$$\therefore \qquad \frac{G\,m\,M}{r^2} = m\left(\frac{2\pi}{T}\right)^2 r.$$

$$\text{or} \qquad T^2 = \left(\frac{4\pi^2}{GM}\right) r^3 \text{ or } T \propto r^{\frac{3}{2}}$$

Examples

1. *If* $G = 6{\cdot}7 \times 10^{-11}$ *SI units, the radius of the earth* $= 6400$ *km and* $g = 9{\cdot}8$ ms^{-2}, *calculate the mean density of the earth.*

Sol $g = \dfrac{GM}{R^2} = \dfrac{G}{R^2} \cdot \dfrac{4\pi}{3} R^3 \rho = \dfrac{4\pi}{3} GR\rho$

or $\rho = \dfrac{3g}{4\pi\,GR} = \dfrac{3 \times 9{\cdot}8}{4\pi \times 6{\cdot}7 \times 10^{-11} \times 6400 \times 10^3} = 5456$ kgm^{-3} **Ans.**

2. *The radius of the earth is* 6370 *km, its mean density* 5500 *kgm*$^{-3}$ *and the gravitational constant* $6{\cdot}66 \times 10^{-11}$ *SI units. Calculate the earth's potential and escape velocity.*

Sol. We know that, $V = \dfrac{GM}{R} = \dfrac{G}{R} \cdot \dfrac{4\pi}{3} R^3 \rho$

or $$V=\frac{4\pi}{3}GR^2\rho$$

$$=\frac{4\pi}{3}\times 6\cdot 6\times 10^{-11}\times(6370\times 10^3)^2\times 5500$$

$$=6\cdot 2\times 10^7 \text{ joule per kg } (\text{Jkg}^{-1}). \textbf{ Ans.}$$

$$v=\sqrt{\frac{2GM}{R}}=\sqrt{\frac{2G}{R}\quad\frac{4\pi}{3}R^3\rho}=\sqrt{\frac{8\pi GR^2\rho}{3}}$$

or $$v=R\sqrt{\frac{8\pi G\rho}{3}}$$

$$=6370\times 10^3\sqrt{\frac{8\pi\times 6\cdot 66\times 10^{-11}\times 5500}{3}}$$

$$=637\times 10^4\times 10^{-4}\sqrt{\frac{8\pi\times 6\cdot 66\times 55}{30}}$$

$$=637\sqrt{\frac{8\pi\times 6\cdot 66\times 55}{30}}$$

$$=1\cdot 11\times 10^4 \text{ms}^{-1} \textbf{ Ans.}$$

3. *Suppose that a tunnel is dug through the earth from one side to the other along a diameter and a stone is dropped into the tunnel. Show that the motion of the stone is simple harmonic and calculate the period of the motion.* $G=6\cdot 6\times 10^{-11}$ *SI units, density of the earth* $=5500\ kgm^{-3}$.

Sol. Let x be the distance of the stone from the centre of the earth at any instant t.

the instantaneous force on the stone $=-\dfrac{G\left(\frac{4\pi}{3}x^3\rho\right)m}{x^2}$

(attraction is to be considered due to inner mass only)

If f is the instantaneous acceleration then

$$-\frac{G4\pi}{3}x\,\rho m=mf$$

or $$f=-\left(\frac{4\pi}{3}G\rho\right)x.$$

Thus $f\propto -x$. Hence the motion is simple harmonic.

In a simple harmonic motion $f=-\omega^2 x$ where ω is the cyclic frequency of the motion.

$\therefore$ the cyclic frequency of the stone $=\sqrt{\frac{4\pi}{3}G\rho}$

$$\text{or} \quad T=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{3}{4\pi G\rho}}=\sqrt{\frac{3\pi}{G\rho}}=\sqrt{\frac{3\pi}{6\cdot6\times10^{-11}\times5500}}$$

$= 84.9$ minute **Ans.**

4. *If the gravitational constant* $G=6\cdot7\times10^{-11}$ *SI units and the radius of the earth is* $6\cdot4\times10^6$ *m, calculate the mass of the earth if the acceleration due to gravity on the surface of the earth is* $9\cdot8 ms^{-2}$.

Sol. We have $mg=\frac{GM\,m}{R^2}$ where m is the mass of any body on or near the surface of the earth

$$\text{or} \quad g=\frac{GM}{R^2}$$

$$M=\frac{gR^2}{G}=\frac{9\cdot8\times6\cdot4^2\times10^{12}}{6\cdot7\times10^{-11}}=6\times10^{24}\ kg. \quad \textbf{Ans.}$$

5. *The mass of the earth is 80 times that of the moon and its radius 4 times the radius of the moon. Calculate the acceleration due to gravity of the moon's attraction; given that* $g=9\cdot8 ms^{-2}$ *on the earth.*

Sol. We have $g_e=\frac{GM_e}{R^2{}_e}$; $g_m=\frac{GM_m}{R^2{}_m}$.

$$\therefore \quad \frac{g_m}{g_e}=\frac{M_m}{M_e}\times\frac{R^2{}_e}{R^2{}_m} \quad \text{or } g_m=\frac{M_m}{80M_m}\times\frac{(4R_m)^2}{R^2{}_m}\times9\cdot8=\frac{9\cdot8}{5}$$

$=1\cdot96 ms^{-2}$ **Ans.**

6. *The Indian satellite Aryabhatta is revolving round the earth at a height of 650 km and completes one revolution in 96 minutes. If the earth's radius be* $6\cdot4\times10^6$ *m, find the acceleration of Aryabhatta towards the centre of the earth.*

Sol. We have mg' (centripetal force on the satellite) $=m\omega^2(R+h)$

$$\text{or} \quad g'=\left(\frac{2\pi}{T}\right)^2(R+h)=\frac{4\pi^2}{96^2\times60^2}\times(6\cdot4\times10^6+650\times10^3)$$

or $g' = \dfrac{4\pi^2}{96^2 \times 60^2} \times 7{\cdot}05 \times 10^6 = 8{\cdot}3 ms^{-2}$ **Ans.**

QUESTIONS

(A)

1. The dimensions of G are
(a) ML^3T^{-2}, (b) $M^2T^3L^{-2}$, (c) $ML^{-3}T^2$, (d) $M^{-1}L^3T^{-2}$.

2. The dimensions of G are
(a) FL^3T^{-3}, (b) $F^{-1}L^4T^{-4}$, (c) FL^4T^{-4}, (d) $F^{-1}L^{-4}T^4$.
when F (force), L (length) and T (time) are the base units.

3. The escape velocity for a planet of mass M and radius R is
(a) $\sqrt{\dfrac{2GM}{R}}$, (b) $\sqrt{\dfrac{2GR}{M}}$, (c) $\sqrt{2GMR}$, (d) $\sqrt{\dfrac{GM}{R}}$.

4. For 'weightlessness' at a height h, a body must revolve at a speed
(a) $\sqrt{\dfrac{gR^2}{R+h}}$, (b) $\sqrt{\dfrac{Rg}{R+h}}$, (c) $\sqrt{\dfrac{g}{R+h}}$, (d) $\sqrt{\dfrac{R}{R+h}}$.

5. The gravitational potential inside a spherical shell of mass M and radius a is
(a) $\dfrac{-GM}{a}$, (b) zero, (c) $\dfrac{-GM}{r^2}$, where r is the distance of the point from the centre of the shell, (d) $\dfrac{-GM}{a^2}$.

6. At the centre of the earth, (a) both field and potential are zero, (b) potential is zero but not the field, (c) potential is not zero but the field is zero, (d) both are finite.

7. At the centre of a spherical shell (a) $g=0$, $V=\dfrac{-GM}{a}$, (b) $g=-\dfrac{GM}{a}$, $V=0$, (c) $g=0$, $V=0$, (d) $g=-\dfrac{GM}{a^2}$, $V=-\dfrac{GM}{a}$.

8. In the expression $F=G\dfrac{mm'}{r^2}$ the quantity G
(a) a constant varying from planet to planet
(b) a universal constant
(c) a constant depending on the medium between the masses
(d) a constant for a given place.

9. The acceleration due to gravity at a height h is
(a) $g \cdot \dfrac{h}{R}$, (b) $g \cdot \dfrac{R^2}{(R+h)^2}$, (c) $g \cdot \dfrac{R}{R+h}$, (d) $g\ \dfrac{R+h}{R}$

10. The acceleration due to gravity at a depth h is

(a) $g \cdot \frac{h}{R}$, (b) $g \cdot \frac{R-h}{R}$, (c) $g \cdot \left(\frac{R-h}{R}\right)^2$, (d) $g \cdot \frac{R}{R-h}$

(Ans : **1.** d, **2.** b, **3.** a, **4.** a, **5.** a, **6.** c, **7.** a, **8.** b, **9.** b, **10.** b.)

(B)

1. Point out the defects of Cavendish's method.

2. 'Cavendish weighed the earth'. Justify.

3. Distinguish between inertial mass and gravitational mass ? Is gravitational mass same as inertial mass ?

4. What is escape velocity ? Calculate the escape velocity for a planet of mass M and radius R.

5. The moon has no atmosphere. Justify.

6. What is a satellite ? Calculate the launching speed of a satellite at a height h from the surface of the earth.

7. Explain 'weightlessness'.

(C)

1. State the law of Universal Gravitation and describe Cavendish's method of determining the Gravitation Constant. **(Ran. 1971, '75; Pat. '76; Bhag. '73; Mag. '73)**

2. State and explain Newton's law of gravitation and describe an accurate method of measuring the gravitation constant. **(Ran. 1970 S)**

[**Hint.** Boy's method]

3. Define gravitational potential and gravitational intensity. What is the relation between gravitational potential and intensity. **(Ran. 1976, '74)**

4. Deduce expression for gravitational potential and field at a point due to a spherical shell. **(Ran. 1974, '78)**

5. Distinguish between 'Gravitation' and 'gravity'. How are they related to each other. **(Ran. 1973; Mag. '77; Bih. '78)**

6. Deduce expressions for the gravitational potential and field due to a solid sphere.

7. Explain how the value of acceleration due to gravity g is affected by the rotation of the earth. **(Mith. 1977)**

(D)

1. Given that $G = 6{\cdot}7 \times 10^{-11}$ SI units and the radius of the earth $= 6{\cdot}4 \times 10^3$ km and its mean density 5500 kgm^{-3}, calculate the acceleration due to gravity at the earth's surface. **(Ans. 9·883ms^{-2})**

2. How far from the earth must a body be along a line toward the sun so that the sun's gravitational pull balances the earth's ? The sun is $1{\cdot}5 \times 10^8$ km away

and its mass $3{\cdot}24\times10^5$ times the mass of the earth. (Radius of *the* earth = 6400km)

(Ans. $2{\cdot}63\times10^5$ km)

3. Masses of 200gm and 800gm are 10cm apart. How much work is needed to move unit mass from the centre to a point 8 cm from either mass. $G=6{\cdot}67\times10^{-11}$ SI units.

[Hint. Work done per unit mass = potential difference (Ans. 5×10^{-10} joule]

4. With what horizontal speed should a satellite be projected at 160 km above the surface of the earth so that it will have a circular orbit about the earth? Take the radius of the earth 6400 km. What will be the period of revolution?

(Ans. 7·9 kms^{-1}; 87 minutes)

5. The force of attraction between two solids, one being three times heavier than the other, is ·2 mgm weight when they are 10 cm apart. Calculate the mass of each solid. ($G=6{\cdot}66\times10^{-11}$ SI units and $g=9{\cdot}8ms^{-2}$)

(Ans. 9·9 kg; 29·7 kg).

6. Assuming that the earth moves in a circular orbit of radius $1{\cdot}5\times10^{11}$ m with a period of 365 days per revolution, find the mass of the sun. $G=6{\cdot}7\times10^{-11}$ $Nm^2\ kg^{-2}$.

(Ans. $1{\cdot}99\times10^{30}$ kg.)

7. Assuming that the moon describes a circular orbit of radius $3{\cdot}84\times10^8$ m in 27·3 days and the outer satellite of Mars describes a circular orbit of radius $2{\cdot}35\times10^7$ m in 1·26 days, find the ratio of the mass of the earth to that of Mars.

(Ans. 9·16 : 1)

(E)

1. Would we have more sugar to the kilogramme at the pole or the equator? (dealing is by spring balance)

2. How does the weight of a body vary en route from the equator to the pole?

3. How does the weight of the body change en route from the earth to the moon? Would mass change?

4. One clock is based on an oscillatory spring, the other on pendulum. Both are taken to the moon. Will they keep the same time there?

5. The gravitational potential inside a hollow spherical shell is the same at all points and is equal to the potential of the surface. True or false?

6. Inside a space ship what difficulties would you encounter in walking? In jumping? In drinking?

7. The gravitational field inside a solid sphere is a constant. True or false?

8. It is said that the source of the Mississipi is nearer to the centre of the earth than its mouth. How can the river flow uphill?

(Ans. 1. at the equator. 2. increases. 3. decreases, becomes zero and again slightly increases; No. 4. No. The spring clock will keep the same time that it kept on the earth but the pendulum clock will go slow. 5. True. 6. In drinking. 7. False. 8. Water flows from higher potential to lower potential. The source though nearer to the centre is at higher potential than the mouth and so the river can flow uphill.)

□□□

CHAPTER 10

THE ACCELERATION DUE TO GRAVITY AND THE PENDULUMS : SIMPLE AND COMPOUND

10.1. The Acceleration due to Gravity

The acceleration due to gravity is the acceleration produced in any body moving under the earth's attractive force. This acceleration is denoted by g and this must not be confused with the universal constant G. The acceleration due to gravity has dimensions LT^{-2} and is a vector and the constant G has the dimensions $ML^{-3}T^{2}$ and is a scalar; where as G is a universal constant, g is not a constant. It varies from place to place due to two causes :

(*i*) *due to the spheroidal shape of the earth*. The equatorial diameter is larger than the polar diameter and consequently the acceleration due to gravity in the equatorial region is less than that in the polar region.

(*ii*) *due to the rotation of the earth about its axis*. Bodies at different latitudes of the earth move in circles of different radii with the same angular speed. This causes an increase of the centrifugal force near the equator than near the polar region. See Art. 6.7 for the deduction of the variation of apparent weight with latitude due to the rotation of the earth about its axis.

$$\text{Apparent weight} = mg - m\omega^2 r\cos^2\lambda.$$

If g' is the acceleration due to gravity at latitude λ then the apparent weight of the body $= mg'$

$\therefore\ mg' = mg - m\omega^2 r\cos^2\lambda$ where r is the radius of the earth and ω is the angular speed of rotation of the earth.

or, $$g' = g - \omega^2 r\cos^2\lambda \qquad \text{.. 10.1}$$

10.2. Relation of g and G

If the earth is supposed to be a homogeneous solid sphere, a simple relation between g and G can be establised.

Consider a body of inertial mass m on the surface of the earth. Let its gravitational mass be m' and that of the earth M.

The force of attraction on the body $= \frac{Gm'M}{R^2}$, where R is the radius of the earth. We know, Force = inertial mass $\times$ acceleration

$$\therefore \quad \frac{Gm'M}{R^2} = mg$$

or

$$g = \frac{Gm'M}{mR^2}$$

Since inertial mass of a body is the same as its gravitational mass,

$$\therefore \quad g = \frac{GM}{R^2} \qquad .. \quad 10.2$$

10.3. Measurement of Acceleration Due to Gravity

In laboratories the acceleration due to gravity is measured accurately by pendulums : simple and compound.

The idea of a pendulum came to the mind of Galileo for the first time. Newton made extensive use of the theory of the pendulum and established the equivalence of gravitational mass and inertial mass.

10.4. The Simple Pendulum

The simple pendulum consists of a heavy particle suspended by **a weightless, inextensible and perfectly flexible thread**. The distance between the point of support and the particle is called the length of the pendulum. An ideal simple pendulum as defined above is impossible to realise. In laboratories a small brass ball is suspended by a long thin cotton thread to construct a simple pendulum. The ball is called *the bob of the pendulum* and the distance between the point of support and the centre of gravity of the bob is called *the effective length of the pendulum.*

Let OA represent a simple pendulum of length l and particle mass m The forces acting on m at any instant are : mg, the gravitational force vertically downwards, and T, the tension in the thread.

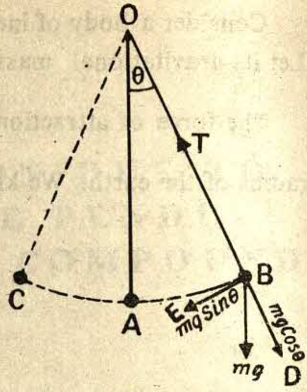


Fig. 10. 1

If the thread makes an angle θ in the anti-clockwise direction with the undisturbed position, then resolving mg along and perpendicular to the thread we have, $mg\cos\theta$ along the thread away from the point of support and $mg\sin\theta$ perpendicular to the thread toward the mean position A. As there is negligible centripetal acceleration along the thread we must have $T = mg\cos\theta$.

The other component $mg\sin\theta$ remains unbalanced. It tends to restore the particle to the equilibrium position. Hence, the restoring force is

$$F = -mg\sin\theta$$

If the angle θ is small (less than 4°), $\sin\theta$ is very nearly equal to θ in radians and the motion of the particle is a straight-line motion. If x be the displacement from the equilibrium position then $x = l\theta$

$$\therefore \qquad F = -mg\theta = -mg\frac{x}{l}$$

$$\because \qquad \text{Force} = \text{mass} \times \text{acceleration}$$

$$\therefore \qquad -\frac{mgx}{l} = mf$$

$$\text{or} \qquad f = -\frac{g}{l}x, \qquad \therefore\ f \propto -x$$

Since acceleration is proportional to displacement and is always directed to A, the motion is simple harmonic.

$$\text{Hence,} \qquad \omega^2 = \frac{g}{l}$$

$$\text{or} \qquad \omega = \sqrt{\frac{g}{l}}$$

$$\text{or} \qquad \frac{2\pi}{T} = \sqrt{\frac{g}{l}}, \qquad \text{or,} \quad T = 2\pi\sqrt{\frac{l}{g}} \qquad \text{.. 10.3}$$

ALTERNATIVE METHOD

We can deduce the above result by considering the particle of the pendulum rotating about the point of support.

The torque on the particle at any instant

$$= mg\,l\sin\theta, \text{ clockwise}$$
$$= -mg\,l\sin\theta, \text{ anticlockwise}$$

Torque = moment of inertia × angular acceleration

$\therefore\ -mg\,l\sin\theta = I\alpha$, where α is the angular acceleration.

or $$\alpha = -\frac{mgl}{I}\sin\theta = -\frac{mgl}{I}\theta, \text{ when } \theta \text{ is small.}$$

Here $I = ml^2$ $$\therefore\ \alpha = -\frac{mgl}{ml^2}\theta = -\frac{g}{l}\theta$$

Thus $\alpha \propto -\theta$. Hence motion is simple harmonic.

Here $$\omega^2 = \frac{g}{l}, \qquad \therefore\ T = 2\pi\sqrt{\frac{l}{g}}.$$

10.5. The Compound Pendulum

Any rigid body mounted so that it can swing in a vertical plane about some horizontal axis passing through any point of it is called a compound pendulum.

Consider a compound pendulum of mass m capable of vibrations about a horizontal axis perpendicular to the plane of the paper passing through O. This point is called the centre of suspension. In the equilibrium position the centre of gravity (G) of the pendulum lies vertically below the centre of suspension. Let the line OG make angle θ in the anticlockwise direction with the downward vertical line through the centre of suspension at any instant. The forces acting on it are mg, the gravitational force

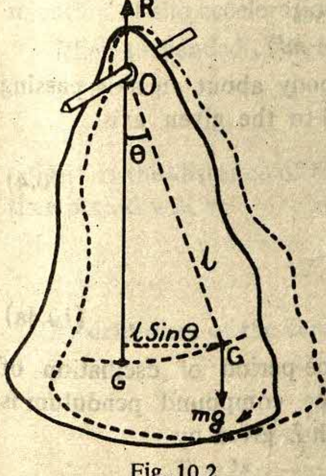


Fig. 10.2

vertically downwards, and R, the reaction of the support vertically upwards. As there is no vertical acceleration, $R=mg$. Thus the body experiences a torque only. If l is the distance of the centre of gravity from the centre of suspension, then

the torque on the body at any instant $= mg\,l \sin\theta$, clockwise
$= -mg\,l \sin\theta$, anticlockwise
$= -mg\,l\,\theta$,,
($\because$ when θ is small, $\sin\theta = \theta$).

If I is the rotational inertia of the body about the axis AB then,

$$-mg\,l\,\theta = I\alpha \qquad \text{(torque = moment of inertia × angular acceleration)}$$

or $$\alpha = -\frac{mgl}{I}\theta.$$

Thus $$\alpha \propto -\theta.$$

Hence the motion is simple harmonic.

Here $$\omega^2 = \frac{mgl}{I}$$

or $$\omega = \sqrt{\frac{mgl}{I}}$$

or $$T = 2\pi\sqrt{\frac{I}{mgl}}.$$

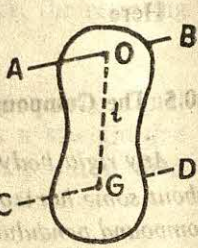


Fig. 10.3

Consider an axis CD through G parallel to AB. Then by the theorem of parallel axes

$$I = I_0 + ml^2 = mk^2 + ml^2$$

where k is the radius of gyration of the body about an axis passing through its centre of gravity and parallel to the given axis.

Then $$T = 2\pi\sqrt{\frac{k^2+l^2}{lg}} \qquad \text{.. (10.4)}$$

or $$T = 2\pi\sqrt{\frac{l+\frac{k^2}{l}}{g}}. \qquad \text{.. (10.4a)}$$

Comparing this expression with the time-period of oscillation of an ideal simple pendulum, we find that the compound pendulum is equivalent to a simple pendulum of length L given by

$$L = l + \frac{k^2}{l}. \qquad \text{.. (10.5)}$$

Such an ideal simple pendulum of which the time-period is the same as that of the given compound pendulum is called an *equivalent simple pendulum* and its length is called the *length of the equivalent-simple pendulum*.

10.6. Centre of Oscillation : Interchangeability of Centre of Suspension and Centre of Oscillation

The length of the equivalent simple pendulum is given by

$$L = l + \frac{k^2}{l}$$

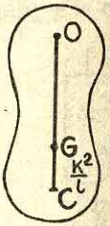


Fig. 10.4

Hence, if the line QG is produced to C such that $GC = k^2/l$ and $OC = l + k^2/l$, then OC gives the length of the equivalent simple pendulum. This point C is defined as the centre of oscillation of the pendulum. Thus *the centre of oscillation of a compound pendulum is a point at a distance k^2/l from the centre of gravity and $l + k^2/l$ from the centre of suspension i. e. at a distance equal to the length of equivalent simple pendulum from the centre of suspension.*

It can be shown that the centre of oscillation is interchangeable with the centre of suspension. This is a very important fact. It is this fact on which compound pendulums are based for the accurate determination of the acceleration due to gravity.

When suspended at O, its time of oscillation

$$T = 2\pi\sqrt{\frac{k^2/l + l}{g}}.$$

If l' is the distance of C, from G then on suspending it at C its time period will be

$$T' = 2\pi\sqrt{\frac{k^2/l' + l'}{g}}$$

By definition of the centre of oscillation

$$l' = k^2/l.$$

$$\therefore \quad T' = 2\pi\sqrt{\frac{\frac{k^2}{k^2/l} + k^2/l}{g}} = 2\pi\sqrt{\frac{l + \frac{k^2}{l}}{g}} = T.$$

10.7. The Bar Pendulum

The simplest and the most convenient form of a compound pendulum is the bar pendulum which is simply a uniform brass bar having holes along its length, symmetrically on either side of its centre of gravity, so that the pendulum may be slipped on to a horizontal knife edge on any one of the holes.

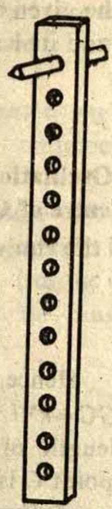


Fig. 10.5

Theory of determination. The time period of a compound pendulum is

$$T=2\pi\sqrt{\frac{\frac{k^2}{l}+l}{g}}=2\pi\sqrt{\frac{L}{g}}$$

where $L=k^2/l+l$, is called the length of the equivalent simple pendulum.

Hence $$g=\frac{4\pi^2 L}{T^2}\ \text{ms}^{-2}. \quad \text{(i)}$$

The expression for time-period of the compound pendulum is a quadratic in l and hence for a given value of T there will be two values of l which means there are two points where the pendulum has the same time-period. Every point has its own corresponding centre of oscillation. Thus on a line through the centre of gravity there are four points about which a pendulum will have the same time-period and the distance between alternate points will be the length of the equivalent simple pendulum.

$$L=k^2/l+l$$

or $$l^2-L.l+k^2=0.$$

If l_1 and l_2 are the roots of this equation then

$$l_1+l_2=L \text{ (by the theory of quadratic equation).}$$

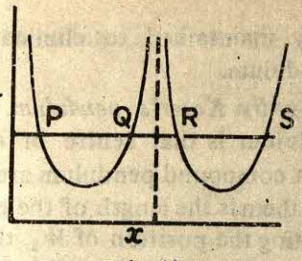


Fig. 10.6

Thus if we measure distances of holes from one end and find the time period about each hole, and finally plot a graph of time period (T) versus distance (x) of holes, the plot will be two identical parabolas as shown in the figure. Draw a horizontal line corresponding to a suitable value of T. This line will cut the plot at four points P, Q, R and S. From the graph PR and QS are estimated. Each of them is L.

$$\therefore \qquad L = \frac{PR+QS}{2} .$$

Finally g is calculated from the relation (i).

Fig. 10.7

10.8. Kater's Pendulum

Captain Kater designed in 1817 an improved form of compound pendulum and he himself used it to determine the value of g at London. This pendulum is known as Kater's pendulum. It consists of a brass rod having a heavy brass cylinder B_1 called the bob of the pendulum at one end and an exactly identical wooden cylinder B_2 at the other end. The latter is attached to maintain symmetry in the shape of the pendulum about its geometrical centre. There are two knife edges K_1 and K_2 at the two ends of the rod near the cylinder. They are parallel and they face each other. A small cylindrical mass W_1 of brass and an identical wooden cylindrical mass W_2 are also there on the rod. These masses can be slided and clamped in any position. Here also the function of the wooden mass is to maintain symmetry in the shape of the pendulum about its geometrical centre. Notice that the masses are in the order, brass-wood-brass-wood from the bob end.

The symmetry in shape is essentially maintained to eliminate the effect of air on the motion of the pendulum.

Theory of the determination of g by Kater's pendulum. The principle of working of Kater's pendulum is that centre of oscillation and the centre of suspension of a compound pendulum are interchangeable and the distance between them is the length of the equivalent simple pendulum. So if by adjusting the position of W_2, the time periods about K_1 and K_2 are made exactly equal then the distance between the knife edges is equal to the length of the equivalent simple pendulum. If this distance is L and T is the time period about either knife edge then

$$T=2\pi\sqrt{\frac{L}{g}} \text{ or } g=\frac{4\pi^2 L}{T^2}.$$

Kater took the trouble to make the two time periods exactly equal by the method of coincidence in which the oscillations of the experimental pendulum are compared with those of a standard electrical seconds pendulum.

10.9. Bessel's Contribution : Computed Time-period

The adjustment of the pendulum to exact equality of periods about the knife edges is an extremely tedious process. Bessel showed that an exact equality of the time-periods about the knife edges was not absolutely necessary and that it was enough to make them only **nearly equal.**

Suppose that t_1 and t_2 are the two nearly equal time periods about k_1 and k_2 respectively and l_1 and l_2 are their distances from the centre of gravity of the pendulum, then we have,

$$t_1=2\pi\sqrt{\frac{k^2+l_1^2}{l_1 g}}$$

or $$k^2+l_1^2=\frac{g}{4\pi^2}\cdot t_1^2 l_1 \qquad \text{.. (i)}$$

and $$t_2=2\pi\sqrt{\frac{k^2+l_2^2}{l_2 g}}$$

or $$k^2+l_2^2=\frac{g}{4\pi^2}\, t_2^2 l_2. \qquad \text{.. (ii)}$$

Subtracting (ii) from (i) to eliminate k^2 we have

$$l_1^2-l_2^2=\frac{g}{4\pi^2}\left(t_1^2l_1-t_2^2l_2\right)$$

or
$$\frac{4\pi^2}{g}=\frac{t_1^2l_1-t_2^2l_2}{(l_1-l_2)(l_1+l_2)}$$

$$=\frac{A}{l_1+l_2}+\frac{B}{l_1-l_2} \quad \text{(say)}$$

$$=\frac{l_1(A+B)-l_2(A-B)}{(l_1+l_2)(l_1-l_2)},$$

$\therefore$ $\quad A+B=t_1^2$

and $\quad A-B=t_2^2,$

whence $\quad A=\dfrac{t_1^2+t_2^2}{2}$ and $B=\dfrac{t_1^2-t_2^2}{2},$

$$\therefore \quad \frac{4\pi^2}{g}=\frac{t_1^2+t_2^2}{2(l_1+l_2)}+\frac{t_1^2-t_2^2}{2(l_1-l_2)}.$$

or
$$\frac{4\pi^2(l_1+l_2)}{g}=\frac{t_1^2+t_2^2}{2}+\frac{t_1^2-t_2^2}{2}\cdot\frac{l_1+l_2}{l_1-l_2},$$

Here (l_1+l_2) is the distance between the knife edges. This distance can be measured accurately. Let it be L.

Then
$$\frac{4\pi^2L}{g}=\tau^2 \text{ (say)} \quad \quad \text{.. (iii)}$$

Where
$$\tau^2=\frac{t_1^2+t_2^2}{2}+\frac{t_1^2-t_2^2}{2}\cdot\frac{l_1+l_2}{l_1-l_2} \quad \quad \text{.. (iv)}$$

This τ is called the *computed time-period*. Only the first term in the expression for τ^2 is important, the other term, being small, is to be treated only as a *Correction term*. So far the first term is concerned it can be measured very accurately by the method of coincidence. To find the '*correction term*', *the pendulum is balanced in a horizontal position on a knife edge.* The distances of the knife edges of the pendulum from this knife (an additional knife edge is used to locate the C. G. of the pendulum) give l_1 and l_2 whence

$(l_1 - l_2)$ is found out. Thus the value of the computed time period can be found out sufficient accurately. Finally g is calculated from the relation (iii).

Examples

1. *The period of a disc of radius* 10 *cm executing small oscillations about a pivot at its rim is* ·784 *s. Find the value of g, the acceleration due to gravity.*

Sol I_0 (Moment of inertia of the disc about its axis)

$$=\frac{1}{2} Mr^2 = Mk^2;$$

$$\therefore \qquad k^2 = \frac{r^2}{2}.$$

Here $$l = r.$$

$$\therefore \qquad T = 2\pi\sqrt{\frac{k^2+l^2}{lg}} = 2\pi\sqrt{\frac{\frac{r^2}{2}+r^2}{rg}}$$

$$= 2\pi\sqrt{\frac{3}{2}\cdot\frac{r}{g}}$$

or $$T^2 = 4\pi^2\,\frac{3}{2}\cdot\frac{r}{g}$$

or $$g = \frac{6\pi^2 r}{T^2} = \frac{6\pi^2 \times \cdot 1}{\cdot 784^2}$$

$$g = 9{\cdot}637 \text{ ms}^{-2}. \textbf{ Ans.}$$

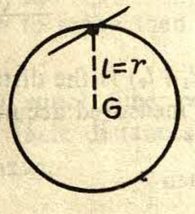


Fig. 10.8

2. *How much faster than its present rate should the earth revolve about its axis so that the weight of a body on the equator may be zero ? (Radius of the earth* = 6400 *km)*

Sol. We have, $g' = g - \omega^2 r \cos^2\lambda$. .. (Eq 10.1)

The apparent weight is zero when $g' = 0$.

$$\therefore \qquad g = \omega^2 r \cos^2 0^\circ$$

or $$\omega^2 = g/r$$

or $$\omega = \sqrt{g/r} = \sqrt{\frac{9{\cdot}8}{6400\times 10^3}} = 1{\cdot}237\times 10^{-3} \text{ rad. sec}^{-1}.$$

$$\text{The present speed} = \frac{2\pi}{86{,}400} = 7 \cdot 2 \times 10^{-5} \text{ rad.sec}^{-1}$$

Hence the required speed is 17 times greater. Ans.

PERIOD OF A SIMPLE PENDULUM IN AN ACCELERATED CAR

3. *A simple pendulum of period 2 s is suspended from the ceiling of a car. The car moves horizontally with acceleration $a = 4{\cdot}9\ ms^{-2}$. Find the period of oscillation of the pendulum.*

Sol. By D' Alembert's principle let us bring the car + pendulum to rest by applying inertia force ($-ma$) on the bob of the pendulum. Let it make angle φ with the vertical in the equilibrium position. For equilibrium we have

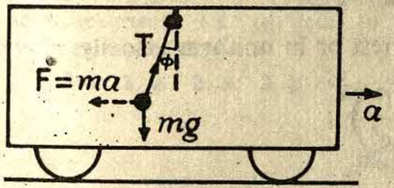


Fig. 10.9

$$T \sin \varphi = ma$$
$$T \cos \varphi = mg$$
$$\therefore \quad \tan \varphi = a/g. \qquad \text{(i)}$$

Now consider a slight displacement θ (clockwise) from the equilibrium position.

The unbalanced torque on the bob

$$= mal\cos(\theta + \varphi) - mgl\sin(\theta + \varphi) \text{ (clockwise)}$$

or the unbalanced torque on the bob

$$= mal\,(\cos\theta \cos\varphi - \sin\theta \sin\varphi) - mgl\,(\sin\theta \cos\varphi + \cos\theta \sin\varphi).$$

Since θ is small we can write $\sin\theta \doteq \theta$ and $\cos\theta = 1$.

$$\therefore \quad \text{Unbalanced torque} = mal(\cos\varphi - \theta \sin\varphi) - mgl(\theta \cos\varphi + \sin\varphi)$$
$$= mal \cos\varphi - mal\, \theta \sin\varphi - mgl\theta \cos\varphi - mgl \sin\varphi;$$

$$\because \quad \tan\varphi = \frac{a}{g} = \frac{ma}{mg},$$

$$\therefore \quad ma \cos\varphi = mg \sin\varphi.$$

$$\therefore \quad \text{Unbalanced torque} = -ml\theta\,(a \sin\varphi + g \cos\varphi).$$

$$\because \quad \text{torque} = \text{rotational inertia} \times \text{angular acceleration}$$

$$\therefore \quad -ml\theta\,(a \sin\varphi + g \cos\varphi) = ml^2 \times \alpha$$

or
$$\alpha = -\frac{a \sin\varphi + g \cos\varphi}{l} \times \theta$$

$$= -\frac{1}{l}\left(a \cdot \frac{a}{\sqrt{a^2+g^2}} + g\frac{g}{\sqrt{a^2+g^2}}\right)\theta \quad (\because \tan\varphi = a/g)$$

or $$\alpha=-\frac{\sqrt{a^2+g^2}}{l}\theta.$$

$\therefore$ $\alpha \propto -\theta$ and hence motion is simple harmonic.

The cyclic frequency of the pendulum is given by

$$\omega^2=\frac{\sqrt{a^2+g^2}}{l} \quad \text{or } \omega=\left(\frac{a^2+g^2}{l^2}\right)^{\frac{1}{4}}$$

or $$T=2\pi\left(\frac{l^2}{a^2+g^2}\right)^{\frac{1}{4}}.$$

When $a=0$ i.e. the car is at rest or in uniform velocity.

$$T=2\text{ s} \quad \therefore \quad 2=2\pi\left(\frac{l^2}{g^2}\right)^{\frac{1}{4}}.$$

When $a=4{\cdot}9$, $$T=2\pi\left(\frac{l^2}{4{\cdot}9^2+g^2}\right)^{\frac{1}{4}}.$$

$\therefore$ $$\frac{T}{2}=\left(\frac{g^2}{4{\cdot}9^2+g^2}\right)^{\frac{1}{4}}=\left(\frac{9{\cdot}8^2}{4{\cdot}9^2+9{\cdot}8^2}\right)^{\frac{1}{4}}$$

or $$T=2.\left(\frac{4}{5}\right)^{\frac{1}{4}}=1{\cdot}895\text{ s} \quad \text{Ans.}$$

QUESTIONS

(A)

1. The acceleration due to gravity is (a) maximum at the pole, minimum at the equator, (b) minimum at the pole, maximum at the equator, (c) constant everywhere, (d) no definite relation with the latitude of the place.

2. The acceleration due to gravity (a) decreases in either way from the surface of the earth, (b) increases in either way from the surface of the earth, (c) increases towards the centre of the earth and decreases away from the earth, (d) decreases towards the centre of the earth and increases away from the earth.

3. If a rigid body is suspended as its centre of gravity, the time period of oscillation is (a) zero, (b) infinity, (c) neither zero nor infinity, (d) exactly 1 s.

4. If the earth suddenly stops rotating, the value of g at a place (a) increases, (b) decreases, (c) remains constant, (d) increases or decreases depending on the position of the earth in its orbit.

5. The orbit of planets around the sun is (a) circular, (b) elliptical, (c) parabolic, (d) hyperbolic.

6. The period of a compound pendulum is minimum when the distance (l) of the centre of suspension from the centre of gravity and the radius of gyration (k) of the pendulum are such that (a) $l=2k$, (b) $k=2l$, (c) $l=k$, (d) $\sqrt{l}=k$.

7. The length of the equivalent simple pendulum is (a) k^2/l, (b) $l+l/k^2$, (c) $l+k^2/l$, (d) $l+k/l$.

8. If a simple pendulum of period T is suspended from the ceiling of an elevator and the elevator starts moving upward with uniform acceleration $g/2$, the period of the pendulum is (a) $\sqrt{2/3}\,T$, (b) $\sqrt{3/2}\,T$, (c) $\sqrt{2}\,T$, (d) $\sqrt{1/2}\,T$.

9. The minimum time period of a compound pendulum of radius of gyration k is (a) $2\sqrt{k/g}$, (b) $2\pi\sqrt{2k/g}$, (c) $2\pi\sqrt{k/2g}$, (d) $2\pi\sqrt{3k/g}$.

10. The time period of a simple pendulum of infinite length is (a) infinite, (b) zero, (c) exactly 1 s, (d) equal to the time of revolution of a satellite just above the surface of the earth.

Ans. **1.** a. **2.** a. **3.** b. **4.** a. **5.** b. **6.** c. **7.** c. **8.** a. **9.** b. **10.** d.

(B)

1. What is a simple pendulum ?

2. How does the periodic time of a simple pendulum vary when taken from the equator to the north pole ?

A simple pendulum has hollow spherical bob. Will its period of oscillation be altered if the bob is (a) entirely filled, and (b) half filled with mercury ?

3. Explain how the axial rotation of the earth affects the acceleration due to gravity.

4. What is computed time period ? Find an expression for it.

5. Distinguish between a simple pendulum and a compound pendulum.

6. Deduce the time period of a simple pendulum.

(C)

1. Distinguish between g and G. Establish the relation between g and G. What are the factors on which g depends ? Derive an expression showing the variation of g with latitude of a place.

2. What is a compound pendulum ? Find an expression for its time period and show that the centre of suspension and the centre of oscillation can be interchanged.

How you would measure the acceleration due to gravity by a bar pendulum ?

3. Describe a Kater's pendulum and explain how you would determine the value of g in your laboratory with it.

4. Explain the theory of Kater's pendulum and explain how you would use it to calculate the acceleration due to gravity in terms of two nearly equal time periods about the two knife edges.

5. Define a simple pendulum. Can it be realised in practice ? Show that the

time period of an ideal simple pendulum is

$t=2\pi\sqrt{\frac{m}{m'}\cdot\frac{l}{g}}$ where m is the inertial mass of the pendulm particle and m' is its gravitational mass.

Refering to Fig. 10.1, the gravitational pull on the bob $=\frac{GMm'}{R^2}=m'g$ where m' is the gravitational mass of the bob.

Taking moment of $m'g$ about the point of suspension in the displaced position we have :

Unbalanced torque $=m'gl\sin\theta$, clockwise

$=-m'gl\sin\theta$, anticlockwise

i.e. in the positive direction.

$\because$ torque = moment of inertia $\times$ angular acceleration

$\therefore$ $-m'gl\sin\theta=ml^2\times\alpha$ where m is the inertial mass

or $$\alpha=-\frac{m'}{m}\cdot\frac{g}{l}\theta \quad (\because \theta \text{ being small, } \sin\theta=\theta)$$

$\therefore$ $\alpha\propto-\theta$ and the motion is simple harmonic. The cyclic frequency of the pendulum is given by

$$\omega=\sqrt{\frac{m'}{m}\cdot\frac{g}{l}}$$

or $$T=2\pi\sqrt{\frac{m}{m'}\frac{l}{g}}.$$

(D)

1. A faulty seconds pendulum loses 20 s a day. Find the required alteration in length so that it may keep correct time. ($g=9{\cdot}8$ ms^{-2})

(Ans. ·046 cm)

2. A simple seconds pendulum loses 2 minutes per day when taken to the top of a mountain. Find the value of g on the top of the mountain. (g on the ground level$=9{\cdot}8$ ms^{-2})

(Ans. 9·773 ms^{-2})

3. A uniform circular disc of radius 25 cm oscillates in its plane about a point on its circumference. Calculate the time period of oscillation. ($g=9{\cdot}8$ ms^{-2})

(Ans. 1·201 s)

4. How far away from the earth does the acceleration due to gravity become one per cent of its value at the earth's surface? (Radius of the earth=6380 km)

(Ans. $5{\cdot}742\times10^4$ km)

5. A uniform cube is free to turn about one edge which is horizontal. Find the length of the edge so that it may execute a complete oscillation in 2s. ($g=9{\cdot}8$ ms^{-2})

(Ans. 1·05 m)

6. A simple pendulum of length l is suspended from the ceiling of a moving lift. Find the period of the pendulum when it is going (i) up with acceleration 'a' (ii) down with acceleration a'.

$\left(\text{Ans. } t=2\pi\sqrt{\frac{l}{g+a}}\,;\, t=2\pi\sqrt{\frac{l}{g-a}}\right).$

7. Show that the period of a simple pendulum of infinite length is given by

$$T=2\pi\sqrt{\frac{R_e}{g}}$$

where R_e is the radius of the earth and g is the acceleration due to gravity at the surface of the earth.

(E)

1. There are..................collinear points about which a compound pendulum has the same time period.

2. Is the inertial mass the same as the gravitational mass ?

3. The lower half of the spherical bob of a simple pendulum is removed. Does its period increase or decrease ?

4. Can a simple pendulum be realised in practice ?

5. The acceleration due to gravity at the equator is minimum on account of the rotation of the earth, or its oblate spherical shape or both ?

6. The period of a simple pendulum decreases or increases at the top of a mountain.

7. The period of a simple pendulum will decrease when taken to the bottom of a mine. True or false ?

(Ans. 1. Four. 2. Yes. 3. decreases. 4. No. 5. both. 6. increases. 7. false)

CHAPTER 11

ELASTICITY

11.1. Elasticity

All solid bodies are made of molecules or atoms held together by strong forces of attraction. For our clear perception we may think as if the molecules or atoms are joined by small springs of very large force constant. So all bodies can, more or less, be deformed by applying large forces. The simple cases of deformation of solids that we come across in our everyday life are : (i) stretching of a rubber cord in which there is change in length (Fig. 11.1a), (ii) compression of a body by forces in all directions so that there is a

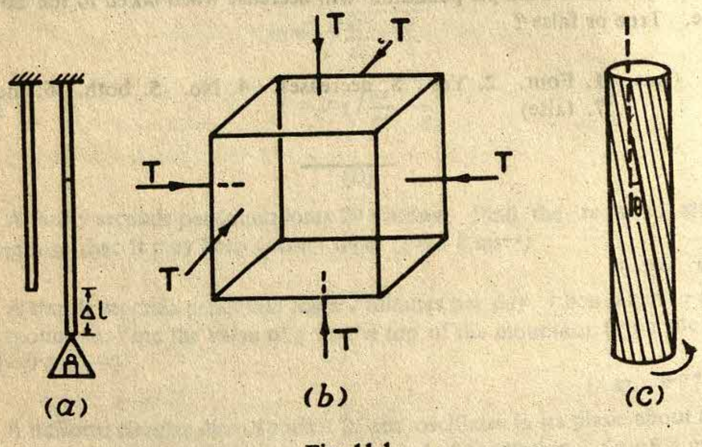


Fig. 11.1

change of volume (Fig 11.1b) and (iii) twisting of a cylinder fixed at one end and twisted by a torque applied at the other end so that there is change in shape only (Fig. 11.1c).

In all these cases when the deforming forces or torque are removed, the bodies spontaneously regain their original length, volume or shape as the case may be. *This property of a material body to regain its original condition, on the removal of the deforming forces or torque is called Elasticity.*

Materials may be divided into two groups according to their

behaviour when the deforming forces are withdrawn. If the body completely regains its original length, volume or shape, as the case may be, when the forces are withdrawn, the body is said to be perfectly *elastic*. If on the other hand, the body retains its altered shape or size, it is called perfectly *plastic*. No body is perfectly elastic or perfectly plastic. Materials differ in their degree of elasticity and plasticity. Quartz and phosphor-bronze are two materials almost perfectly elastic.

Origin of elastic properties :

The origin of elastic properties is the fundamental fact that a system tends to lie in the position of minimum potential energy which is the same as the position of zero force. The plot of potential energy of interatomic interactions against interatomic separation is shown in the figure here. The atoms of a solid are located at positions $r = r_0$ which corresponds to the minimum potential energy. When an external force is applied, the atoms are displaced from the potential minima to positions of higher potential energy. If the force is released the atoms move back to the equilibrium position in compliance with the above principle. This is how elastic property arises in matter.

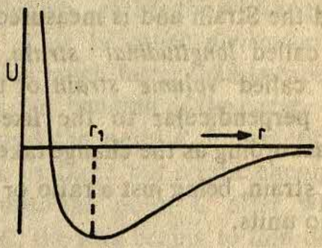


Fig. 11.1 (d)

11.2. Stress : Strain

Stress. When a deforming force is applied to a body, forces of reaction are called into play from within due to the relative displacement of its molecules, tending to restore the body to its original condition. *This internal force per unit area is called Stress.* The internal force or the restoring force as we may call it, depends on the degree of deformation. The greater the deformation, the greater is the restoring force from within. This is the reason for the fact that different loads cause different amounts of change in length,

volume or shape. In the equilibrium position which hardly takes any time to be established, the externally applied force is balanced by the internal force. Hence, in practice, stress is measured by the externally applied force per unit area.

$$\therefore \qquad \text{Stress} = \frac{\text{Restoring force}}{\text{Area}} \qquad .. \quad (11.1)$$

$$\text{or} \qquad \text{In practice, Stress} = \frac{\text{Force Applied}}{\text{Area}} \qquad .. \quad (11.1a)$$

If the force be inclined to the surface, its resolved component perpendicular to the surface, measured per unit area, is called the *normal stress*, and the resolved component acting along the surface, per unit area, is called *tangential stress*. Further, a stress may be compressive or expansive (also called tensile) according as there is decrease or increase in length or volume of the deformed body. Stress has the dimension $ML^{-1}T^{-2}$ or FL^{-2}. Its unit is newton per square metre (Nm^{-2}) or pascal (Pa).

Strain. The degree of deformation produced in a strained body is called the Strain and is measured by the change in length per unit length called *longitudinal strain*, the change in volume per unit volume called *volume strain* or the angle through which a line originally perpendicular to the fixed end is turned called *shearing strain*, according as the change takes place in length, volume or shape.

The strain, being just a ratio or angle, is dimensionless and so it has no units.

According to the three types of changes that may take place there are three strains.

These are :

(i) Longitudinal strain $= \dfrac{\text{change in length } (\triangle l)}{\text{original length } (l)}$

(ii) Volume strain $= \dfrac{\text{change in volume } (\triangle v)}{\text{original volume } (v)}$

(iii) Shearing strain $=\theta$, angle by which a line perpendicular to the fixed end is turned.

11.3. Elastic Limit : Permanent Set : Tenacity : Elastic Fatigue : Elastic Hysteresis

The behaviour of a wire when the load on it is gradually increased is very interesting. Fig.

11.2 shows the stress-strain diagram for weld-iron. The plot shows that with gradual increase of stress, the strain increases proportionally upto the point A. This point is called *elastic limit* because if the wire is deloaded it will completely regain its original length. This means within this range the property of elasticity of the material of the wire is fully retained. After passing the point A, the strain increases rather faster than that between O and A. If the wire is deloaded after passing the point A, it will shrink no doubt but will not completely regain its original length, rather the wire acquires a permanent strain given by OO'. This is called *permanent set.* Beyond B, there begins a great extension much faster than between A and B which is shown by slight downward bending of the curve. The point B is called the *yield point*. Beyond the point B, for practically little or no increase in stress, there is large but erratic change in strain shown by a wavy (zig-zag) line upto the point C after which the wire becomes plastic Beyond C, the wire begins to thin down with increase of stress at some section determined by accidental circumstances where the wire ultimately breaks. The maximum stress before rupture which corresponds to the point D is called the *tenacity* or *breaking stress* of the material.

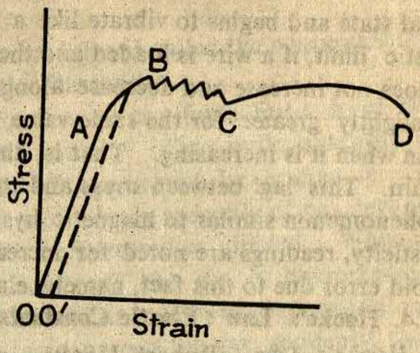


Fig. 11.2

Within the elastic limit different materials show different behaviour with regard to their resumption of the normal state when the deforming forces are withdrawn. A *quartz* fibre recovers its original condition almost immediately after the deforming forces are withdrawn while a *glass fibre* takes several hours before it regains its original condition. *The delay in recovering the normal condition on the removal of the deforming forces* is called *the elastic-after-effect.*

It was observed by Lord Kelvin that vibrations of a torsional pendulum died away much faster in the case of a wire kept vibrating continuously for sometime than in that of a fresh wire, as if the wire got tired and could not keep on vibrating for a long time. This property was termed by Kelvin as elastic fatigue.

If the wire is allowed to rest for some time, it recovers its original state and begins to vibrate like a fresh wire. Even within the elastic limit, if a wire is loaded and then deloaded, strictly speaking, it does not increase and decrease along the same path. The strain is slightly greater for the same value of stress, when it is decreasing than when it is increasing. That is, there is lag between stress and strain. This 'lag' between stress and strain is called *elastic hysteresis*, a phenomenon similar to magnetic hysteresis. In all experiments on elasticity, readings are noted for increasing and decreasing loads to avoid error due to this fact, namely, elastic hysteresis.

11.4. Hooke's Law : Elastic Constants

Hooke's Law. Robert Hooke, in 1679, gave the fundamental law of elasticity known after his name as Hooke's Law. In Hooke's words the law is : 'Ut tensio sic vis' meaning 'as the tension, so the strain.' The law is now formally stated as.

Within elastic limit, stress is proportional to strain.

Since stress is proportional to strain

$$\frac{\text{stress}}{\text{strain}} = \text{a constant} \qquad \text{.. (11.2)}$$

This constant is called modulus of elasticity or coefficient of elasticity :

Elastic Constants. Corresponding to the three types of strain, there are three different moduli of elasticity as follows :

(i) Modulus of longitudinal elasticity

or $$\text{Young's modulus } (Y) = \frac{\text{longitudinal stress}}{\text{longitudinal strain}}$$

(ii) Modulus of volume or bulk elasticity

or $$\text{bulk modulus } (K) = \frac{\text{volume stress}}{\text{volume strain}}$$

(iii) Modulus of shear elasticity

or $$\text{Rigidity } (n) = \frac{\text{shearing stress}}{\text{shearing strain}}$$

The increase in length of a wire is always accompanied by a sideways contraction. This sideways contraction is directly proportional to the longitudinal extension. *The ratio of the lateral strain to the longitudinal strain* is called Poisson's ratio (σ). Thus

$$\text{Poisson's ratio } (\sigma) = \frac{\text{lateral strain}}{\text{longitudinal strain}}$$

The above four constants Y, K, n and σ are called elastic constants.

The reciprocal of the bulk modulus is called the *compressibility*.

11.5. Isothermal and Adiabatic Elasticities

A solid possesses all the four elastic constants but a fluid (liquid and gas) possesses only *bulk elasticity*. Thus bulk modulus is the only modulus of elasticity which is possessed by matter in all the states. The change in volume of a body specially of gas depends on the condition under which it is compressed or dilated. When a gas is compressed heat is generated. If the gas is compressed in a cylinder having perfectly non-conducting walls, heat generated remains fully lodged in the gas or if we compress the gas suddenly, the heat produced has hardly any time to pass away. Such a process in which heat cannot enter the system from its environment nor leave the system to the environment is called an adiabatic process. On the other hand if the gas is compressed slowly in a cylinder of perfectly conducting walls, the heat of compression gradually passes away to the environment. In this process, the temperature of the system remains constant. Such a process is called an isothermal process. Corresponding to the two processes a substance possesses two bulk elasticities : isothermal bulk elasticity (K_T) and adiabatic bulk elasticity (K_S).

(*a*) *Isothermal bulk elasticity of a perfect gas.* When a perfect gas expands isothermally, its expansion is governed by Boyle's law

$$PV = \text{a constant.}$$

Differentiating with regard to V

$$P + V\frac{dP}{dV} = 0 \quad \text{or} \quad P = -\frac{dP}{\left(\dfrac{dV}{V}\right)}$$

Here dP measures the compressive stress and $\dfrac{dV}{V}$ the volume strain, the negative sign shows that an increase in pressure causes a decrease in volume

$$\therefore \qquad K_T = \frac{\text{volume stress}}{\text{volume strain}} = \frac{dP}{-\dfrac{dV}{V}} = P$$

Thus the isothermal bulk modulus of a perfect gas is equal to its normal pressure.

(*b*) *Adiabatic bulk elasticity of a perfect gas.* When a perfect gas expands adiabatically, its expansion is governed by the equation.

$$PV^{\gamma} = \text{a constant.}$$

where γ is a constant of the gas depending on the atomicity of the molecules of the gas and is the ratio of the specific heat capacity of the gas at constant pressure to the specific heat capacity at constant volume.

For a monotomic gas $\gamma = 1{\cdot}66$, $\gamma = 1{\cdot}41$ for a diatomic and $\gamma = 1{\cdot}33$ for triatomic molecules.

Differentiating with regard to V

$$P\gamma V^{\gamma-1} + V^{\gamma}\,\frac{dP}{dV} = 0$$

or
$$\gamma P + V\,\frac{dP}{dV} = 0$$

or
$$\gamma P = -\,\frac{dP}{\frac{dV}{V}} = E_S$$

Thus the adiabatic bulk elasticity of a gas is γ times its normal pressure.

$$\text{Further, } \frac{E_S}{E_T} = \frac{\gamma P}{P} = \gamma$$

or
$$E_S = \gamma E_T$$

or Adiabatic modulus of elasticity $= \gamma$ times isothermal modulus of elasticity.

11.6. Strain Potential Energy of a Strained Body

(a) *Work done in streching a wire.* In deforming a body work is done by the external agent against the internal force. There is flow of energy from the agent to the body. Thus the work done against the internal force is stored as energy of the body.

Suppose a wire of length l is stretched through $\triangle l$ by a force F. Let f be the internal force when the stretch is x.

Then,
$$Y = \frac{\frac{f}{A}\,(\text{stress})}{\frac{x}{l}\,(\text{strain})} = \frac{fl}{Ax}$$

or $f=\frac{YAx}{l}$ where A is the area of cross-section of the wire.

The elementary work done in stretching the wire further through dx is

$$\text{elementary work done}=f\,dx=\frac{YAx}{l}dx$$

$\therefore$ the work done in stretching from $x=0$ to $x=\triangle l$

$$=\frac{YA}{l}\int_0^{\triangle l} x\,dx$$

$$=\tfrac{1}{2}\frac{YA}{l}\triangle l^2$$

Now, when $x=\triangle l$, $\quad f=F$

$\therefore$ $$Y=\frac{Fl}{A\triangle l}$$

$\therefore$ the work done in stretching the wire through $\triangle l$

$$=\frac{1}{2}\left(\frac{YA\triangle l}{l}\right)\triangle l=\frac{1}{2}F\triangle l.$$

$$=\tfrac{1}{2}\times\text{stretching force}\times\text{stretch}.$$

The work done is stored as potential energy of the body.

$\therefore$ The strain potential energy of a strained body

$$=\frac{1}{2}\,\frac{YA}{l}\,\triangle l^2\text{ joule or }\frac{1}{2}F\triangle l\text{ joule.}$$

The strain potential energy per unit volume

$$=\frac{\frac{1}{2}F\triangle l}{Al}=\frac{1}{2}\left(\frac{F}{A}\right)\left(\frac{\triangle l}{l}\right)$$

$$=\tfrac{1}{2}\times\text{stress}\times\text{strain}.$$

(b) *Work done in compression.* Suppose that a body of volume V

is compressed by $\triangle V$. Let p be the stress when volume strain is $\frac{v}{V}$.

$$\text{Then } K=\frac{p\text{ (Stress)}}{\frac{v}{V}}=\frac{pV}{v} \text{ or } p=\frac{Kv}{V}$$

The elementary work done in further compressing it through

$$dv=pdv=\frac{Kv}{V}\,dv.$$

$\therefore$ the work done in compressing from $v=0$ to $v=\triangle V$.

$$=\frac{K}{V}\int_0^{\triangle V} vdv=\frac{1}{2}\,\frac{K}{V}\triangle V^2$$

When $\quad v=\triangle V, p=P$

$\therefore \quad P=\frac{K\triangle V}{V}$

$\therefore$ the work done in compressing by $\triangle V=\frac{1}{2}P\triangle V$.

The work done is stored as the strain potential energy of the body.

$\therefore$ The strain potential energy of the body $=\frac{1}{2}\times$ stress $\times$ change in volume

and the strain potential energy per unit volume

$$=\frac{\frac{1}{2}P\triangle V}{V}$$

$$=\frac{1}{2}P\left(\frac{\triangle V}{V}\right)$$

$$=\frac{1}{2}\times\text{stress}\times\text{strain}.$$

11.7. Determination of Young's modulus for a Wire

(a) *A simple way of measuring Young's modulus by stretching a*

wire is as follows. Two long wires AB and CD of the same material are suspended vertically close to each other from the same support. One of the wires AB carries a fixed scale at the lower end. It is straightened by a heavy cylindrical weight. The second wire carries the vernier of the scale A hanger is suspended from the vernier. The slotted half kilogramme weights can be slipped on to the hanger to load the wire gradually.

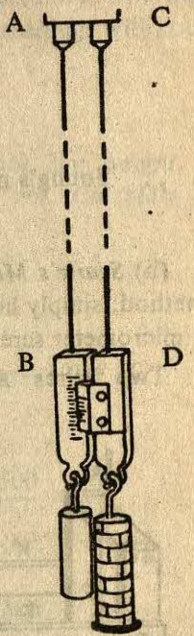


Fig. 11.5

First it is ascertained to which load the wire can be loaded to keep it within elastic limit. The diameter of the wire is measured at several points by a screw gauge and hence the area of cross-section of the wire is computed. The area of cross section multiplied by the tenacity (breaking stress) of the wire gives the breaking load of the weight. Half the breaking load is the permissible load for the wire.

Now place this permissible load on the hanger and after waiting for a few minutes remove the greater part of the weight leaving only one or two kilogrammes. This load is called the 'dead load'. This keeps the wire taut and free from 'kinks'. Read the main and vernier scale.

Now increase the load by a half kilogramme and again read the scale. Go on increasing the load by steps of half kg and read the scale up to the permissible load. Now remove the load by half kg till the dead load is reached, noting the reading of the scale at every step. Take the mean of the readings for increasing and decreasing loads. Subtract the first mean reading from all other mean readings to obtain the elongation for 0, $\frac{1}{2}$ kg, 1 kg, $1\frac{1}{2}$ kg........etc loads. Plot a graph 'load' versus 'elongation'. The graph should be a straight line passing through the origin. Hooke's law is verified, if the graph is a straight line.

From the graph find out the elongation l corresponding to any suitable load, say, M kg. Measure the length of the wire from the point of suspension up to the point where the vernier is attached to it. Let it be L.

$$\text{Then stress} = \frac{Mg}{\pi r^2}$$

$$\text{and strain} = \frac{l}{L}$$

$$\therefore \quad \text{Young's modulus } (Y) = \frac{\frac{Mg}{\pi r^2}}{\frac{l}{L}} = \frac{MgL}{\pi r^2 l} \; Nm^{-2}$$

(b) *Searle's Method.* This method is the same as the vernier method; simply here the elongation is measured more accurately by a micrometer screw.

Two wires *A* and *B* of the same material, length and area of cross-section are suspended from a rigid support and carry, at their lower ends, two brass frames *C* and *D*. One end of a spirit level is pivoted in the frame *C* and the other end rests on a micrometer screw *S* working in the frame *D*. The frames are kept together by links *K*,*K′*. They prevent the frames from twisting the wires relative to each other but they freely allow the vertical relative motion of the frames as they are hinged, not clamped.

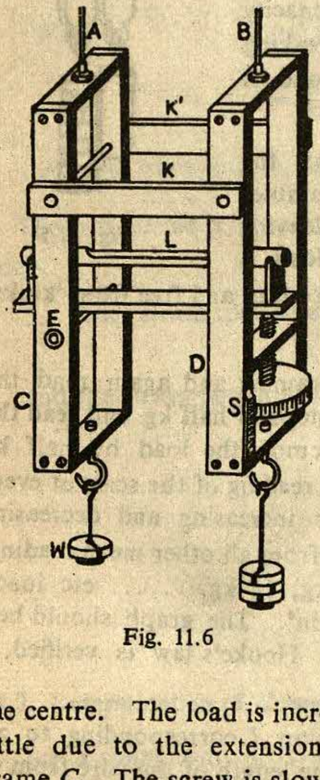


Fig. 11.6

From the lower end of the frame *C*, a heavy weight is suspended to straighten the wire *A*. From the lower end of the frame *D*, a hanger is suspended. Slotted weights are slipped on to the hanger to load the wire *B* suitably. The screw is worked up or down until the air bubble in the spirit-level is just in the centre. The load is increased when the frame *D* moves down a little due to the extension of the wire, and the bubble shifts to the frame *C*. The screw is slowly worked to bring the bubble back to its central position. The difference between the initial and final reading gives the elongation of the wire. For full details of experi-

mental procedure see the vernier method. As the two wires are made of the same material, any variation of temperature will affect both the wires by equal amounts and so the readings will not be affected by a change of temperature.

Examples

1. *A rubber cord of* 1 *cm diameter is loaded with 2kg weight. A length of* 50 *cm is found to be extended to* 50·1 *cm. Calculate the Young's modulus of rubber.*

Sol. Strain produced $= \frac{\cdot 001 \text{ m}}{\cdot 50 \text{ m}} = \frac{1}{500}$

$$\text{Stress} = \frac{2 \times 9{\cdot}8}{\pi(\cdot 005)^2} = \frac{19{\cdot}6}{25\pi} \times 10^6 \text{ Nm}^{-2}$$

$$\therefore \qquad Y = \frac{\text{Stress}}{\text{Strain}} = \frac{\frac{19{\cdot}6}{25\pi} \times 10^6}{\frac{1}{500}} = \frac{19{\cdot}6 \times 500}{25\pi} \times 10^6$$

$$\text{or} \qquad Y = \frac{19{\cdot}6}{5\pi} \times 10^8 = 1{\cdot}25 \times 10^8 \text{ Nm}^{-2} \textbf{ Ans.}$$

2. *Find the work done in stretching a wire of cross-section* 1 *sq. mm and length* 2 *m through* ·1 *mm, if Young's modulus for the material of the wire is* 2×10^{11} Nm^{-2}.

Sol. Work done $= \frac{1}{2} \times F \times \triangle l$

$$\text{Again } Y = \frac{\frac{F}{A}}{\frac{\triangle l}{l}} = \frac{Fl}{A \triangle l} \qquad \text{or} \qquad F = \frac{YA\triangle l}{l}$$

$$\therefore \text{ Work done} = \frac{1}{2} \frac{YA}{l} \triangle l^2 = \frac{1}{2} \times \frac{2 \times 10^{11} \times 1 \times 10^{-6}}{2} (\cdot 1 \times 10^{-3})^2$$

$$= \tfrac{1}{2} \times 10^{11} \times 10^{-6} \times 10^{-8}$$
$$= \cdot 5 \times 10^{-3}$$
$$= 5 \times 10^{-4} \text{ joule } \textbf{Ans.}$$

3. *Show that a small uniform volume strain is equivalent to three mutually perpendicular linear strains each equal to one third of the volume strain.*

Sol. Imagine a unit cube to be compressed equally and unifor-

mly on all sides, so that the length of each edge decreases by l and its volume by v.

Then clearly, volume strain $= \frac{v}{1} = v$ and linear strain along each

$$\text{edge} = \frac{l}{1} = l.$$

Since it is a unit cube, the reduced length of each edge is $(1-l)$ and the reduced volume is $(1-v)$

$\therefore \quad 1-v=(1-l)^3=1-3l$, neglecting l^2, l^3. . .terms

or $\quad v=3l \quad$ or $\quad l=\frac{v}{3}$ **Proved.**

QUESTIONS

(A)

1. Stress has the dimensions of (a) $ML^{-1}T^{-2}$, (b) $ML^{-2}T^{-2}$, (c) MLT^{-1}, (d) $ML^{-1}T$.

2. Young's modulus has the dimensions of (a) $ML^{-1}T^{-2}$, (b) $ML^{-2}T^{-2}$, (c) MLT^{-1}, (d) $ML^{-1}T$.

3. The work done by a stretching force F on a wire in stretching it by l is (a) Fl, (b) $\frac{1}{2}Fl$, (c) Fl^2, (d) $\frac{1}{2}Fl^2$.

4. The unit of modulus of elasticity is (a) Nm^{-1} (b) Nm^{-2}, (c) Nm, (d) Nm^2

5. If E_T and E_S stand for the isothermal and adiabatic elasticities of a perfect gas then (a) $E_T=E_S$, (b) $E_T=\gamma E_S$, (c) $E_S=\gamma E_T$, (d) $E_S=(\gamma-1)E_T$.

6. If two wires of the same material and length but of unequal diameters $(d_1=2d_2)$ are stretched by the same force then their elongation l_1 and l_2 are such that (a) $l_1=l_2$, (b) $l_1=2l_2$, (c) $l_2=2l_1$, (d) $4l_1=l_2$.

7. If two wires of the same material and length but of unequal diameters $(d_1=2d_2)$ are stretched by the same amount then works done W_1 and W_2 on them are such that (a) $W_1=2W_2$, (b) $W_2=2W_1$, (c) $W_1=4W_2$, (d) $W_2=4W_1$.

8. Poisson's ratio is. (a) longitudinal strain/lateral strain
(b) longitudinal stress/lateral strain
(c) lateral strain/longitudinal strain
(d) lateral stress/lateral strain.

9. The substance which shows almost hundred percent elasticity is (a) copper, (b) quartz, (c) rubber, (d) wood.

10. We note readings for increasing and decreasing loads to avoid error due to (a) elastic fatigue, (b) elastic after-effect, (c) elastic hysteresis, (d) erratic change in length.

Ans. 1. a, 2. a, 3. b, 4. b, 5. c, 6. d, 7. c, 8. c, 9. b, 10. c

(B)

1. Show that the strain potential energy per unit volume is $\frac{1}{2}\times$stress$\times$strain.

2. Explain the behaviour of a wire when it is subjected to increasing load.

3. Explain the terms 'permanent set,' 'elastic limit', 'elastic fatigue' and 'elastic after effect'.

4. Explain (i) stress, (ii) strain, (iii) elastic limit.

(C)

1. State and explain Hooke's law. Describe an experiment to verify it.

2. What are the different moduli of elasticity ? Describe how Young's modulus can be determined experimentally.

3. Explain stress, strain and Young's modulus. Describe Searle's method of determining Young's modulus of the material of a wire. Why are two wires of the same length, diameter and material used in this method ?

(D)

1. Find the force necessary to stretch by 1 mm a rod of iron 1 m long and 2 mm in diameter. (Y for iron$=2\times10^{11}$ Nm^{-2})

(Ans. 628 newton)

2. A 10 m long rubber cord is suspended from a rigid support. What will be the increment in its length by its own weight ? (Y for rubber$=4{\cdot}9\times10^{8}$ Nm^{-2} and density of rubber$=1500$ kgm^{-3}.)

(Ans. 15 cm).

3. Find the greatest length of copper wire that can hang vertically without breaking. Breaking stress of copper$=2{\cdot}9\times10^{8}$ Nm^{-2}. Density of copper $=8900$ kgm^{-3}.

(Ans. 3320 m)

4. A steel wire whose diameter is 2 mm just under tension between two points at 30°C. If the temperature falls to 20°C, calculate its tension. (Coefficient of linear expansivity of steel$=11\times10^{-6}$ $°C^{-1}$ and Young's modulus of elasticity of steel$=2{\cdot}1\times10^{11}$ Nm^{-2}).

(Ans. 72·6 N)

5. A body of mass 10 gm fastened to one end of a rubber cord of cross-section 5×10^{-6} m^2 revolve in a horizontal circle with uniform speed about the other end which is fixed. It executes 2000 revolutions per minute. If the radius of the circle is 12 cm, find the unstretched length of the cord. Young's modulus of the cord$=5\times10^{8}$ Nm^{-2}.

(Ans. 117 m)

6. Calculate the energy stored up in a wire 5 m long and 1 mm in radius when stretched by 1 mm. Young's modulus of the material of the wire 2×10^{11} Nm^{-2}.

(Ans. ·063 joule)

7. A unit cube is sheared by 1′ by a tangential force of 23×10^{7} N. Calculate the rigidity of the material of the cube.

(Ans. $7{\cdot}9\times10^{9}$ Nm^{-2}).

(E)

1. The unit of modulus of elasticity is......

2. The work done by a stretching force **F** on a wire in stretching it by l is......

3. A solid possesses all the elastic constants. False or true ?

4. A liquid can have all the four elastic constants. False or true ?

5. The compressibility of a liquid is the reciprocal of......

6. The tenacity of a wire depends on its area of cross-section. True or false ?

Ans. 1. Nm^{-2}, **2.** $\frac{1}{2} Fl$, **3.** True, **4.** False, **5.** Bulk Modulus, **6.** True.

CHAPTER 12

SURFACE TENSION

12.1. Evidences and Experiments that Show Membrane like Behaviour of Liquid Surfaces

The surface of a liquid behaves somewhat like a stretched elastic membrane Just as a stretched elastic membrane has a natural tendency to contract and occupy a minimum area, so also the surface of a liquid has got the natural tendency to contract and occupy a minimum possible area as permitted by the circumstances of the liquid mass. We have many evidences in support of this fact and a direct experimental demonstration of this fact :

(a) We very often find that a small quantity of a liquid spontaneously takes a spherical shape, e.g., rain drops, small quantities of mercury placed on a clean glass plate etc. Now, for a given volume, a sphere has the least surface area. Thus, this fact shows that a liquid always tends to have the least surface area.

(b) If we immerse a camel-hair brush in water, its hairs spread out, but the moment it is taken out of water, they all cling together as though bound by some sort of elastic thread.

Experimental demonstration. Make a circular loop of wire and dip it in a soap solution. Take it out of the solution and find a thin film of soap solution across the loop. Place a moistened cotton loop gently on the film. The thread will lie across the film in an irregular manner (Fig. 12.1 a). Now prick the film inside the cotton thread loop by a pin. At once the thread will spontaneously take the shape of a circle (Fig. 12.1 b). The thread has a fixed perimeter. For a given perimeter a circle has got the *maximum* area. Hence the remaining portion of the film occupies the *minimum* area.

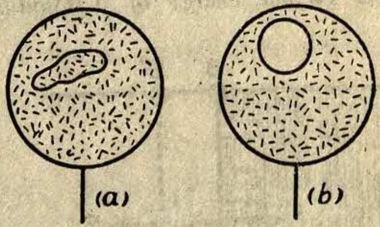


Fig. 12.1

12.2. Surface Tension and Surface Energy

Surface tension. The above evidences and experiement prove beyond doubt that the surface of a liquid behaves to some extent like a stretched elastic membrane having a natural tendency to contract and occupy a minimum possible area as permitted by the circumstances of the liquid mass. *This property of the surface of a liquid by virtue of which it tends to contract and occupy the minimum possible area is called surface tension and is measured by the force per unit length of a line drawn in the liquid surface, acting perpendicularly to it and tangentially to the surface of the liquid and tending to pull the surface apart along the line.*

Surface tension has the dimensions MT^{-2} or FL^{-1}. Its unit is newton per metre (Nm^{-1}). It depends on temperature. The surface tension of all liquids decreases linearly with temperature, over small ranges, so that the surface tension at t°C is given by $T = T_0(1-\alpha t)$, where T_0 is the value at 0°C and α is a constant called the *'temperature coefficint'* of surface tension.

Surface energy. Any strained body possesses potential energy which is equal to the work done in bringing it to the present state from its initial unstrained state. The surface of a liquid is also a strained system and hence the surface of a liquid also has potential energy which is equal to the work done in creating the surface. *This energy per unit area of the surface is called surface energy.* Its unit is joule per square metre (Jm^{-2}).

The surface energy of a liquid is numerically equal to its surface tension. To prove this consider a rectangular frame of wire having one side AB free to slide over the parallel sides AD and BC. Form a soap film across $ABCD$. The side AB is pulled to the left due to surface tension. To keep the wire in position a force F, equal and opposite to the force due to surface tension has to be applied to the right. If T is the surface tension and l is the length of AB, then the

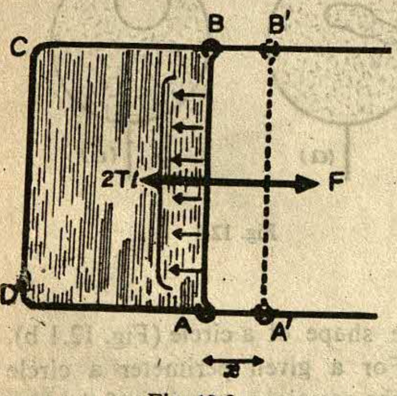


Fig. 12.2

force due to surface tension over AB is $2lT$ to the left because the film has two surfaces.

Since the film is in equilibrium

$$F=2lT$$

Now, if the wire AB is pulled to the right, energy will flow from the agent to the film and this energy is stored as potential energy of the surface created just now. Let the wire be pulled slowly through x.

Then the work done = energy added to the film from the agent

$$=F.x=2lTx$$

The area of the film created $=2lx$.

$\because$ Surface energy of the film $=\dfrac{2lTx}{2lx}=T.$

Thus surface energy of the film is numerically equal to its surface tension.

12.3. Theory of Surface Tension

The surface tension of a liquid arises out of the *inter attraction* of its molecules. Molecules of a fluid (liquid and gas) attract one another with a force varying inversely as some high power of the distance and thus this force decreases rapidly with distance. The distance up to which the force of attraction between two molecules is appreciable is called the *molecular range*, and is generally of the order of 10^{-9} metre. A sphere of radius equal to the molecular range drawn around a molecule is called *sphere of influence* of the molecules lying within the sphere of influence.

Consider three molecules A, B and C having their spheres of influence as shown in Fig. 12.3. The sphere of influence of A is well inside the liquid, that of B partly outside and that of C fifty-fifty.

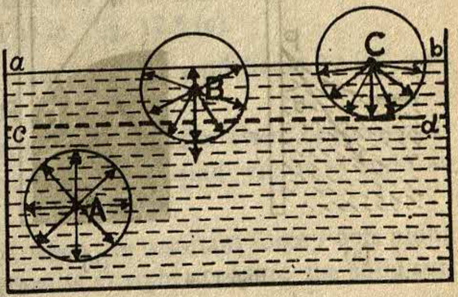


Fig. 12.3

Molecules like A do not experience any resultant force as they are attracted equally in all directions. Molecules like B or C will experience a resultant force

directed inward. Thus the molecule well inside the liquid will have only kinetic energy but the molecules near the surface will have kinetic energy as well as potential energy which is equal to the work done in placing them near the surface against the force of attraction directed inward.

Imagine a plane *cd* inside the liquid parallel to the free surface *ab* at a distance equal to the molecular range. The layer of liquid between two planes is called the surface film. Clearly, all the molecules in the film will have their sphere of influence partly outside and hence will have potential energy. Let u be the average potential energy per molecule, n number of molecules per unit volume and A is the area of the film.

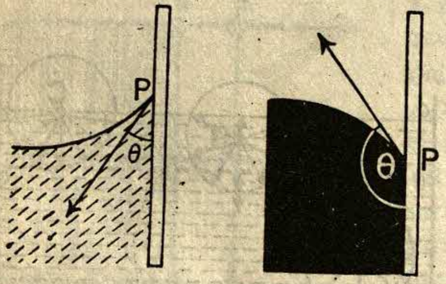
Then the potential energy of the surface film is

$$U = \mathrm{Arn}\bar{u} \qquad \text{.. (12.1)}$$

where r = molecular range = thickness of the surface film. Let us now recall the universal principle of stable equilibrium of a system, that is, a system is in stable equilibrium when its potential energy is a minimum. According to this principle a system tends to occupy a position in which its potential energy is minimum. Hence the surface film must also tend to occupy that position in which its potential energy is a minimum. Since r, n and $\bar{u}$ are constants, $U \propto A$. Therefore the surface film must have a natural tendency to occupy a minimum area. This is exactly what is the surface tension of liquid surface.

12.4. Angle of Contact : Shape of Liquid Surface near a Solid Surface

When a solid body in the form of a tube or plate is immersed in a liquid, the surface of the liquid near the solid is, in general, curved. *The angle between the tangents to the liquid surface and the solid surface at the point of contact, inside the liquid, is called the angle of contact* for that pair of solid and liquid.



(a) (b)
Fig. 12.4

Theory of angle of contact. The angle of contact arises due to

adhesive and cohesive forces on the molecules of the liquid which lie near the solid surface. Forces of attraction between molecules of different substances are called *adhesive forces* and the forces of attraction between molecules of the same substance are called *cohesive forces.*

Consider a liquid molecule near the solid surface. This molecule is attracted by the molecules of the solid wall.

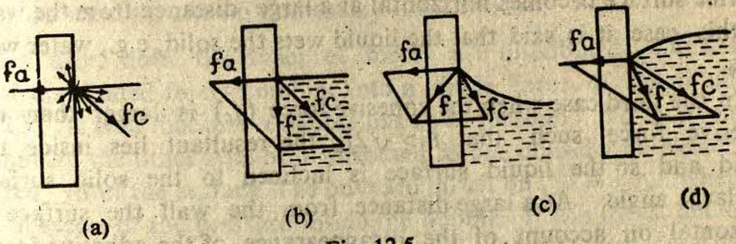


Fig. 12.5

These forces are adhesive and are distributed over 180° and hence their resultant acts at right angles to the solid wall. The forces of cohesion, i.e., attraction by liquid molecules, are distributed over 90° and hence their resultant will be inclined at 45° to the solid wall. Let f_a be the resultant adhesive force and f_c be the resultant cohesive force. Then the angle between f_a and f_c is 135°. Let f be their resultant making angle θ with f_a. Then

$$\tan \theta = \frac{f_c \sin 135^\circ}{f_a + f_c \cos 135^\circ} = \frac{f_c}{\sqrt{2}\, f_a - f_c}$$

Case I. If $\sqrt{2}\, f_a = f_c$, then $\tan \theta = \infty$ or $\theta = 90^\circ$, i.e., resultant will lie along the solid surface (Fig. 12.5 (b).

Case II. If $\sqrt{2} f_a > f_c$, $\tan \theta$ is $+ve$ and hence $\theta < 90^\circ$, i.e , the resultant lies inside the solid as in Fig. 12.5 (c).

Case III. If $\sqrt{2} f_a < f_c$, $\tan \theta$ is $-ve$ and hence $\theta > 90^\circ$, i.e., the resultant lies inside the liquid (Fig. 12.5 d).

A liquid cannot permanently withstand a shearing force as it has no rigidity. Hence the free surface of a liquid will be at right angles to the resultant force.

Thus, in the first case, when the resultant force f acts along the solid surface, the liquid surface is at right angles to the solid surface.

In the second case when the cohesive force is smaller the adhesive force such that $f_c < \sqrt{2} f_a$, the resultant is inside the solid and so the

liquid surface is inclined to the solid surface at a small angle. As the distance of the particle from the wall increases, the adhesive force becomes smaller in magnitude and the cohesive force spreads over a larger angle. At a point remote from the wall the adhesive force becomes vanishingly small and the cohesive forces spread over 180° and the resultant cohesive force f_c acts vertically downward. The free surface of the liquid is, in consequence, concave upwards near the wall. The concavity of the surface decreases gradually and the free surface becomes horizontal at a large distance from the wall. In this case it is said that the liquid wets the solid, e.g., water wets glass.

In the third case when the cohesive force (f_c) is larger than the adhesive force such that $f_C > \sqrt{2} f_a$ the resultant lies inside the liquid and so the liquid surface is inclined to the solid surface at a large angle. At a large distance from the wall the surface is horizontal on account of the disappearance of the adhesive force. Thus in this case the liquid surface is convex upwards near the wall. In this case it is said that the liquid surface does not wet the solid, e.g., mercury and glass.

12.5. Pressure Inside a Soap Bubble

The pressure inside a soap bubble or a liquid drop must be in excess of the pressure outside the bubble drop because without such pressure difference a bubble or a drop cannot be in stable equilibrium. Due to surface tension the bubble or drop has got the tendency to contract and disappear altogether. To balance the tendency to contract, there must be an excess of pressure inside the bubble.

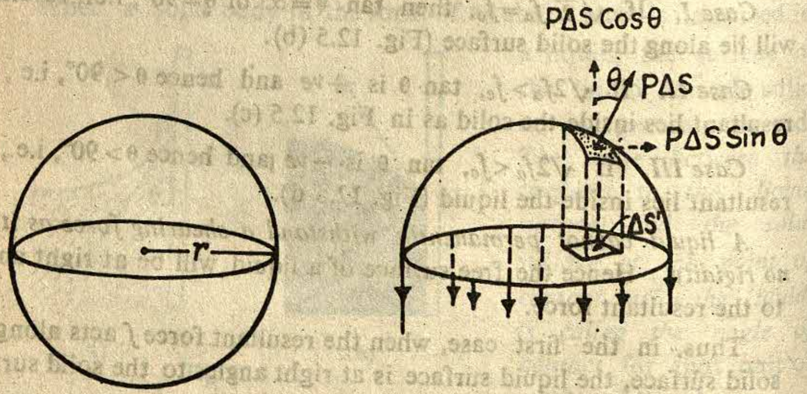


Fig. 12.6

To obtain a relation between the excess of pressure and the surface tension, consider a soap bubble of radius r and surface tension T. Divide the bubble into two halves by a horizontal plane passing through its centre and consider the equilibrium of one-half, say, the upper half. The forces acting on it are :

(i) forces due to surface tension distributed along the circumference of the section.

(ii) outward thrusts on elementary areas of it.

Obviously, both the types of forces are distributed. The first type of distributed forces combine into a single force of magnitude $2\pi r \times 2T$. It is $2T$ because a bubble has two surfaces To find the resultant of the other type of distributed forces, consider an elementary area $\triangle S$ of the surface. The outward thrust on $\triangle S = p \triangle S$ where p is the excess of the pressure inside the bubble. If this thrust makes an angle θ with the vertical, then it is equivalent to $\triangle Sp \cos\theta$ along the vertical and $\triangle Sp \sin\theta$ along the horizontal. The resolved component $\triangle Sp \sin\theta$ is ineffective as it is perpendicular to the resultant force due to surface tension. The resolved component $\triangle Sp\cos\theta$ contributes to balancing the force due to surface tension.

The resultant outward thrust $= \Sigma \triangle Sp \cos\theta$

$$= p\Sigma \triangle S \cos\theta$$

$$= p\Sigma \triangle S' \text{ where } \triangle S' = \triangle S \cos\theta$$

= area of the projection of $\triangle S$ on the horizontal dividing plane

$$= p \times \pi r^2 \quad (\because \; \Sigma \triangle S' = \pi r^2)$$

For equilibrium of the bubble we have

$$\pi r^2 p = 2\pi r . 2T$$

or
$$p = \frac{4T}{r} \quad \ldots \quad (12.2)$$

If it is a drop, the resultant force due to surface tension is $2\pi rT$. Hence for equilibrium of a drop we have

$$\pi r^2 p = 2\pi rT$$

or
$$p = \frac{2T}{r} \quad \ldots \quad (12.2\text{ a})$$

12.6. Measurement of Surface Tension

(*i*) *Capillary Rise Method.* When a glass tube of very fine bore called a capillary tube is dipped in a liquid like water, the liquid

immediately rises up into it due to surface tension. This phenomenon of rise of a liquid in a narrow tube is known as *Capillarity*.

Suppose that a capillary tube of radius r is dipped vertically in a liquid. The liquid surface meets the wall of the tube at some inclination θ called the angle of contact. Due to surface tension a force $\triangle lT$ acts on an element $\triangle l$ of the circle of contact along which the liquid surface meets the solid surface and it is tangential to the liquid surface at inclination θ to the wall of the tube. This force is exerted by the liquid on the wall of the tube. By the third law of motion, the tube exerts the same force on the liquid in the opposite direction. Resolving this latter force along and perpendicular to the wall of the tube, we have $\triangle lT\cos\theta$ along the tube vertically upward and $\triangle lT\sin\theta$ perpendicular to the wall. The latter component is ineffective. It simply compresses the liquid against the wall of the tube. The vertical component $\triangle lT\cos\theta$ pulls the liquid up the tube.

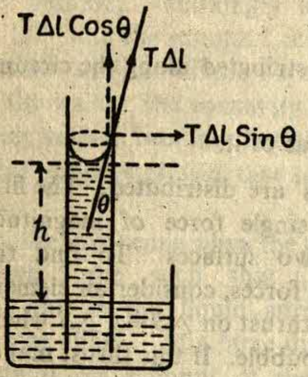


Fig. 12.7

The total vertical upward force $= \Sigma \triangle lT\cos\theta$

$$= T\cos\theta\Sigma\triangle l$$

$$= T\cos\theta 2\pi r.$$

$$(\because \quad \Sigma\triangle l = 2\pi r).$$

Due to this upward pull liquid rises up the capillary tube till it is balanced by the downward gravitational pull. If h is the height of the liquid column in the tube up to the bottom of the meniscus and v is the volume of the liquid above the horizontal plane touching the meniscus at the bottom, the gravitational pull, i. e, weight of the liquid inside the tube is $(\pi r^2 h + v)\rho g$.

For equilibrium of the liquid column in the tube

$$2\pi rT\cos\theta = (\pi r^2 h + v)\rho g. \tag{12.3}$$

The small volume of the liquid above the horizontal plane through the lowest point of the meniscus can be calculated if θ is given or known. For pure water and glass $\theta = 0°$ and hence the meniscus is hemispherical

$\therefore \quad v$ = volume of the cylinder of height r − volume of hemisphere.

$$=\pi r^2 r-\frac{1}{2}\,\frac{4\pi}{3}\,r^3$$

$$=\pi r^3-\tfrac{2}{3}\pi r^3=\tfrac{1}{3}\pi r^3$$

$\therefore$ For water and glass

$$2\pi r\,T=(\pi r^2 h+1/3\,\pi r^3)\,\rho\,g.$$

$$T=\tfrac{1}{2}\,\rho g r(h+r/3)\ \text{newton per meter (Nm}^{-1}) \quad \ldots \quad (12.3\text{ a})$$

$\because \quad h \gg r$, we have $hr=\dfrac{2T}{\rho g}$ = a constant. Thus $h \propto \dfrac{1}{r}$.

This is known as **Jurin's law.**

Three or four capillary tubes of different bores are taken and they are carefully washed first with sulphuric acid and then with a strong solution of caustic soda and finally by passing a stream of tap water. The liquid (water) is taken in a trough. The capillary tubes and a needle are fixed on a glass plate by wax. The tubes are then dipped into the liquid such that the lower end of the needle just touches the surface of the liquid. A travelling microscope is then focused on the meniscus of the liquid inside the tube. The horizontal cross-wire of the microscope is set tangential to the lowest point of the meniscus and its reading on the vertical scale is noted.

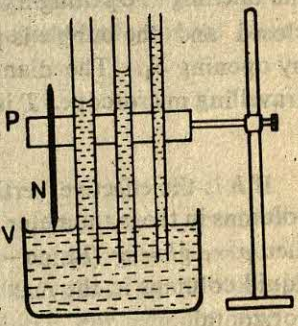


Fig. 12.8

The microscope is next focused on the top of the needle and its reading is noted as before. The difference between the two readings added to or subtracted from the length of the needle according as the meniscus lies above or below the tip of the needle gives h. The same process is repeated with other tubes. The diameters of the tubes are then measured by the microscope and finally T is calculated from the above formula.

(ii) *The Bubble Method.* The expression for the excess of pressure, that is, $p=\frac{4T}{r}$ affords a simple method of measuring the surface tension of a soap solution. A vertical glass tube AB having a fine orifice at its lower end is connected to a manometer and a side tube. The manometer contains a liquid of low density such as xylene and one of its two limbs instead of being vertical is inclined at certain small angle to the horizontal. Closing stop-cocks S_1 and S_2, the lower end of the tube is dipped into the soap solution when a thin film is formed at the opening. Opening S_2 the film is then blown into a bubble. S_2 is closed and the bubble is put in communication with the manometer by opening S_1. The diameter of the bubble is measured by a travelling microscope. T is calculated from the formula,

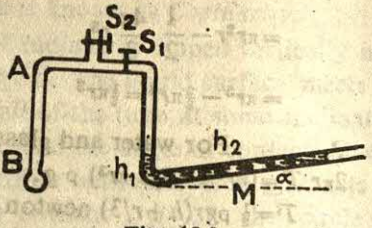


Fig. 12.9

$$T=\frac{pr}{4}\text{Nm}^{-1}$$

If h is the effective vertical difference in the heights of the liquid columns in the manometer and ρ is the density of the manometer liquid then $p=\rho gh=\rho g\,(h_2\sin\alpha-h_1)$ where h_1 and h_2 are the lengths of liquid columns in the manometer tube and α is the inclination to the horizontal.

$$\therefore \quad T=\tfrac{1}{4}\,\rho ghr\ \text{Nm}^{-1}=\tfrac{1}{4}\,\rho gr\,(h_2\sin\alpha-h_1)\ \text{Nm}^{-1}$$

12.7. Some Phenomena Due to Surface Tension

(*a*) *Kerosene oil spreads over water spontaneously.* Let us consider the equilibrium of a liquid I floating over liquid II. At the point of contact there are three forces : (a) surface tension T_1 of the liquid I along the tangent to its surface, (b) surface tension T_2 of the liquid II along the tangent to its surface, (c) the surface tension between I and II along the tangent to the surface in contact. For equilibrium of the liquid I over liquid II, T_1, T_2 and T_3 must be represented by the three sides of a triangle. This triangle of forces is called Newmann's triangle.

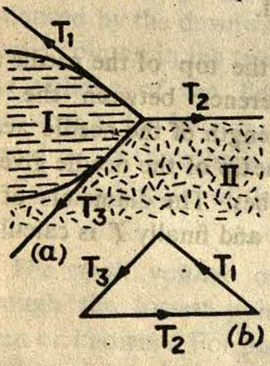


Fig. 12.10

For a kerosene oil drop over water Newmann's traingle is not possible and so kerosene spreads spontaneously over a water surface.

(*b*) *Water spreads over a clean glass plate but not mercury.* Now consider the equilibrium of a liquid drop over a solid surface. Here also there are three forces : T_1 for air-liquid, T_2 for air-solid and T_3 for liquid-solid surface.

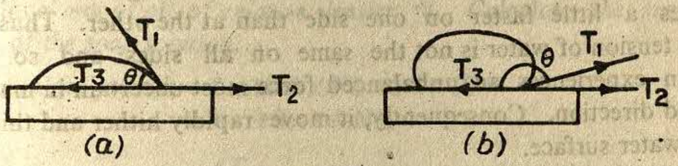


Fig. 12.11

Here T_3 and T_2 are always opposite to each other. The resolved component of T_1 along the surface will, therefore, decide the equilibrium of the drop over the solid. If θ be the angle of contact then condition for equilibrium is

$$T_3 + T_1 \cos\theta = T_2$$

or
$$\cos\theta = \frac{T_2 - T_3}{T_1}$$

So long $T_2 - T_3 < T_1$ i.e. $\cos\theta < 1$ equilibrium is possible. The moment $T_2 - T_3 > T_1$ i.e. $\cos\theta > 1$, which is absurd, equilibrium is impossible. For water and clean glass, equilibrium is not possible and hence water spreads over glass. For mercury equilibrium condition is satisfied and hence mercury does not spread over glass.

(*c*) *A needle can float over water.* If an ordinary sewing needle is carefully placed on the surface of water, it is found to float. The experiment can be performed by placing a needle on a piece of blotting paper and floating the paper on water. The paper soon gets wet and sinks, while the needle floats on water. The vertical component of the surface tension balances the weight of the needle and consequently it floats.

(*d*) *Small floating bodies attract.* Small floating bodies such as straw or match sticks are found to attract each other when they are close to each other. When two straw particles or match sticks come sufficient near to each other, the liquid between them rises as in a capillary tube. The pressure between them falls short of the atmosphere pressure and so the atmospheric pressure pushes them towards each other from the sides.

(*e*) *A toy camphor scorpion scampers on water.* Just for fun pieces of camphor are arranged together in the shape of a scorpion and floated in water when it is found to scamper haphazardly. The reason is that as camphor dissolves in water, the solution has a smaller surface tension than pure water. Probably due to irregularity in the shape of the pieces and the local condition of water, the camphor dissolves a little faster on one side than at the other. Thus the surface tension of water is not the same on all sides and so the scorpion experiences an unbalanced force most uncertain in magnitude and direction. Consequently, it moves rapidly hither and thither on the water surface.

(*f*) *Taming of wind by oil.* By pouring oil on a vast sheet of water the adverse situation due to wind can be avoided to some extent. The wind carries away the surface film containing oil and leaves behind a clean sheet of water. The surface tension of the oil contaminated water is less than that of pure water. Thus the pull back on a floating body is greater than the pull forward and so carrying away of floating bodies by the wind is reduced to some extent.

(*g*) *Will water flow out in a vertical capillary tube of insufficient length* ? No. If θ be the angle of contact between the liquid (water here) and the tube, and R, the radius of the liquid menicus in the tube, we have $r=R\cos\theta$, where r is the radius of the tube. When $\theta=0°$, $r=R$. We have assumed this in the deduction of the Eq. 12.3 a. However, if θ is not zero then

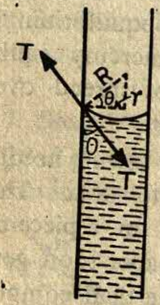


Fig. 12.12

$$2\pi rT\cos\theta=\pi r^2h\rho g \text{ or } 2T\cos\theta=rh\rho g$$

$$\text{or} \quad 2T\cos\theta=R\cos\theta\, h\rho g$$

$$\text{or} \quad Rh=\frac{2T}{\rho g}=\text{a constant.}$$

Thus in an insufficiently long tube, R will change till the above relation is satisfied. In a sufficiently long tube R being fixed, it is the value of h that changes to satisfy the above relation.

Examples

1. *Calculate the work done in blowing a bubble of radius 5 cm. The surface tension of soap solution is* ·024 Nm^{-1}.

Sol. The work done in blowing a bubble is stored as the potential energy of the bubble.

...e surface energy is numerically equal to the surface tension.

$\therefore$ The work done $= 4\pi(\cdot05)^2 \times \cdot024 \times 2$ (multiplied by 2 because there are two surfaces)

$$= 4\pi \times 25 \times 24 \times 10^{-7} \times 2$$
$$= 15\cdot1 \times 10^{-4} \text{ joule Ans.}$$

2 *The pressure inside a soap bubble of radius* 1 *cm balances* 1·4 *mm of a column of oil of relative density* ·8. *Calculate the surface tension of the soap solution.*

Sol. Here $p = \rho gh = \cdot8 \times 1000 \times 9\cdot8 \times 1\cdot4 \times 10^{-3}$
$$= \cdot8 \times 9\cdot8 \times 1\cdot4 \text{ Nm}^{-2}$$

$$p = \frac{4T}{r}$$

or
$$T = \frac{pr}{4} = \frac{1}{4} \times \cdot8 \times 9\cdot8 \times 1\cdot4 \times \cdot01$$

$$= \cdot02744 \text{ Nm}^{-1} \text{ Ans.}$$

3. *A capillary tube is dipped in water. Water rises to a height* 12 *cm. If the angle of contact is zero and the radius of the tube is* ·1 *mm, what is the surface tension of water ?*

Sol. $T = \frac{1}{2} \rho gr\,(h + r/3)$

$$= \frac{1}{2} \times 1000 \times 9\cdot8 \times \cdot1 \times 10^{-3}\left(\cdot12 + \frac{\cdot1 \times 10^{-3}}{3}\right)$$

$$(\because \rho = 1000 \text{ kg m}^{-3})$$

$$= \cdot05 \times 1\cdot2 \times 9\cdot8 = \cdot0588 \text{ Nm}^{-1} \text{ Ans.}$$

4. *What is the pressure inside a small air bubble of* ·1 *mm radius ? The surface tension of water* $= \cdot072$ Nm^{-1} *and the atmosphere pressure* $= 1\cdot013 \times 10^5$ Nm^{-2}

Sol. $\because P_{inside} = P_{outside} + \dfrac{2T}{r}$ as it is a bubble in water.

$\therefore$ Pressure inside the bubble $= 1\cdot013 \times 10^5 + \dfrac{2 \times \cdot072}{\cdot1 \times 10^{-3}}$

$$= 1\cdot013 \times 10^5 + 1440$$
$$= 1\cdot013 \times 10^5 + \cdot0144 \times 10^5$$
$$= 1\cdot0274 \times 10^5 \text{ Nm}^{-2} \text{ Ans.}$$

QUESTIONS

(A)

1. The excess of pressure inside a drop of water is (a) $4T/r$, (b) $2T/r$, (c) T/r, (d) $2T/r^2$.

2. The excess of pressure inside a soap bubble is (a) $4T/r$, (b) $2T/r$, (c) T/r, (d) $2T/r^2$.

3. The area enclosed by a loop of given perimeter is maximum when it is (a) rectangular, (b) elliptical, (c) circular, (d) of any shape.

4. The surface area of a body of given volume is minimum when it is (a) cubic, (b) spherical, (c) ellipsoidal, (d) paraboidal in shape.

5. A liquid wets a solid, if the angle of contact is (a) obtuse, (b) acute, (c) 90°, (d) 180°

6. A liquid drop takes a spherical shape because of the phenomenon of (a) elasticity, (b) gravitation, (c) surface tension, (d) viscosity.

7. The surface of a liquid is convex, if cohesive force F_c and adhesive force F_a on a liquid molecule near the solid surface are such that (a) $\sqrt{2}\,F_a=F_c$, (b) $F_a=F_c$, (c) $\sqrt{2}\,F_a>F_c$, (d) $\sqrt{2}\,F_a<F_c$.

8. The angle of contact of a liquid surface with a solid surface is less than 90°. Is (a) $F_a=F_c$, (b) $\sqrt{2}\,F_a=F_c$, (c) $\sqrt{2}\,F_a<F_c$, (d) $\sqrt{2}\,F_a>F_c$?

9. If ϱ is the diameter of the sphere of influence of a liquid, then the thickness of its surface film is (a) ϱ, (b) $\varrho/2$, (c) 2ϱ, (d) 3ϱ.

Ans. 1. b, 2. a, 3. c, 4. b, 5. b, 6. c, 7. d, 8. d, 9. b.

(B)

1. Show that the excess of pressure inside a soap bubble of radius r is $\frac{4T}{r}$ where T is the surface tension or the soap solution.

2. Explain why the surface of liquid near solid is curved.

3. What is Newmann's triangle ?

4. Explain why water spreads over a clean glass plate but not mercury.

(C)

1. What do you understand by surface tension and surface energy ? Explain the origin of surface tension in liquids.

2. What is meant by capillarity ? Explain why the surface of water in a glass tube is concave while that of mercury is convex.

3. Deduce an expression for the rise of liquid in a capillary tube. Describe an experiment to find the surface tension of water.

4. What is angle of contact ? Describe with theory a method of determining the surface tension of water.

5. Describe how you will determine the surface tension of a soap bubble. Deduce the formula used.

(D)

1. Water rises in a capillary tube to a height of 12 cm. If the surface tension of water is ·07 Nm^{-1}, find out the radius of the capillary tube.

(Ans. $1{\cdot}119\times10^{-4}$ m).

2. How high does water rise in a capillary tube whose inner diameter is $4{\cdot}4\times10^{-4}$ m if the angle of contact is negligible ? Surface tension of water is ·073 Nm^{-1}.

(Ans. $6{\cdot}7\times10^{-2}$ m)

3. If the angle of contact is 30°, the density of the liquid 890 kgm^{-3} and the surface tension ·03 Nm^{-1}, to which height will the liquid rise in a tube the bore of which has a diameter of ·3 mm ?

(Ans. $3{\cdot}96\times10^{-2}$ m)

4. Find the work done in breaking a ·5 cm radius water drop into drops of 1 mm radius each. (Surface tension of water = ·07 Nm^{-1})

(Ans. $8{\cdot}79\times10^{-5}$ joule)

5. The pressure of air in a soap bubble of ·7 cm diameter is 8 mm of water above the atmospheric pressure. Calculate the surface tension of the soap solution.

(Ans. ·0686 Nm^{-1})

6. Calculate the excess of pressure inside a drop of mercury of diameter 3 mm. (Surface tension of mercury = ·44 Nm^{-1}).

(Ans. 586·7 Nm^{-2}).

7. Calculate the depression of mercury in a capillary tube of radius ·2 mm if the surface tension of mercury is 55 Nm^{-1} and the angle of contact of mercury with glass is 120°. The relative density of mercury = 13·6. (Ans. 2·06 cm.)

[Hint. Neglecting v in the Eq. 12.3

$$2\pi rT\cos\theta=\pi r^2 h\rho g$$

or

$$2T\cos\theta=\rho ghr.\ ]$$

8. Calculate the work done in blowing a bubble of radius 10 cm (T = ·03 Nm^{-1}). What additional work will be performed in further blowing it to double the radius ?

(Ans. $7{\cdot}54\times10^{-3}$ J; $2{\cdot}26\times10^{-2}$ J).

(E)

1. The pressure inside a soap bubble is............(greater or less) than the pressure outside the bubble.

2. A liquid wets a solid when its angle of contact is............(acute or obtuse).

3. The force of attraction between molecules of different substances is called ....(adhesive or cohesive) force.

4. The force of attraction between molecules of the same substance is called ....(adhesive or cohesive) force.

5. Small floating bodies attract when they come closer. This is due to............ (surface tension, elasticity).

6. A liquid rests in equilibrium with another liquid when their surface tensions relative to air and also surface tension of one relative to the other are such that a triangle may be constructed. Is this true or false ?

7. A liquid does not wet a solid when its angle of contact in...............(acute or obtuse).

8. A soap bubble and a drop of soap water of the same radius have the same tendency to contract. True or false ?

9. Captain Haddock was pouring his favourite drink into his thirsty mouth from a bottle in a space-ship while his co-pilot inadvertently switched off the anti-gravity device. What happens to his drink ?

Ans : 1. greater, 2. acute, 3. adhesive, 4. cohesive, 5. surface tension, 6. true, 7. obtuse, 8. false, the bubble has greater tendency to contract as it has two surfaces. 9. Drink in between his mouth and bottle will turn into spherical bubble and the rest in the bottle will not fall into his mouth.

CHAPTER 13

HYDROSTATICS (or FLUID STATICS)

13.1. Fluids : Liquids and Gases

Hydrostatics (or Fluid Statics) deals with behaviour of fluids at rest. A fluid is a substance that can flow and therefore the term fluid includes liquids and gases. *A solid can sustain tangential forces (shearing stresses) but a fluid cannot sustain a tangential force* (or shearing stress). When a tangential force is applied to a solid body fixed at one end, it is sheared, but when a tangential force is applied to a fluid its layers would simply slide over one another. In fact, it is the inability of fluids to resist such tangential forces (or shearing forces) that gives them the ability to change their shape or to flow. *A liquid surface always rests at right angles to the force to which it is subjected due to this property.* This is why normally the free surface of liquid is horizontal.

13.2. Pressure and Density

The force acting on a fluid is described by the term *pressure which is defined as the magnitude of the normal force per unit surface area.* If $\triangle F$ is the force exerted by the fluid on an elementary area $\triangle S$, then the pressure of the fluid at the surface is defined as

$$p = \lim_{\triangle S \to 0} \frac{\triangle F}{\triangle S} \quad \text{.. (13.1)}$$

The total force exerted by a fluid on the whole of the area in contact with it is called the *thrust*.

Thus,

thrust = pressure × area

Pressure is a scalar quantity. Its unit is newton per square meter (Nm^{-2}) or pascal (Pa)

The density of a fluid is defined as the mass per unit volume of that fluid. It $\triangle m$ is the mass of the fluid in an elementary volume $\triangle v$ surrounding a point, then the density of the fluid at the point is defined as

$$\rho = \lim_{\triangle v \to 0} \frac{\triangle m}{\triangle v}$$

In a homogeneous fluid at rest the density is the same at all points. The unit of density is obviously kilogramme per cubic meter (kg m^{-3}) The density of a substance relative to water at 4°C is called the relative density or specific gravity. The relative density of a substance is a pure number. It has no unit. In SI the density of water is 1000 $kg\ m^{-3}$.

$\therefore$ *The density of any substance = The relative density of the substance × density of water at* 4°C.

13.3. Pressure at a Point in a Liquid

If a liquid is at rest every portion of the liquid is in equilibrium. Let us consider a small element of the liquid volume submerged within the body of the liquid. Let y be the depth of the element from the free surface of the liquid. The thickness of the element is dy and the top and the bottom have each an area A. The mass of this element is $\rho A\, dy$ and its weight is $\rho Adyg$. The forces exerted on the element by the surrounding liquid are perpendicular to its surfaces. The horizontal thrusts on the element neutralise but the vertical thrusts on the top and bottom faces are not equal in magnitude. If p be the pressure on the upper face and $p+dp$ the pressure on its lower face, the downward force is pA and the upward force is $(p+dp)\ A$.

For equilibrium of the element, the net upward force = the net downward force

or $\quad (p+dp)\ A = pA + \rho\ Adyg$

or $\quad dp = \rho\ gdy.$

If p_0 be the pressure at the free surface where $y=0$ and p be the pressure at a depth y then

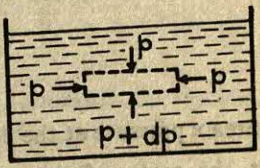


Fig. 13.1

$$\int_{p_0}^{p} dp = \int_{0}^{y} \rho\ gdy$$

or $\quad p - p_0 = \rho\ gy$

or $\quad p = p_0 + \rho\ gy \quad \text{.. (13.2)}$

Due to the liquid only, the pressure is

$$p = \rho\ gy \quad \text{.. (13.2a).}$$

13.4. Centre of Pressure

The thrust on the walls and the base of a vessel containing a liquid is the resultant of the thrusts on its elementary areas. The point at which the resultant thrust is effective is called the *centre of pressure*.

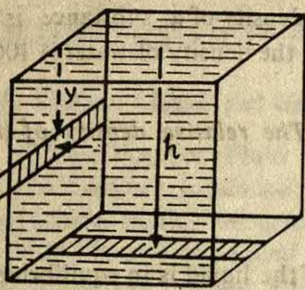


Fig. 13.2

Consider a cubical vessel h on each side and of area S on each surface. Consider a thin strip of area $\triangle S$ at the base.

The thrust on $\triangle S = \triangle S \times \rho\, gh$.

$\therefore$ the total thrust on the base $= \Sigma \triangle S\, \rho\, gh = \rho\, gh\, \Sigma \triangle S = \rho\, gh\, S$,

= pressure at the base $\times$ area of the base.

To find the thrust on the side wall consider a thin strip of width dy at a depth y.

the thrust on the strip = area $\times$ pressure.

$$= (dyh)\, (\rho\, gy) = \rho\, ghydy.$$

$\therefore$ the total thrust on the entire wall $= \int_0^h \rho\, ghydy$.

$$= \tfrac{1}{2}\, \rho g h^3.$$

$$= \left(\rho\, g . \frac{h}{2}\right) h^2.$$

$$= (\tfrac{1}{2}\, \rho g h) \times S.$$

= area $\times$ pressure at the centre of the wall.

Let the centre of pressure on the side wall be at a depth h'. It is a fact that the sum of moments of the thrusts on elementary areas about the top edge is equal to the moment of the resultant thrust about the same edge.

the moment of the thrust on the elementary area = force $\times$ perpendicular distance.

$$= \rho\, ghydy \times y = \rho\, ghy^2\, dy.$$

$\therefore$ the total moment of the elementary thrusts

$$= \int_0^h \rho\, ghy^2\, dy$$

$$= \tfrac{1}{3}\,\rho\, gh^4$$
$$= \tfrac{1}{3}\,\rho\, gh^2\, S \;(\because\;\; S = h^2)$$
$$\therefore\;\; (\tfrac{1}{2}\,\rho\, gh\, S) \times h' = \tfrac{1}{3}\, gh^2\, S$$

or $$h' = 2/3h \qquad .. \quad (13.3)$$

13.5. Laws of Fluid Pressure

(*a*) *A fluid at rest exerts pressure equally in all directions.*

(*b*) *The pressure at any point on the same horizontal line in a fluid at rest is the same.*

(*c*) *The pressure at a point of a liquid at rest is proportional to the depth of the point from the free surface of the liquid.*

13.6. Pascal's Law : The Hydraulic Press (Bramah's Press)

One of the natural consequences of the fundamental laws of fluid mechanics is Pascal's law which states that the *pressure exerted anywhere in a mass of a confined liquid is transmitted undiminished in all directions throughout the mass so as to act undiminished at right angles to the surface of the vessel exposed to the liquid.*

This is simply a consequence of the law, namely, Eq. 13.2 and the incompressibility of liquids. We have seen that the pressure inside a liquid at a depth h is given by

$$p = p_0 + \rho\, gh$$

where p_0 is the external pressure.

Let us increase the external pressure by an arbitrary amount $\triangle p_0$. Since liquids are virtually incompressible, the density ρ remains essentially constant during the process.

$\therefore\;\; \triangle p = \triangle p_0$, i. e., the increase in external pressure is equal to the increase in pressure at any point inside the liquid. According to this law a small force applied somewhere on a confined mass of liquid will appear as a very big force on the wall of the container. This is exactly what is done in the machine—the Hydraulic Press (Bramah's Press).

The machine consists of two stout metallic cylinders A and B connected by a stout metallic tube D. In the cylinder A a small piston Q is moved up and down by a lever L. In the cylinder B there is a thick solid cylinder R capable of moving up and down. In fact, this cylinder works as a ram. Any material intended to be compressed (to be rammed) is placed on the top of the platform and is

compressed against a fixed girder G supported on strong pillars. In the cylinder A two valves work : one at the bottom and the other on the side. The valve V_1 at the bottom allows water to flow from the tank T to the cylinder A, but not from A to T. The other valve V_2

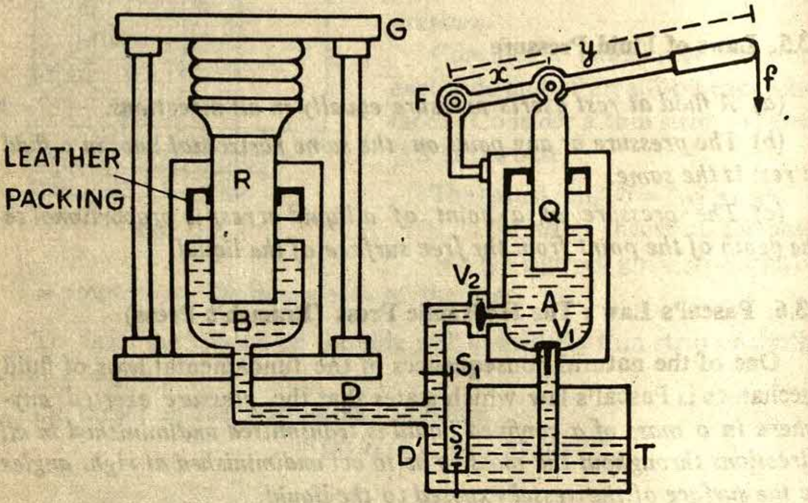


Fig. 13.3

allows water to flow from A to B but not from B to A. The connecting tube D is connected to the tank by a second tube D'. There is stop-cock (S_2) near the water tank and another (S_1) near the cylinder A. To make R and Q work water-tight, a leather packing having the form of an inverted cup is inserted in the annular recess in the body of the cylinders A and B. Leakage of water under pressure was a great problem in this machine. The idea of leather packing to make the pistons water-tight was given by the engineer Bramah and so the press is sometimes called Bramah's press.

Working. In the beginning the ram R is at its lowest position. The cylinders A and B and the connecting tubes are full of water. The material, such as cotton bales, jute bales etc. to be compressed is stacked on the top-platform of the ram. The lever L is then moved up and down manually or electrically. When the piston Q is raised by the lever, the pressure inside A decreases and so water from the tank enters into A pushing open the valve V_1. During the downstroke the valve V_1 closes, preventing the flow of water from the cylinder A back to the tank T. The valve V_2 opens under pressure and water is forced into the cylinder B through the connecting pipe

D. The ram with the load on its platform rises slowly towards the girder. When the material on the platform touches the girder, the actual operation of the machine starts. Now, as the piston Q is worked, Pascal's law becomes fully effective. Before this stage Pascal's law was not hundred per cent effective, i. e., the pressure was transmitted no doubt but not undiminished. Now since water is closed from all sides, i. e., water is really confined in a vessel of constant volume, the pressure exerted by the piston is transmitted undiminished on to the ram and so the latter presses (rams) the material on its platform with a great force against the girder.

To release the cotton bales after compression, the stop-cock S_2 is opened, when water from B rushes into the tank T under pressure and the ram comes down by its own weight.

Mechanical advantage of the machine. The ratio of the thrust generated by the machine to the force applied to the machine is called its mechanical advantage.

Let f be the force applied at the end of the lever of the machine and f' is the force generated on the piston Q, then the moment of f about the fulcrum is the same as the moment of f' about the fulcrum. If y is the length of the lever and x is the distance of the point at which the piston Q is attached to the lever, then

$$f \times y = f' \times x$$

or
$$f' = \frac{fy}{x}$$

If α be the area of cross-section of the piston Q, then the pressure

exerted on the water $= \frac{f'}{\alpha} = \frac{f}{\alpha} \cdot \frac{y}{x}$

This pressure is transmitted undiminished on to the ram (Pascal's law). If β be the area of cross-section of the ram, then

the thrust generated on the ram $= \beta \frac{f}{\alpha} \frac{y}{x}$

$$= f \cdot \frac{\beta}{\alpha} \frac{y}{x}$$

$\therefore$ Mechanical advantage of the machine $= \dfrac{\text{thrust generated}}{\text{force applied}}$

$$= \frac{f \cdot \frac{\beta}{\alpha} \cdot \frac{y}{x}}{f}$$

or $$\text{Mech. advantage} = \frac{\beta}{\alpha} \cdot \frac{y}{x} \quad \ldots (13.4)$$

13.7. Archimedes' Principle : Buoyancy

Another natural consequence of the laws of fluid statics is Archimedes' principle.

PRINCIPLE :

When a body is wholly or partly immersed in a fluid at rest, the body is buoyed up with a force equal to the weight of the fluid displaced by the body.

When a body is wholly or partly immersed in a fluid at rest, the fluid exerts pressure on every part of the body's surface in contact with the fluid. The pressure is greater on the parts immersed more deeply (the third law of fluid pressure). So the thrusts on all sides result in an upward force called *buoyancy*. As a result of this the body is buoyed up, i. e., made lighter.

Consider a solid rectangular block inside a liquid with its top face parallel to the free surface of the liquid at a depth h. The thrusts on the vertical faces of the block are equal and opposite and so they neutralise each other. The top surface experiences a thrust downward and the bottom surface experiences an upward thrust.

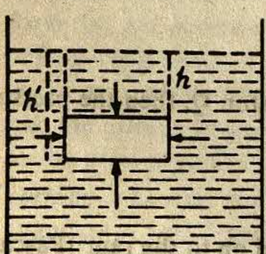


Fig. 13.4

The downward thrust on the top surface

$$= \text{pressure} \times \text{area}$$

$$= (p_0 + \rho\, gh) \times A$$

The upward thrust on the bottom surface

$$= (p_0 + \rho gh') \times A$$

$\therefore$ The resultant upward thrust

$$= (p_0 + \rho gh')A - (p_0 + \rho gh)\, A$$

$$= \rho g\, (h' - h)A$$

But $A(h'-h)$ is the volume of the block; so the resultant upward thrust, that is, the buoyancy is equal to the weight of the liquid displaced by the block.

The most useful application of this principle is the determination of the volume of a body of irregular shape.

Example

1. *A body weighs 2 kg in air and 1·75 kg. in water. Find the volume and density of the solid.*

Sol. Let V be the volume of the solid

$$\text{Buoyancy} = V \times 1000\, g. \quad (\because \text{ density of water} = 1000 \text{ kgm}^{-3})$$
$$= \text{Loss in weight}$$
$$= 2\, g - 1{\cdot}75\, g = {\cdot}25\, g$$

or $$1000\, V = {\cdot}25$$
$$V = 25 \times 10^{-5} \text{ cubic metre} \quad \textbf{Ans.}$$

$$\text{Density} = \frac{\text{Mass}}{\text{Volume}} = \frac{2}{25 \times 10^{-5}} = 8000 \text{ kgm}^{-3} \quad \textbf{Ans.}$$

13.8. Some Interesting Cases on 'Buoyancy' and Downward Thrust

1. A beaker containing water is weighed on a balance. Now a body is immersed in water and suspended by a thread from an external support. What happens to the beam of the balance ?

The beam tilts on the side of the beaker. When the solid is immersed in water, it is buoyed up by water. By Newton's third law the body in its turn exerts an equal force on the water in the downward direction. This accounts for the tilting of the beam on the side of the beaker.

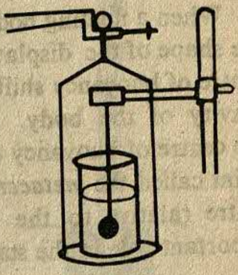


Fig. 13.5

2. A beaker containing water is placed on the pan of a balance. A piece of stone is placed on the same pan outside the beaker and the two are counterpoised. Now the stone is put inside the beaker. What happens to the equilibrium of the balance ? The equilibrium of the balance remains undisturbed. The buoyancy of the stone is balanced by the reaction of the buoyancy.

3. In the above example 2, if a lump of candy sugar is used instead of a stone and is put in the beaker after counterpoising the

two, what happens to the equilibrium of the balance ?

In the beginning the equilibrium of the balance remains undisturbed. As sugar melts the beam will tilt more and more to the side of the beaker.

4. In the example 1 if the body be a lump of sugar candy, what happens to the equilibrium of the beam ? The beam tilts at once on the side of the beaker. As sugar melts, it will tilt more and more because when sugar melts, its buoyancy vanishes; but it becomes a part of the sugar solution in the beaker. So its weight is added to the weight of the beaker and water and consequently the system (beaker + water + sugar) weighs more.

13.9. Equilibrium of a Floating Body : Its Stability and Metacentre

CONDITIONS (LAWS) OF EQUILIBRIUM OF A FLOATING BODY

1. The weight of the floating body is equal to the weight of the liquid displaced.

2. The centre of gravity of the body and the centre of gravity of the displaced liquid (called the centre of buoyancy) must lie on the same vertical line.

The first condition is needed for translational equilibrium and the second condition is needed for rotational equilibrium of the body.

STABILITY OF EQUILIBRIUM AND METACENTRE

When a floating body is tilted a little from its equilibrium position, the shape of the displaced liquid is changed and consequently the centre of buoyancy shifts to a new position B' but not the centre of gravity of the body. The vertical line through the new position of the centre of buoyancy cuts the central line of the body at some point called the *metacentre* of the body The position of the metacentre ralative to the centre of gravity of the body plays a very important role in the stability of equilibrium of the body.

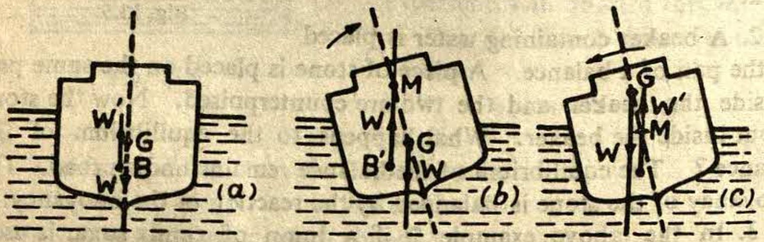


Fig. 13.6

When the metacentre M lies above the centre of gravity of the body (Fig. 13.6 b) the couple formed by the weight (W) of the body and the upward thrust (W') tends to restore the body to its position of equilibrium (Fig. 13.6a) Hence in this case when the metacentre lies above the centre of gravity of the body, the equilibrium of the body is *stable*.

When the metacentre M lies below the centre of gravity of the body (Fig. 13.6 c), the couple formed by W and W' tends to overturn the body. Hence the equilibrium in this condition is unstable.

13.10. Atmospheric Pressure : Fortin's Barometer and Aneroid Barometer

The gaseous layer which surrounds the earth is called the *atmosphere*. It is held to the earth by the action of gravity. The atmosphere is a mechanical mixture of gases mainly nitrogen and oxygen. The pressure of the gaseous mixture of the atmosphere is called the *atmospheric pressure*. Since gravity is the only force on the atmosphere, *the pressure of the atmosphere is obviously the weight of the vertical column of air of unit cross-section and height equal to that of the atmosphere.*

A way to measure atmospheric pressure was given by Evangelista Torricelli (1607-1647). A long glass tube about 1 m long is filled with mercury and then inverted in a dish of mercury as in Fig. 13.7. The mercury stands in the tube after falling a little. Generally, at the sea-level it stands up to 76 cm. This column of mercury is forced to stand in the tube by the atmospheric pressure. To see this, draw a horizontal line immediately below the free surface of mercury in the dish and consider two points A and B : A outside the tube and B inside the tube. According to second law of fluid mechanics, namely, *the pressure at any two points on the same horizontal line in a fluid at rest is the same.*

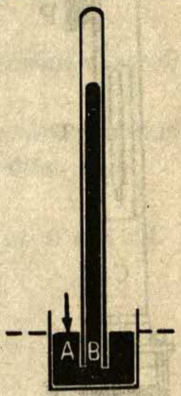


Fig. 13.7

Pressure at A = Pressure at B.

Pressure at A = Atmospheric pressure (P)

and Pressure at B = pressure due to the column of mercury + pressure of mercury vapour from top.

The space above the mercury column contains only mercury vapour whose pressure is so small at ordinary temperatures that it can be neglected. Hence the pressure at $B = \rho gh$ where ρ = density of mercury, h = height of mercury column in the tube.

$$\therefore \quad P \text{ (atmospheric pressure)} = \rho gh \text{ Nm}^{-2} \quad \ldots \quad (13.5)$$

Thus this simple arrangement gives the measurement of the atmospheric pressure. This is called a *Simple barometer*.

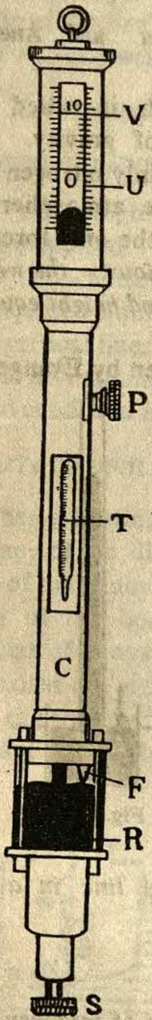


Fig. 13.8

Fortin's Barometer. The Fortin's barometer is an improved form of the simple barometer. In this barometer the tube AB is enclosed within a long brass casing C. A rectangular slit is cut in the upper part of the brass casing so that the level of mercury in the tube may be visible from outside. A scale in cm is engraved on the edge of the slit. The vernier V of the scale is moved up and down by turning the knob P of the rack and pinion arrangement.

The cistern containing mercury is made of a glass cylinder fitted in a wooden or metal cylinder, whose lower end is closed by chamois leather. A wooden block resting on the tip of a screw is attached to the leather. This arrangement makes the cistern a vessel of adjustable volume. The level of mercury can be raised or lowered by working the screw and made to touch an ivory pointer F hanging from the ceiling of the cistern. The zero of the scale engraved on the top starts from here. The barometer tube is of wide bore at the upper portion so that the effect of surface tension is avoided. The tube bulges near the lower end and then gradually tapers to a point It is the 'bulge' which rests on a padded seat and keeps the tube in position. The tapering of the lower end prevents the

oscillations of the mercury level during adjustment. The whole instrument is fitted on a wooden board and hung against a vertical wall from a peg in the laboratory.

To read the barometer, the base screw S is worked until the mercury surface in the cistern just touches the tip of the ivory pointer. Then the vernier V is moved along the main scale until its lower edge appears tangential to the convex mercury surface. The total reading (main scale reading + vernier scale reading) of the scale gives the barometer reading at the time of observation.

This reading requires correction on account of : (a) expansion of the brass scale, (b) change of density of mercury, (The standard pressure is the pressure due to zero-degree cold mercury), (c) correction for the sea-level. As the acceleration due to gravity decreases with height above the sea level and the calculation of pressure requires g ($P=h\rho g$), the standard place for g has been fixed as the sea-level at latitude 45°.

Let h cm be the observed height at t°C and the brass scale be correct at 0°C.

Then 1 cm of the scale at 0°C is exactly 1 cm

and 1 cm of the scale at t°C is exactly $(1+\alpha t)$

where α is the linear expansivity of brass.

$\therefore$ h cm at t°C is exactly $h(1+\alpha t)$.

Thus the correct height (after correction of expansion of scale)

$$=h(1+\alpha t)$$

Let h be the height of zero-degree cold mercury corresponding to the atmospheric pressure (P) at the time of observation.

$$\text{Then } P=h_o\rho_o g=h(1+\alpha t)\rho_t g$$

where ρ_o and ρ_t are the densities of mercury at 0 and t°C respectively.

$$\text{or} \qquad h_o=h(1+\alpha t)\frac{\rho_t}{\rho_o}$$

If γ be the cubical expansivity of mercury then

$$\rho_o=\rho_t(1+\gamma t)$$

$$\therefore \quad h_o=h(1+\alpha t)\frac{\rho_t}{\rho_t(1+\gamma t)}=h(1+\alpha t)(1+\gamma t)^{-1}$$

or $h_o=h(1+\alpha t)(1-\gamma t)$ (neglecting terms containing higher powers of γ)

$= h(1+\alpha t-\gamma t)$ (again neglecting the $\alpha\gamma t^2$ term as it is very small)

or $\quad h_o = h[1-(\gamma-\alpha)t]$ $\quad$ (13.6)

The correction for sea-level at 45° latitude is beyond the scope of the book. Here we note down the result :

The standard height of barometer

$$= h[1-(\gamma-\alpha)t]\,[1-{\cdot}00264405\cos 2\lambda - 3\times 10^{-7}H]$$

where λ = latitude of the place and H = height of the place above the sea-level in metre. The acceleration due to gravity at the sea-level, 45° latitude is 9·80616 ms^{-2}.

The Aneroid Barometer. The Fortin's barometer is very sensitive, but is very bulky and so it is not portable. In parallel to this we have the aneroid barometer which is small in size and so is easily portable, though not so sensitive. It uses no liquid and hence the

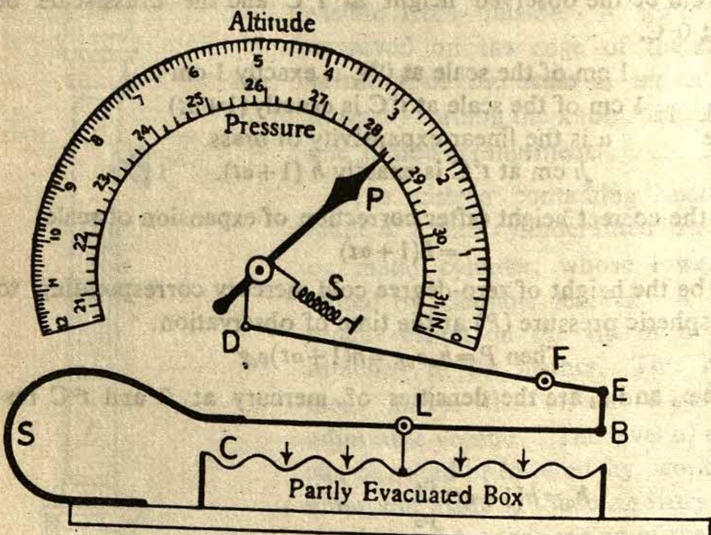


Fig. 13.9

name aneroid barometer (aneroid means no liquid). It consists of a cylindrical box which is partially exhausted of air and closed from the top by a corrugated tin diaphragm so that it may yield easily to changes of external pressure. The depression or elevation of the diaphragm takes place in proportion to the changes of atmospheric

pressure and the change is indicated by a pointer connected to a multiplying system of levers.

Examples

1. ***Calculate the atmospherie pressure in absolute units (Nm^{-2}) when the barometer height is 76 cm.***

Density of mercury $= 13{\cdot}6 \times 10^3$ kgm^{-3} and $g = 9{\cdot}8$ ms^{-2}.

Sol. Atmospheric pressure = Pressure due to 76 cm of mercury.

$= {\cdot}76 \times 9{\cdot}8 \times 13{\cdot}6 \times 10^3$ ($\because P = \rho g h$).

$= 101292{\cdot}8$

$= 1{\cdot}013 \times 10^5$ Nm^{-2} **Ans.**

This pressure, i.e., $1{\cdot}013 \times 10^5$ Nm^{-2} is often termed as '*one atmosphere*' (atm).

2. ***A U-tube is partly filled with water. Another liquid, which does not mix with water, is poured into one side until it stands 5 cm above the water level on the other side, which has meanwhile risen a distance 10 cm. Find the density of the liquid.***

Sol. Consider a horizontal line *AB* through the point of contact.

By the second law of fluid pressure

Pressure at A = Pressure at B

or ${\cdot}25 \times \rho \times 9{\cdot}8 = {\cdot}20 \times 1000 \times 9{\cdot}8$

($\because$ density of water $= 1000$kgm^{-3})

or $\rho = \dfrac{{\cdot}20}{{\cdot}25} \times 1000$

$= 800$ kgm^{-3} **Ans.**

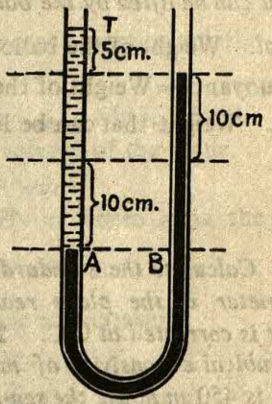


Fig. 13.10

3. ***The two limbs of an inverted U-tube provided with a stopcock in the side tube near the bend are dipped in two breakers, one containing water and the other a liquid. A little air is then sucked out by opening the stop-cock which is subsequently closed. The water and the liquid rise in the limbs up to 20 cm and 16 cm from their respective free surface. Calculate the relative density of the liquid.***

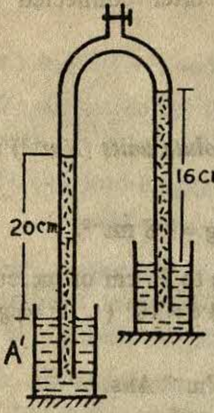


Fig. 13.11

Sol. Considering equilibrium of water column we have

P = (atmospheric pressure)

$= p_o + \cdot 20 \times \rho_w g$

where p_o = pressure of the enclosed air.

Considering equilibrium of the liquid column, we have

or $P = p_o + \cdot 16 \times \rho g$

$\therefore \quad p_o + \cdot 20 \rho_w g = p_o + \cdot 16 \times \rho g$

or $\dfrac{\rho}{\rho_w} = \dfrac{20}{16} = 1 \cdot 25$ **Ans.**

This example illustrates the way of finding the density of a liquid miscible with water. The arrangement is known as Hare's appratus.

4. *The volume of a balloon is* 500 m^3. *It is filled with hydrogen whose density is* ·089 kgm^{-3}. *The density of air is* 1·25 kgm^{-3}. *What weight can be lifted by the balloon ?*

Sol. Weight of the balloon $= 500 \times \cdot 089\ g$ newton

Buoyancy = Weight of the displaced air $= 500 \times 1 \cdot 25\ g$ newton

$\therefore$ Weight that can be lifted $= 500 \times 1 \cdot 25\ g - 500 \times \cdot 089\ g$

$= 500 \times (1 \cdot 25 - \cdot 089)\ g$ newton

$= 500 \times 1 \cdot 161$ kg.

$= 580 \cdot 5$ kg. **Ans.**

5. *Calculate the standard height at a place of latitude* 23° *when the barometer of the place reads* 70 *cm. at* 30°*c. The scale of the barometer is corrected at* 0°C. *The linear expansivity of brass* 18×10^{-6} *and cubical expansivity of mercury* 18×10^{-5} *and the height of the place is* 450 *m above the sea-level.*

Sol. The standard formula for correction is

$$h_{standard} = h[1 - (\gamma - \alpha)t]\ [1 - 2 \cdot 644 \times 10^{-3} \cos 2\lambda - 3 \times 10^{-7} \times H]$$

or $h_{standard} = \cdot 70[1 - (180 - 18) \times 10^{-6} \times 30]$

$$[1 - 2 \cdot 644 \times 10^{-3} \cos 46° - 3 \times 10^{-7} \times 450]$$

$$= \cdot 7[1 - 162 \times 3 \times 10^{-5}]\ [1 - 2 \cdot 644 \times \cdot 6947 \times 10^{-3} - 135 \times 10^{-6}]$$

$$= \cdot 7[1 - 486 \times 10^{-5}]\ [1 - 1 \cdot 8368 \times 10^{-3} - \cdot 135 \times 10^{-3}]$$

$$= \cdot 7[1-486\times10^{-5}]\,[1-1\cdot9718\times10^{-3}]$$
$$= \cdot 7[1-486\times10^{-5}-1\cdot9718\times10^{-3}+486\times1\cdot9718\times10^{-8}]$$
$$= \cdot 7[1-486\times10^{-5}-197\cdot18\times10^{-5}+\cdot96\times10^{-5}]$$
$$= \cdot 7[1-682\cdot18\times10^{-5}]$$
$$= \cdot 692 \text{ m. } \textbf{Ans.}$$

6. *A boat loaded with a stone floats on the surface of still water of a lake. When the stone is transferred from the boat to the lake water, what will happen to the water-level?*

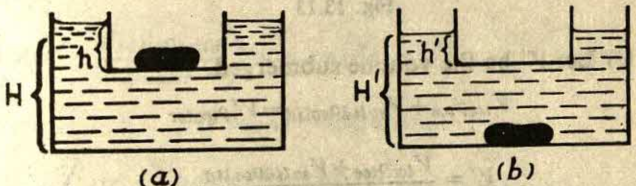


Fig. 13.12

Sol. Let H and H' be the height of the water level from the datum plane before and after the stone is transferred to the lake water.

Let h and h' be the length of the submerged portion of the boat in the two cases.

In the first case $(M_{boat}+V\rho_{stone})g = h\alpha\rho_{water}g$ where α is the area of cross-section of the boat and V is the volume of the stone.

or $$M_{boat}+V\rho_{stone} = h\alpha\rho_{water} \qquad \text{(i)}$$

Now the volume of water in lake $= AH-\alpha h$ where A is the area of cross-section of the lake.

After the stone is removed,

$$M_{boat} = h'\alpha\rho_{water} \text{ by law of floatation.} \qquad \text{(ii)}$$

The volume of water in the lake $= AH'-\alpha h'-V$.

But the volume of water in the lake remains the same.

$\therefore$ $$AH-\alpha h = AH'-\alpha h'-V$$

or $$A(H-H') = \alpha h-\alpha h'-V$$

$$= \left(\frac{M_{boat}}{\rho_{water}}+\frac{V\rho_{stone}}{\rho_{water}}\right)-\frac{M_{boat}}{\rho_{water}}-V$$

or $(H-H') = \dfrac{V}{A}\left(\dfrac{\rho_{stone}}{\rho_{water}}-1\right)$. This is always positive.

$\therefore$ $H > H'$, i.e., the level will fall. **Ans.**

7. *A piece of ice containing a solid in it floats in a glass of water. How will the level of water be affected if the solid is heavier than water and if it is lighter than water ?* **(I.I.T. 1973).**

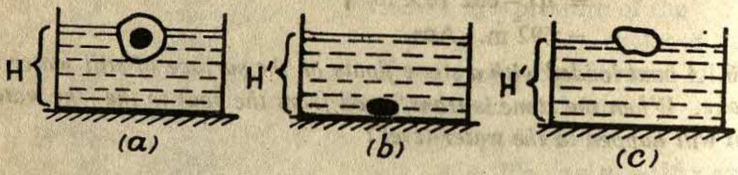


Fig. 13.13

Sol. (*i*) Let V' be the volume submerged.

Then $$V_{ice}\rho_{ice}+V_{solid}\rho_{solid}=V'\rho_{water}$$

or $$V'=\frac{V_{ice}\rho_{ice}+V_{solid}\rho_{solid}}{\rho_{water}} \quad \dots \quad \text{(i)}.$$

Mass of water in the glass $=(AH-V')\,\rho_{water}$ where A is the area of cross-section of the glass.

When the ice melts, the mass of water in the glass

$$=(AH'-V_{solid})\,\rho_{water}$$

Also the mass of water in the glass after the ice melts = previous mass of water + mass of water formed by conversion of ice into water.

$$=(AH-V')\,\rho_{water}+V_{ice}\rho_{ice}$$

$\therefore \quad (AH-V')\,\rho_{water}+V_{ice}\rho_{ice}=(AH'-V_{solid})\,\rho_{water}$

or $\quad AH-V'+\dfrac{V_{ice}\rho_{ice}}{\rho_{water}}=AH'-V_{solid}$

or $\quad A(H-H')=V'-\dfrac{V_{ice}\rho_{ice}}{\rho_{water}}-V_{solid}$

$$=\frac{V_{ice}\rho_{ice}+V_{solid}\rho_{solid}}{\rho_{water}}-\frac{V_{ice}\rho_{ice}}{\rho_{water}}-V_{solid}$$

or $\quad A(H-H')=V_{solid}\left(\dfrac{\rho_{solid}}{\rho_{water}}-1\right)$. This is always positive.

$\therefore \quad H-H'>0$ or $H>H'$, i.e., level will go down. **Ans.**

(ii) When the solid is lighter than water, the solid will float after the ice melts.

Let V'' be the volume of the submerged (Fig 13.16 c) portion of the solid.

Now the mass of water in the glass $=(AH'-V'')\,\rho_{water}$

By the law of floatation $V''\rho_{water}=V_{solid}\rho_{solid}$.

$\therefore$ mass of water in the glass $=(AH'-V'')\,\rho_{water}$

$$=\left(AH'-\frac{V_{solid}\rho_{solid}}{\rho_{water}}\right)\rho_{water}$$

$$\therefore\quad \left(AH'-\frac{V_{solid}\rho_{solid}}{\rho_{water}}\right)\rho_{water}=(AH-V')\,\rho_{water}+V_{ice}\rho_{ice}$$

or $$AH'\rho_{water}-V_{solid}\rho_{solid}$$

$$=\left(AH-\frac{V_{ice}\rho_{ice}+V_{solid}\rho_{solid}}{\rho_{water}}\right)\rho_{water}+V_{ice}\rho_{ice}$$

or $$AH'\,\rho_{water}-V_{solid}\rho_{solid}$$

$$=AH\rho_{water}-V_{ice}\rho_{ice}-V_{solid}\rho_{solid}+V_{ice}\rho_{ice}$$

or $$H'=H$$

i.e., there will no overflow of water. **Ans.**

8. *Water stands at a depth D behind the vertical upstream face of a dam. Find the resultant force exerted on the dam by the water and the torque about O by this force. Take W as the width of the dam.*

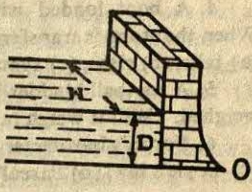


Fig. 13.14

Sol. The average pressure on the dam = pressure at the centre of the submerged portion of it.

$=\rho g\dfrac{D}{2}$ $\left(\text{because the centre of the submerged portion of the dam is at a depth } \dfrac{D}{2}\right)$.

The resultant force on the dam $=$ area $\times$ average pressure

$$=(D\times W)\times\tfrac{1}{2}\rho gD$$
$$=\tfrac{1}{2}\rho gD^2W.\quad \textbf{Ans.}$$

This force is effective at a distance 1/3 D up from the bottom.

$\therefore$ The torque about O = Force $\times$ perpendicular distance

$$=(\tfrac{1}{2}\rho gD^2W)\times\tfrac{1}{3}D$$
$$=\tfrac{1}{6}\rho gD^3W.\quad \textbf{Ans.}$$

9. *A swimming pool has the dimensions* $24m \times 9m \times 2{\cdot}4m$. *When it is filled with water, what is the force on the bottom ? On the sides ? On the ends ?*

Sol. Thrust on the bottom = area × pressure

$$=(24\times 9)\times(1000\times 9{\cdot}8\times 2{\cdot}4)=5{\cdot}08\times 10^6\text{N. Ans.}$$

$$\text{Thrust on the sides}=(24\times 2{\cdot}4)(1000\times 9{\cdot}8\times \frac{2{\cdot}4}{2})={\cdot}68\times 10^6\text{N. Ans.}$$

$$\text{Thrust on the ends}=(9\times 2{\cdot}4)(1000\times 9{\cdot}8\times \frac{2{\cdot}4}{2})={\cdot}254\times 10^6\text{N. Ans.}$$

QUESTIONS

(A)

1. The equilibrium of a floating body is stable if (a) the centre of gravity of the body is below the metacentre, (b) the metacentre is below the centre of gravity, (c) the metacentre is coincident with the centre of gravity, (d) the centre of buoyancy is above the metacentre.

2. The centre of pressure on a vertical wall of height h immersed in water is at a depth (a) 1/3 h, (b) 2/3 h, (c) 3/4 h, (d) 1/2 h from the free surface of the water.

3. If the effort arm is 8 times the load arm of the lever of a hydraulic press and the area of the ram is 15 times the area of the pistion, then the mechanical advantage of the machine is (a) 15/8, (b) 8/15, (c) 120, (d) 90.

4. A boat loaded with a stone floats on the surface of still water of a lake, When the stone is transferred from the boat to the lake water, the water level (a) falls, (b) rises, (c) remains unaffected.

5. A football bladder is weighed on a balance. It is then inflated and is again weighed. Will it weigh (a) more, (b) less, (c) the same ?

6. Fortin's barometer works on (a) Hooke's law, (b) Fluid pressure law. (c) Boyle's law. (d) Pascal's law.

(**Ans. 1. a. 2. b. 3. c. 4. a. 5. c. 6. b.**)

(B)

1. Define pressure and density at a point.
2. Explain what is meant by 'buoyancy'. How does it arise ? (**Bih. 1970, '75**)
3. Explain what do you understand by the term 'atmospheric pressure'.
4. Define the centre of pressure. Show that the centre of pressure on a vertical wall is at a depth two-thirds of the full depth of the liquid.
5. What corrections are necessary in the readings of a Fortin's barometer ?
6. Describe an aneroid barometer.

(C)

1. Explain with a neat diagram the principle and action of a Hydraulic (Bramah) press.

What is its mechanical advantage ?

Does it violate the principle of conservation of energy ? Justify your statement.

2. Discuss the stability of equilibrium of a floating body. Apply your result to the case of a uniform sphere of wood floating on water. (Bih. 1969 A; Mag '73 S)

3. Describe with a neat diagram a Fortin's barometer and explain how you would measure the atmospheric pressure with it.

Why is a thermometer attached to a standard barometer ?

(D)

1. The neck and bottom of a bottle are 1·25 cm and 10 cm in diameter respectively. When the bottle is full of oil, the cork in the neck is pressed with a force of 1 kg. wt. What force is exerted on the bottom of the bottle ?

(Ans. 64 kg f where 1 kg $f=g$ newton)

2. The cross-sections of the two pistons of a Bramah's press are 1·5 cm^2 and 60 cm^2. The piston of the pump is operated by a lever whose arms are 5 cm and 70 cm. If in each stroke the end of the lever rises by 30 cm , find the number of strokes required to raise the piston by 2 cm. (Ans 37·3)

3. Express the normal pressure in absolute units. ($g=9{\cdot}8\ ms^{-2}$.) (Mag. 1975S)

(Ans. $1{\cdot}013\times10^5\ Nm^{-2}$)

4. A faulty Fortin's barometer reads 750 and 748 mm when the true atmospheric pressure are 760 and 756 mm respectively. Calculate the true pressure when it reads 730 mm. (Ans. 732·85 mm)

5. A cube of wood supporting a 200 gm mass just floats in water. When the mass is removed, the cube rises by 2 cm. What is the size of the cube ?

(I. I. T. 1978)

(Ans $l=10$ cm)

6. A rod of length 6 m has a mass of 12 kg. It is hinged at one end at a distance of 3 m below a water surface. (i) What weight must be attached to the other end of the rod so that 5 meters of the rod is submerged ? (ii) Find the magnitude and direction of the force exerted by the hinge on the rod. (The specific gravity of the material of the rod = ·5). (I. I. T. 1976)

(Ans. 2·33 kg; 5·67 kg wt downward)

7. A large block of ice 5 m thick has a vertical hole drilled through it and is floating in the middle of a lake. What is the minimum length of a rope required to scoop up a bucketful of water through the hole ? (Density of ice = 900 kgm^{-3})

(I. I. T. 1975)

(Ans. ·5 m)

8. A cubical block of steel (density 7800 kgm^{-3}) floats on mercury (density 13600 kgm^{-3}) with its sides vertical. Assuming the side of the cube to be ·1 m (a) What lengh of the block is above the mercury surface ? (b) If water is poured on the mercury surface, what will be the height of the water column when the water surface just covers the top of the steel block ? (I. I. T. 1971)

(Ans. ·0426 m; ·046 m)

(E)

1. A piece of ice is floating in a beaker filled to the brim with water. Will there be an overflow of water as the ice melts ?

2. A sudden decrease in the barometer reading indicates................(storm fine weather).

3. An ice cube containing lead shot in it floats in a glass of water. How will the level of water be affected when the ice cube melts ? (I. I. T.)

4. A piece of ice floating in a beaker filled to the brim with a liquid of density 1500 kgm^{-3}. Will there be an overflow of water as the ice melts ? (I. I. T. 1977)

5. A piece of cork is embedded inside an ice block which floats in water. What will happen to the level of water when all the ice melts ? (I. I. T. 1976)

6. A man in a boat drinks water from a lake. Will the level of water of the lake fall or rise ?

7. An empty boat collects stones lying at the bottom of a lake. Will the level rise or fall ?

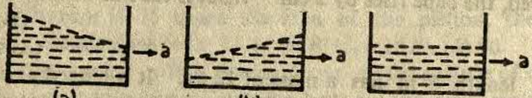
8. Does Archimedes' law hold in a vessel in free fall ? In a satellite moving in a circular orbit ?

9. Two bodies of equal weight and volume and having the same shape, except that one has an opening at the bottom and the other is sealed, are immersed to the same depth in water. Is less work required to immerse one than the other ? If so, which one and why ?

10. A soft plastic bag weighs the same when empty as when filled with air at atmospheric pressure. Why ?

11. Very often a sinking ship will turn over as it becomes immersed in water.

12. A vessel containing water is given a constant horizontal acceleration '*a*' towards the right along a straight path. Will the free surface turn clockwise, anticlockwise or remain horizontal ? Why ? (I. I. T. 1981)



(a) (b) (c)

(Ans. 1. No. 2. Storm. 3. the level will fall. 4. the liquid will overflow. 5. No overflow. 6. No change in the level of water. 7. the level will rise. 8. No. No. 9. The one having a hole because water will enter and the buoyancy will be less. 10. Because the buoyancy of air will equal the weight of the air pumped into the bag. 11. The c. g. will go above the centre of buoyancy—a condition for unstable equilibrium. 12. It will turn clockwise : the inertial force *ma* to the left and gravitational force *mg* on a particle of mass *m* on the surface will give rise to a resultant force on the particle having inclination $\tan^{-1} g/a$ with the horizontal. A liquid level always rests perpendicular to the force (resultant) to which it is subjected.)

CHAPTER 14

VISCOSITY

14.1. Viscosity : Coefficient of Viscosity

If water in a tub is whirled and then left to itself, the motion of the water subsides. This is a very common observation. What stops the motion ? There is no external force to stop it. A natural conclusion is, therefore, that whenever there is relative motion between parts of a fluid, internal forces are set up in the fluid which oppose the relative motion between the parts in the same way as forces of friction operate when a block of wood is dragged along the ground. This is why to maintain relative motion between layers of a fluid an external force is needed. The moment the external force is withdrawn, on account of the internal force, frictional in nature, the motion is destroyed.

This property of a fluid by virtue of which it opposes the relative motion between its different layers is known as Viscosity and the force that is called into play is called the Viscous force.

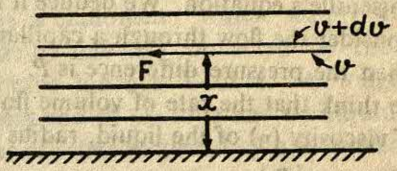


Fig. 14.1

Consider the slow and steady flow of a fluid over a fixed horizontal surface. Let v be the velocity of a thin layer of the fluid at a distance x from the fixed solid surface. Then according to Newton, the viscous force acting tangentially to the layer is proportional to the area of the layer and the velocity gradient at the layer. If F is the viscous force on the layer then

$$F \propto A \text{ where } A \text{ is the area of the layer}$$

$$\propto -v$$

$$\propto \frac{1}{x}$$

The -ve sign is put to account for the fact that the viscous force

is opposite to the direction of motion

or $$F=-\eta\frac{Av}{x} \quad .. \quad (14.1)$$

or $$F=-\eta A\frac{dv}{dx} \quad .. \quad (14.1)$$

where η is a constant depending upon the nature of the liquid and is called the coefficient of viscosity and

$$\frac{v}{x}=\text{velocity gradient }=\frac{dv}{dx}\text{ (by Calculus).}$$

If $A=1$ and $\frac{dv}{dx}=1$, we have $F=-\eta$.

Thus the coefficient of viscosity of a liquid may be defined as the viscous force per unit area of the layer where there is unit velocity gradient.

The coefficient of viscosity has the dimension $ML^{-1}T^{-1}$ or $FL^{-2}T$. Hence its unit is newton second per square metre (Nsm^{-2}) or kilogramme per metre per second ($kgm^{-1}s^{-1}$).

14.2. Poiseuille's Equation for Flow of a Liquid Through a Capillary Tube

When a liquid flows *slowly* and *steadily* through a capillary tube, the flow is streamline. The rate of this streamline flow through the tube is given by equation deduced by Poiseuille and is known as Poiseuille's equation. We deduce it here by the method of dimensions. Consider the flow through a capillary tube of radius R and length l when the pressure difference is P. From the physical considerations we think that the rate of volume flow will depend on the coefficient of viscosity (η) of the liquid, radius (R) of the tube and the pressure gradient $\left(\frac{P}{l}\right)$ and not on the actual value of the pressure difference because if the tube is halved and simultaneously the pressure difference is also halved, the rate of volume flow remains unchanged. So we can write

$$V=k\left(\frac{P}{l}\right)^{x}\eta^{y}R^{z}$$

where k is a dimensionless constant.

By taking dimensions of both sides, we have

$$L^{3}T^{-1}=(ML^{-2}T^{-2})^{x}(ML^{-1}T^{-1})^{y}L^{z}$$
$$=M^{x+y}L^{-2x-y+z}T^{-2x-y}$$

$\therefore \quad x+y=0, \; -2x-y+z=3$ and $-2x-y=-1.$

Solving we have, $\quad x=1, y=-1, z=4.$

$$\therefore \quad V=k\left(\frac{P}{l}\right)\eta^{-1}R^4=\frac{kPR^4}{\eta l}.$$

Experimentally, $\quad k=\frac{\pi}{8}.$

or
$$V=\frac{\pi PR^4}{8\eta l}. \quad \text{.. (14.2)}$$

The rate of mass flow $(M)=V\rho$

or
$$M=\frac{\pi PR^4\rho}{8\eta l}. \quad \text{.. (14.2a)}$$

We owe this deduction to Poiseuille and hence this type of flow of a fluid through a capillary tube is called Poiseuille's flow.

14.3. Determination of the Viscosity of a Liquid

The above principle affords a method of measuring the coefficient of viscosity of a liquid by observing the rate of volume flow of the liquid through a capillary tube.

A capillary tube of known length and radius is fitted horizontally to the bottom of a vessel having an over-flow arrangement which maintains the pressure difference between the ends of the tube. The pressure difference is measured by a manometer which simply consists of two glass tubes placed side by side and connected by rubber tubes to the ends of the capillary tube through T-tubes. A weighed beaker is placed below the delivery end of the tube and the liquid is collected in it at a certain interval of time recorded by a stop-watch.

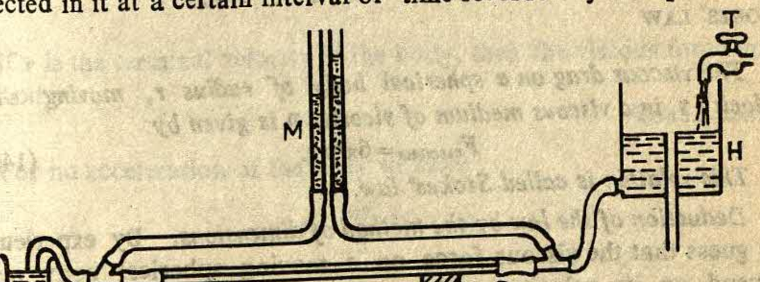


Fig. 14.3

The beaker containing water is weighed again. The difference gives the mass of liquid collected.

There are two important sources of error in the above formula : (a) part of the thrust, due to the difference of pressure between the ends of the flow-tube, imparts kinetic energy to the liquid and the whole of it, therefore, is not used in overcoming the viscous resistance of the liquid; (b) the liquid is accelerated immediately after it enters into the tube, with the result that the velocity of flow is not uniform throughout the tube.

The correct formula for η after these two sources of errors are taken into account is :

$$\eta=\frac{\pi PR^4}{8V(l+1{\cdot}64R)}-\frac{V\rho}{8\pi(l+1{\cdot}64R)}.$$

14.4. Motion of a Solid Body Through a Viscous Medium : Stokes' Law : Terminal Velocity

When a solid body moves through a viscous medium, its motion is opposed by a viscous force depending on the velocity and shape and size of the body. The energy of the body is continually wasted in overcoming the viscous resistance of the medium. This is why cars, aeroplanes etc. are shaped streamline to minimise the viscous resistance on them. Let the body be driven by a constant force. In the beginning the viscous drag on the body is small as its velocity is small and so the body is accelerated through the medium by the driving force. With the increase of velocity of the body the viscous drag on it will also increase and eventually when it becomes equal to the driving force the body will acquire a constant velocity. This velocity is called the *terminal velocity of the body.*

STOKES' LAW

The viscous drag on a spherical body of radius r, moving with velocity v, in a viscous medium of vicosity η is given by

$$F_{viscous}=6\pi\eta rv. \tag{14.3}$$

This relation is called Stokes' law.

Deduction of the law by the method of dimensions. By experience we guess that the viscous force on a moving spherical body may depend on its velocity, radius and coefficient of viscosity of the medium. We may then write

$$F=k\ v^a r^b \eta^c$$

where k is a constant (dimensionless) and a, b and c are the constants to be determined.

By taking dimensions of both sides, we have

$$MLT^{-2}=(LT^{-1})^a L^b (ML^{-1}T^{-1})^c$$

or
$$MLT^{-2}=M^c L^{a+b-c} T^{-a-c}.$$

Equating powers of M, L and T we have

$$c=1$$

$$a+b-c=1$$

and
$$-a-c=-2.$$

Solving we have $a=1,\ b=1$ and $c=1$

$$F=k\eta rv$$

Experimentally k is found to be 6π;

$\therefore$
$$F=6\pi\eta rv.$$

14.5. Terminal Velocity of a Spherical Body Falling Under Gravity Through a Viscous Medium

Consider the downward motion of a spherical body through a viscous medium such as rain drops falling through air. If r is the radius of the body and ρ the density of the material of the body then the weight of the body $=\frac{4\pi}{3}r^3\rho g$ downward and the buoyancy of the body $=\frac{4\pi}{3}r^3\sigma g$ upward where σ is the density of the medium.

The net downward driving force $=\frac{4\pi}{3}r^3(\rho-\sigma)g$.

If v is the terminal velocity of the body, then the viscous force on the body is

$$F=6\pi\eta rv \qquad \text{... Stokes' law.}$$

For no acceleration of the body we have

$$6\pi\eta rv=\frac{4\pi}{3}r^3(\rho-\sigma)g$$

or
$$v=\frac{2}{9}\cdot\frac{r^2g(\rho-\sigma)}{\eta}. \qquad \text{... (14.4)}$$

14.6. Determination of the Coefficient of Viscosity of a Liquid (Stokes' Method)

The above principle affords another method of determining the coefficient of viscosity of a liquid by observing the fall of a spherical body (steel balls) through a viscous medium. The method is specially useful for very viscous liquids such as castor oil, glycerol etc.

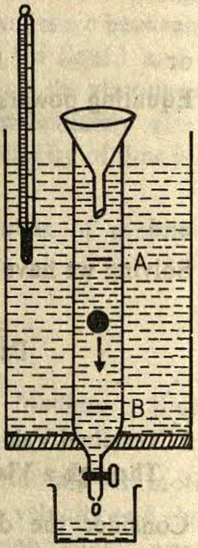


Fig. 14.4

The experimental liquid is taken in a long vertical glass tube. Two marks *A* and *B* well below the top are made on the tube. The upper mark must be essentially well below the free surface of the liquid to ensure the acquirement of terminal velocity by the steel balls. One of the steel balls is taken and is gently dropped at the top after measuring its diameter by a screw-gauge. The moment it reaches *A*, a stop-watch is started and when it crosses the second mark *B*, the watch is stopped. The distance *S* between *A* and *B* divided by the time recorded gives the terminal velocity of the ball. Knowing the terminal velocity in this way, the coefficient of viscosity is calculated from the formula

$$\eta = \frac{2}{9} \cdot \frac{r^2(\rho - \sigma)g}{v}.$$

In deducing this formula it is assumed that the liquid at a large distance from the moving body is at rest, that is, the walls of the container are assumed to be at infinity. In practice the walls of the tube are not at an infinite distance from the moving steel balls. Hence it requires a correction. This is called the wall effect. Another correction is necessary for the bottom not being at an infinite distance. This is called the end effect. After introducing the necessary corrections, the final working formnla is

$$\eta = \frac{2}{9} \frac{r^2(\rho - \sigma)g}{v(1 + 2{\cdot}4r/R)(1 + 3{\cdot}3r/H)}$$

where *R* is the radius of the glass tube and *H* is the full depth of the liquid column in the tube.

14.7. Importance of Viscosity

Like friction, viscosity affects our daily lives in many ways. The viscosity of water takes away much of the power developed by the engine of a ship. In the same way viscosity of air diminishes the power of car or aeroplane. The quality of a fountain-pen ink depends largely on its viscosity. The normal circulation of blood through our veins and arteries depends on the viscosity of blood. It is the viscosity of air that slows down rain-drops and saves us from being hit too hard by raindrops.

Examples

1. *A flat plate of area* 10 *sq. cm. is separated from a larger plate by a thin layer of glycerine* 1 *mm thick. If the viscous coefficient of glycerine is* 2 $kgm^{-1}s^{-1}$, *what force is required to keep the plate moving with a velocity of* ·01 ms^{-1} ?

Sol. To keep it moving with constant velocity only the viscous force is to be overcome. Hence the force required is equal to the viscous force.

$$\text{Velocity gradient} = \frac{\cdot 01}{\cdot 001} = 10;$$

$$\text{Force needed} = \eta A \frac{dv}{dx} = 2 \times 10 \times 10^{-4} \times 10$$

$$= \cdot 02 \text{ newton. } \textbf{Ans.}$$

2. *Water at* 25°C ($\eta = \cdot 001\ kgm^{-1}\ s^{-1}$) *is escaping from a tank by a horizontal capillary tube* 20 *cm long and* 1·2 *mm diameter. The water stands* 1 *m above the tube. At what rate is the water escaping ?*

Sol. $$V = \frac{\pi P r^4}{8\eta l}.$$

Here, $$P = h\rho g = 1 \times 1000 \times 9{\cdot}8 = 9800;$$

$$V = \frac{\pi \times 9800 \times (\cdot 6 \times 10^{-3})^4}{8 \times \cdot 001 \times \cdot 2} = \frac{\pi \times 98 \times 6^4}{8 \times 2} \times 10^{-10}$$

$$= 2{\cdot}5 \times 10^{-6} \text{m}^3\text{s}^{-1}. \textbf{ Ans.}$$

3. *A gas bubble of diameter* 2 *cm rises steadily through a solution of density* 1750 kgm^{-3} *at the rate of* ·35 *cm per second. Calculate the coefficient of viscosity of the solution.*

Sol. We have, $v=\frac{2}{9}\cdot\frac{r^2g(\rho-\sigma)}{\eta}$.

Here, ρ=density of air is negligible.

$$\therefore\quad v=-\frac{2}{9}\cdot\frac{r^2g\sigma}{\eta}.$$

The negative sign shows that the velocity is upward

$$\therefore\quad \eta=\frac{2}{9}\cdot\frac{r^2g\sigma}{v}=\frac{2}{9}\cdot\frac{\cdot01^2\times9\cdot8\times1750}{\cdot0035}$$

$$=109\ \text{kgm}^{-1}\text{s}^{-1}\text{ or Nsm}^{-2}.\ \textbf{Ans.}$$

QUESTIONS

(A)

1. For a flow to be streamline the velocity should be (a) less than the critical velocity, (b) greater than the critical velocity, (c) equal to the critical velocity, (d) $\sqrt{2}$ times the critical velocity.

2. The terminal velocity is (a) the velocity at the end of the motion, (b) the velocity with which a body is dropped in a liquid, (c) the constant velocity of a solid body in a liquid, (d) the average velocity of a body in a iquid.

3. The viscous force on a spherical body of radius r moving with velocity v through a liquid of viscosity η is (a) $\pi r^2\eta v$, (b) $2\pi\eta rv$, (c) $6\pi\eta rv$, (d) $6\pi\eta r^2v$.

4. When a rain-drop falls through air its velocity will (a) go on increasing, (b) go on decreasing, (c) be constant for sometime and then will increase, (d) go on increasing for sometime and then will become steady.

5. The terminal velocity of rain-drops is proportional to (a) r, radius of the drop, (b) r^2, (c) $\sqrt{r}$, (d) is the same for all drops.

6. The rate of flow of a liquid through a capillary tube of radius r is proportional to (a) r, (b) r^2, (c) r^3, (d) r^4.

Ans. 1. a, 2. c, 3. c, 4. d, 5. b, 6. d.

(B)

1. Explain what do you mean by viscosity of a liquid. (Pat. 1970, Mag. '78)
2. What is fugitive elasticity ? Explain.
3. What do you mean by terminal velocity ?
4. Define coefficient of viscosity and give its unit. Obtain its dimensions.

(C)

1. Describe Poiseuille's method for the determination of the coefficient of viscosity of water.

2. What is Stokes' law ?

Describe a method to show how with the help of this law the coefficient of viscosity of a thick liquid is determined.

(D)

1. A drop of water of radius 10^{-5} m falling through air attains a velocity of $1{\cdot}2\times10^{-2}$ ms^{-1}, Calculate the coefficient of viscosity of air if the density of air is $1{\cdot}2$ kgm^{-3}.

[Hint— $\eta=\frac{2}{9}\cdot\frac{r^2}{v}(\rho-\sigma)g$] (Ans. $1{\cdot}8\times10^{-5}$ SI units)

2. A glass ball of diameter 2×10^{-3} m and density 2000 kgm^{-3} falls in a jar filled with oil of density 800 kgm^{-3}. After attaining the terminal velocity it traverses a distance of 6 cm in 6 s. Find the coefficient of viscosity of the oil. ($g=9{\cdot}8$ ms^{-2}).

(Ans. ·26 SI unit)

3. An air bubble of 10^{-2} m radius is rising at a steady rate of $2{\cdot}5\times10^{-3}$ ms^{-1} through a large column of a liquid of density 1500 kgm^{-3}. Calculate the coefficient of viscosity of the liquid.

(Ans. $1{\cdot}3\times10^{-2}$ SI units)

4. Find the terminal velocity of an oil drop of density 950 kgm^{-3} and radius 10^{-6} m falling through air of density $1{\cdot}3$ kgm^{-3}, if the viscosity of air is $1{\cdot}8\times10^{-5}$ SI units.

(Ans. $1{\cdot}13\times10^{-4}$ ms^{-1})

5. Two equal drops of water one falling through the air with a steady velocity of 5×10^{-2} ms^{-1}. If the drops coalesce, what will the new velocity be ?

(Ans. $7{\cdot}94\times10^{-2}$ ms^{-1})

(E)

1. The property by virtue of which retarding forces are called into play when there is relative motion between layers is known as..................

2. The rate of volume flow through a capillary tube is..................

3. The terminal velocity acquired by rain-drops is proportional to.............. (r, r^2) where r is the radius of a drop.

4. The flow of ink in a fountain-pen depends on..................of ink.

(Ans. viscosity, 2. $\frac{\pi P r^4}{8\eta l}$, 3. r^2, 4. viscosity)

CHAPTER 15

FLUID DYNAMICS (HYDRODYNAMICS) : BERNOULLI'S THEOREM

15.1. Hydrodynamics

Hydrodynamics (or Fluid Dynamics) deals with the behaviour of fluids in motion. When a fluid is in motion, its flow is characterised by the following :

(i) *Fluid flow can be steady (or streamline) or nonsteady (turbulent).* When a fluid flows in such a way, that its velocity at any point does not change with time in magnitude and direction, its flow is said to be *steady* (or streamline or laminar). That is, every particle passing through the point momentarily acquires the velocity $\vec{v}$ of that point. When that particle passes on to the next point, it forgets the velocity of the previous point and acquires momentarily the velocity of the second point and so on. In other words, if each particle follows exactly the *same path* and the *same changes of direction and magnitude of velocity as its predecessor,* the flow is said to be steady or streamline. These conditions can be achieved only at low speeds, e.g. the flow of water through a capillary tube is streamline only at a small hydrostatic pressure difference between its ends. In non-steady or turbulent flow the velocities are not fixed in time. In such a flow the velocities vary erratically from point to point as well as from time to time. A tidal bore, the stream of a water fall, water currents in mountainous rivers are all examples of turbulent motion.

(ii) *Fluid flow can be rotational or irrotational.* If the element of fluid at each point has no net angular velocity about that point, the fluid flow is said to be irrotational. If in the irrotational flow of a fluid we imagine a small paddle wheel, it will move on without rotating. In rotational flow the wheel will move as well as rotate. Rotational flow is characterised by the formation of whirl pools or eddies

(iii) *Fluid flow can be compressible or incompressible.* When there is no change in density of the fluid as it flows, it is said to be an incompressible flow; otherwise it is compressible.

(iv) *Finally, fluid flow can be viscous or nonviscous.* As we have seen in the previous chapter, viscosity in fluid motion is the analog of friction in the motion of solids over solids. Viscosity introduces tangential forces between layers of fluid and causes dissipation of mechanical energy.

In this text we shall consider *steady*, *irrotational*, *incompressible* and *nonviscous* flow of fluids and that too of liquids only.

15.2. Streamlines : Tubes of Flow : The Equation of Continuity

In a steady flow every fluid particle follows the same path as its predecessor. If we consider the path along which a particle moves, the direction of the line at any point is the direction of the velocity of the fluid at that point. Such a line is called a *streamline*. Thus we may define *streamline as a curve, the tangent to which at any point gives the direction of flow of the fluid at that point.*

Streamlines are just like magnetic or electric lines of force. No two streamlines can ever cross one another, for if they did, an oncoming fluid particle could go either one way or the other, and the flow could not be steady.

Fig. 15.1

In principle we can draw a streamline through every point in the fluid. If we consider a number of streamlines through points distributed over a closed curve drawn in the field of flow of the fluid, the streamlines will define the surface of a three dimensional figure tubular in form (Fig. 15.1). This is called a tube of flow. The boundary of such a tube consists of streamlines, everywhere parallel to the velocity of the fluid particles. *No fluid particle can cross the boundaries of a tube of flow.* The fluid (incompressible) that enters at one end of a tube of flow must leave it at the other end.

The equation of continuity. Consider a tube of flow of a fluid and two transverse sections of it (Fig. 15.1). Let a_1 and a_2 be their cross-sectional areas (shown by dotted circles) and v_1 and v_2 be the velocities of fluid particles at the two sections respectively.

The rate of mass flow called the mass flux at the first section

$$=a_1 v_1 \rho_1$$

and the mass flux, i.e., the rate of mass flow through the second section $$=a_2 v_2 \rho_2$$

Since no fluid particles can cross the walls of the tube and there are no "sources" or "sinks" of fluid inside the tube, the mass crossing each section of the tube in each second must be the same.

$$\therefore \quad a_1 v_1 \rho_1 = a_2 v_2 \rho_2$$

or

$$a v \rho = a \text{ constant} \tag{15.1}$$

This is known as the *equation of continuity* in the simplest form.* If the fluid is incompressible, then $\rho_1 = \rho_2$ and the equation of continuity is further simplified to

$$a v = a \text{ constant} \tag{15.1a}$$

The product av is called the volume flux or the volume flow rate.

15.3. Critical Velocity and Reynolds Number

The flow of a liquid through a narrow tube is steady when the velocity of flow of the liquid is small. In all experiments designed to measure the co-efficient of viscosity, streamlines are assumed to be present, and therefore it is important to consider the condition favourable to its production. As the velocity of flow of a liquid through a tube is gradually increased by increasing the pressure difference between the ends of the tube, the motion continues to remain steady up to a certain velocity which marks the transition from streamline motion into turbulence. This velocity is called the *critical velocity*. It will obviously depend on the radius of the tube through which the liquid flows, the viscocity of the liquid, and the density of the liquid. Hence we may write

$$v_c = k r^x \eta^y \sigma^z$$

where k is a constant (dimensionless) and x, y and z are powers of r, η and σ respectively, to be determined.

Taking dimensions of both sides we have,

$$LT^{-1} = L^x (ML^{-1}\,T^{-1})^y\,(ML^{-3})^z$$

or

$$LT^{-1} = L^{x-y-3z}\,M^{y+z}\,T^{-y}$$

Equating powers of M, L and T we have

$$y+z=0$$
$$x-y-3z=1$$

and

$$y=1$$

*In general the equation of continuity is, $div\,(\rho\vec{v}) + \frac{d\rho}{dt} = s$ where s is the rate of creation of mass per unit volume at the point where the instantaneous velocity is v and the instantaneous density is ρ. The point is a 'source' if s is positive and a "sink" if s is negative.

Solving we have $x=-1$, $y=1$ and $z=-1$

$$\therefore \qquad v_c = kr^{-1}\eta^1\sigma^{-1}$$

or,
$$v_c = \frac{k\eta}{r\sigma} \qquad \text{.. (15.2)}$$

Osborne Reynolds showed, by experiment, that the constant k is nearly 1000. This is called Reynolds Number. Thus,

$$v_c = \frac{1000\eta}{r\sigma} \qquad \text{.. (15.2a)}$$

15. 4. Pressure Energy

The energy of any system is its capacity of doing work. The kinetic energy of a body is the capacity for doing work by virtue of its motion. The gravitational potential energy is its capacity for doing work by virtue of its position in the earth's gravitational field. An elementary mass of a liquid in motion possesses the capacity for doing work not only by virtue of its motion and position in the gravitational field but also by virtue of the pressure of the liquid.

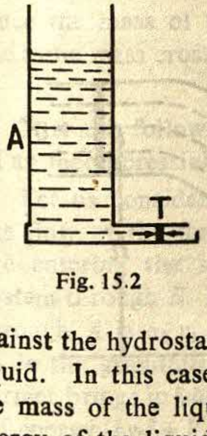


Fig. 15.2

Consider a tank A containing a liquid of density ρ, provided with a side tube T, of cross-sectional area a. Suppose this tube is provided with a frictionless piston. The thrust on the piston due to pressure of the liquid is 'ap'. To keep the piston in position an external agent must apply the same force on the piston in the opposite direction. If the piston is moved a little by Δx to the left, work will be done by the external agent against the hydrostatic thrust without imparting any velocity to the liquid. In this case energy will flow from the external agent to the mass of the liquid forced into the vessel and this is stored as energy of the liquid. If the piston is allowed to move back to its previous position, work will be done by the liquid against the external force. This time energy will flow back to the external agent. Thus it is clear that due to pressure a liquid can do work. This energy of liquid possessed by virtue of its pressure is called pressure energy.

The work done on the liquid by the external agent

$$=(ap)\triangle x = ap\triangle x.$$

The mass of the liquid forced into the vessel $= a\triangle x\rho$

The work done is stored as energy of the mass forced into the vessel.

$\therefore$ The pressure energy per unit mass $= \dfrac{a.\triangle xp}{a\triangle x\rho}$.

or, $$u=\frac{p}{\rho} \qquad \text{.. (15.3)}$$

where u stands for pressure energy per unit mass.

15.5. Bernoulli's Theorem

The most fundamental equation in fluid dynamics is given by Bernoulli's theorem. It was first presented by Daniel Bernoulli in his Hydrodynamica in 1738. In the basic form Bernoulli's theorem states that *in the nonviscous steady flow of an imcompressible fluid in a gravitational field* $\frac{1}{2}v^2+gh+\frac{p}{\rho}=a$ *constant where* v *is the velocity of flow,* h *is the height,* p *is the pressure and* ρ *is the density of the fluid.* If we take recognition of pressure energy it may be stated as :

In the nonviscous, steady and incompressible flow of a fluid the total energy of an elementary mass of the fluid remains constant throughout the displacement.

Consider an elementary mass $\triangle m$ of a fluid flowing through a tube. At any point in the path of flow let v, p and h be its velocity, pressure and height respectively. According to Bernoulli's theorem, the kinetic energy of $\triangle m$ + its potential energy + its pressure energy = a constant.

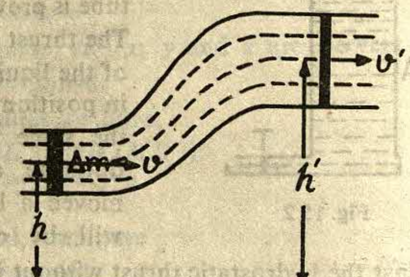


Fig. 15.3

or, $$\frac{1}{2}\triangle mv^2+\triangle mgh+\quad+\triangle m\frac{p}{\rho}=\text{a constant.}$$

or, $$\frac{1}{2}v^2+gh+\frac{p}{\rho}=\text{a constant} \quad (\because \text{ mass is a constant.})$$

$$\text{or,}\quad \frac{1}{2}v^2+gh+\frac{p}{\rho}=C \qquad \text{..} \quad (15.4)$$

$$\text{or,}\quad h+\frac{p}{\rho g}+\frac{1}{2}\ \frac{v^2}{g}=C' \qquad \text{..} \quad (15.4a)$$

Now h is called the gravitational head, $\frac{p}{\rho g}$ the pressure head and $\frac{1}{2}\ \frac{v^2}{g}$ the velocity head. Thus Bernoulli's theorem may also be stated in this way : *In the nonviscous, steady and incompressible flow of a fluid, the sum of the gravitational head, the pressure head and the velocity head remains constant throughout.*

Deduction. Like all equations in fluid mechanics (statics+dynamics) it is not a new principle but is derivable from the most fundamental principle namely, the principle of conservation of energy. In fact it is the statement of the principle of conservation of energy for fluid motion.

Imagine a tube of flow. Let p_1 be the pressure and v_1 the velocity at a point A where the transverse cross section of the tube is α_1 and whose height above the ground is h_1. The values of the same quantities at the other point B are represented by the suffix 2. Then, since the mass of the fluid contained between A and B is constant, the same mass crosses every section in unit time, or

$$\alpha_1 v_1=\alpha_2 v_2 \qquad \text{..} \quad \text{(i)}$$

This also follows directly from the equation of continuity Eq. 15.1 of an incompressible fluid.

Let us consider the fluid between A and B as a 'system'. With the flow of the fluid through the tube kinetic and potential energies are entering the system through section A and also leaving the system through B. In a short interval of time $\triangle t$. the mass entering through A is $\alpha_1\ v_1\ \rho\triangle t$ and that leaving through B is $\alpha_2\ v_2\ \rho\triangle t$. If V_1 is the gravitational potential at A and that at B is V_2, then the former brings in energy $(\frac{1}{2}v_1{}^2+V_1)\alpha_1 v_1\rho\triangle t$ and that leaving B carries off energy $(\frac{1}{2}v_2{}^2+V_2)\alpha_2 v_2\rho\triangle t$.

Hence the gain of energy by the system in $\triangle t$ is

$$(\tfrac{1}{2}v_1{}^2+V_1)\alpha_1 v_1\rho\triangle t-(\tfrac{1}{2}v_2{}^2+V_2)\alpha_2 v_2\rho\triangle t.$$

$\therefore$ the rate of gain of energy

$$=(\tfrac{1}{2}v_1{}^2+V_1)\alpha_1 v_1\rho-(\tfrac{1}{2}v_2{}^2+V_2)\alpha_2 v_2\rho.$$

According to the principle of conservation of energy the rate of gain of energy must be equal to the rate of doing work by the system against viscous forces and pressure forces. As we are dealing with the nonviscous flow of fluid, the work done against the viscous forces is zero.

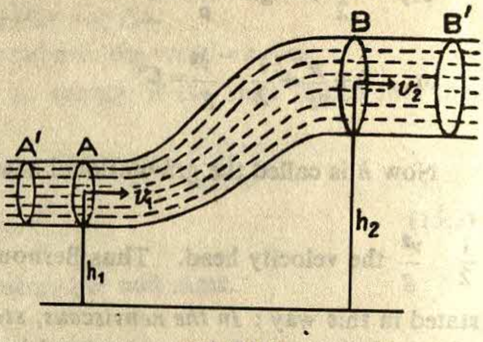


Fig. 15.4

To calculate the work done by the system, consider two similar systems : $A'A$ anterior to AB and BB' posterior to AB. At A, $A'A$ exerts a pressure force $\alpha_1 p_1$ on AB to the right and AB exerts the same force on $A'A$ to the left. Similarly at B, AB exerts a pressure force $\alpha_2 p_2$ on BB' to the right and BB' exerts the same force AB to the left. Thus in the actual flow of fluid AA' does work on AB, AB does work on BB' and so on.

If Δx_1 be the displacement of the end A in time Δt and that of B is Δx_2 in the same time then,

the work done on AB by the anterior system in time

$$\Delta t = \alpha_1 p_1 \Delta x_1$$

and the work done by AB on the posterior system in time

$$\Delta t = \alpha_2 p_2 \Delta x_2.$$

$\therefore$ the net work done by the system AB in time

$$\Delta t = \alpha_2 p_2 \Delta x_2 - \alpha_1 p_1 \Delta x_1$$

$$= \alpha_2 p_2 \frac{\Delta x_2}{\Delta t} \Delta t - \alpha_1 p_1 \frac{\Delta x_1}{\Delta t} \Delta t$$

$$= \alpha_2 p_2 v_2 \Delta t - \alpha_1 p_1 v_1 \Delta t$$

$$\left(\because \ \frac{\Delta x_2}{\Delta t} = v_2 \ \text{ and } \ \frac{\Delta x_1}{\Delta t} = v_1\right)$$

The rate of doing work by $AB = \alpha_2 p_2 v_2 - \alpha_1 p_1 v_1$.

By the principle of conservation of energy

$$(\tfrac{1}{2}v_1^2 + V_1)\alpha_1 v_1 \rho - (\tfrac{1}{2}v_2^2 + V_2)\alpha_2 v_2 \rho = \alpha_2 p_2 v_2 - \alpha_1 p_1 v_1.$$

Dividing throughout by $\alpha_1 v_1 \rho (= \alpha_2 v_2 \rho)$, we have

$$\left(\frac{1}{2} v_1^2 + V_1\right) - \left(\frac{1}{2} v_2^2 + V_2\right) = \frac{p_2}{\rho} - \frac{p_1}{\rho}.$$

Now, V_1 = gravitational potential at height $h_1 = V_0 + gh_1$ where V_0 = gravitational potential at the ground level.

and $$V_2 = V_0 + gh_2.$$

$$\therefore \quad \frac{p_1}{\rho} + \frac{1}{2} v_1^2 + V_0 \quad gh_1 = \frac{p_2}{\rho} + \frac{1}{2} v_2^2 + V_0 + gh_2.$$

$$\text{or} \quad \frac{p_1}{\rho} + \frac{1}{2} v_1^2 + gh_1 = \frac{p_2}{\rho} + \frac{1}{2} v_2^2 + gh_2 \quad \text{or} \quad \frac{p}{\rho} + \frac{1}{2} v^2 + gh = \text{a constant.}$$

15.6. Applications of Bernoulli's Theorem

1. *The Venturimeter*. This is a gauge to measure the flow speed of a liquid. It is simply a pipe of known cross-sectional area. In the middle its area is reduced and a manometer tube is attached between this constriction and any other point of the tube.

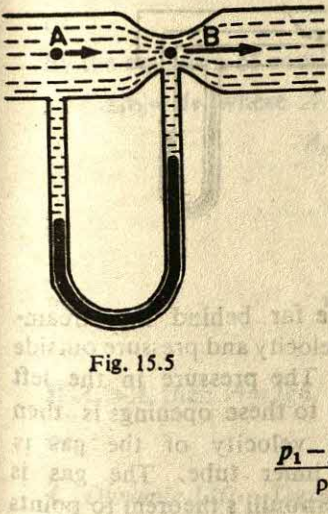


Fig. 15.5

The tube is held horizontally and the liquid is passed through it. The manometer indicates the pressure difference between the two points. Applying Bernoulli's theorem to the two points A and B (see Fig. 15.5).

$$\frac{p_1}{\rho} + \frac{1}{2} v_1^2 + gh = \frac{p_2}{\rho} + \frac{1}{2} v_2^2 + gh$$

$$\frac{p_1 - p_2}{\rho} = \frac{1}{2} v_2^2 - \frac{1}{2} v_1^2$$

By the equation of continuity we have

$$\alpha_1 v_1 = \alpha_2 v_2.$$

$$\therefore \quad \frac{p_1 - p_2}{\rho} = \frac{1}{2} v_2^2 - \frac{1}{2} \cdot \frac{\alpha_2^2}{\alpha_1^2} \cdot v_2^2$$

$$\text{or,} \quad v_2 = \sqrt{\frac{2(p_1 - p_2)\alpha_1^2}{\rho(\alpha_1^2 - \alpha_2^2)}} = \alpha_1 \sqrt{\frac{2(p_1 - p_2)}{\rho(\alpha_1^2 - \alpha_2^2)}}.$$

If we want the volume flux or flow rate, we simply multiply v_2 by α_2.

$$\therefore \qquad V = a_1 a_2 \sqrt{\frac{2(p_1 - p_2)}{\rho(a_1^2 - a_2^2)}} \qquad .. \quad (15.5)$$

If σ is the density of the liquid in the manometer, then $p_1 - p_2 = \sigma gh$, where h is the difference in the liquid levels.

$$\therefore \qquad V = a_1 a_2 \sqrt{\frac{2\sigma gh}{\rho(a_1^2 - a_2^2)}}.$$

or $V \propto \sqrt{h}$. The manometer can thus be calibrated to read the value of V directly.

2. *The Pilot tube.* This is a device to measure the flow speed of a gas. It consists of a narrow tube surrounded by a wider tube.

Fig. 15.6

There are openings at a in the outer tube far behind the stream-lined junction A of the tubes so that the velocity and pressure outside the openings have the free-stream values. The pressure in the left arm of the manometer, which is connected to these openings is then the pressure in the gas stream. The velocity of the gas is reduced to zero at b, a point in the inner tube. The gas is stagnant at that point. Applying the Bernoulli's theorem to points a and b

$$p_a + \tfrac{1}{2}\rho v^2 = p_b \text{ where } \rho \text{ is the density of air.}$$

$$\text{or.} \qquad p_b - p_a = \tfrac{1}{2}\rho v^2 \quad \text{or} \quad \rho' gh = \tfrac{1}{2}\rho v^2$$

where ρ' is the density of the manometer liquid and h is the difference of liquid levels in the manometer.

$$\text{or} \qquad v = \sqrt{\frac{2\rho' gh}{\rho}} \qquad .. \quad (15.6)$$

The device is calibrated to read v directly.

3. *Velocity of efflux.* When a tank is full of a liquid and a small pipe is fitted near the bottom, the liquid comes out with speed. This is called the velocity of efflux. Applying Bernoulli's theorem to a point Q which is immediately below the free surface and another P which is immediately out side the pipe, we have

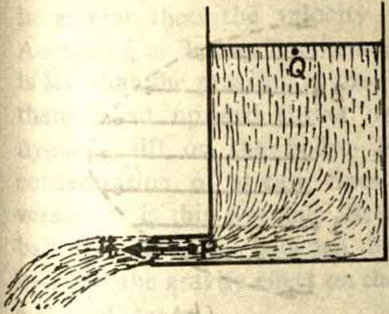


Fig. 15.7

$$\frac{1}{2}v_0^2+gh+\frac{p}{\rho}=\frac{1}{2}v^2+0+\frac{p}{\rho}$$

where p is the atmosphere pressure.

or $$v_0^2+2gh=v^2.$$

By the equation of continuity

$A_0v_0=Av$ where A = area of cross-section of the tube

A_0 = area of the tank.

$$\frac{A^2v^2}{A_0^2}+2gh=v^2.$$

or, $$v^2=\frac{2gh}{1-\dfrac{A^2}{A_0^2}} \qquad .. \quad (15.7)$$

If $A_0 \gg A$, then $v^2=2gh$.

or, $$v=\sqrt{2gh}. \qquad .. \quad (15.7a)$$

4. *Dynamic Lift.* Dynamic lift is the force that acts on a body such as an airplane wing, a hydrofoil, a spinning ball, the spinning shot of a rifle by virtue of its motion through air.

Fig. 15.8 (a) shows the motion of a sphere (non-spinning) through air. The streamlines due to the motion of the sphere are shown in Fig. 15.9 (b).

As the sphere moves through air it drags some air along with it. Hence the air particles in touch with it will be dragged almost with the same velocity as that of the body. The velocity of the distant particles will gradually be less and the air at large distance from the moving sphere will be at rest. This is shown in Fig. 15.8 b. The

velocity of the air particles at corresponding points above and below the sphere, such as 1 and 2, is the same.

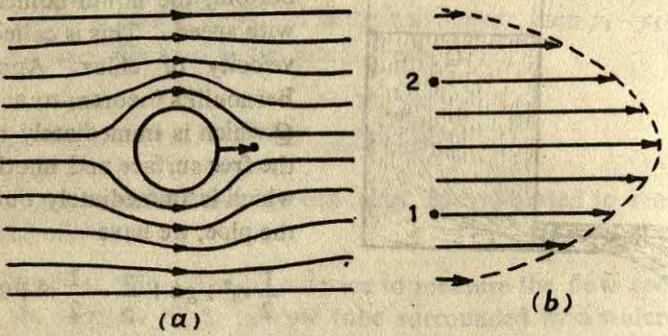


Fig. 15.8

Now suppose that the sphere spins about an axis perpendicular

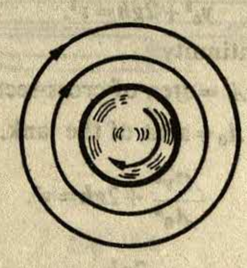


Fig. 15.9

to the plane of the paper. The streamlines associated with the motion are shown in Fig. 15.9.

Finally, let us consider the motion of a spinning sphere through air. Fig. 15.10 (a) shows the resulting streamlines and Fig. 15.10 (b)

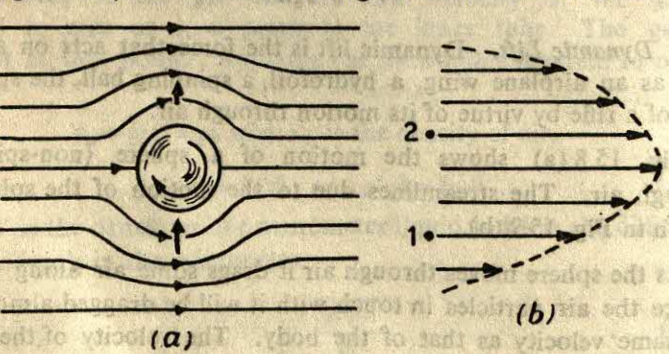


Fig. 15.10

shows the distribution of velocity of the air particles above and

below the sphere. The velocities at points above add, while those at points below subtract. Hence the velocity of the air particles above is greater than the velocity at the corresponding point below. According to Bernoulli's theorem, then, the pressure at a point above is less than the pressure at the corresponding point below so that there is an upward thrust on the spinning sphere. This is the dynamic lift on the body. Note that where there is a greater concentration of stream lines, there the pressure is less and vice versa. It is this dynamic lift that changes the course of a spinning ball bowled to a batsman by a spin bowler; it is this lift that balances the gravity effect on the shot of a rifle and so the shot covers a large distance.

Dynamic lift on airplane. We have seen above that the dynamic lift is always associated with an unsymmetrical set of streamlines. Where streamlines are close together, there the pressure is low and where streamlines are far apart there pressure is high. In the case of spinning balls the unsymmetrical distribution of streamlines is due to the spin of the ball. But in the case of an airplane wing an unsymmetrical pattern of streamlines is obtained by properly shaping the wing and properly orienting it in the airstream. The streamlines are closer together above the wing than they are below and so there is an upward dynamic lift on the wing.

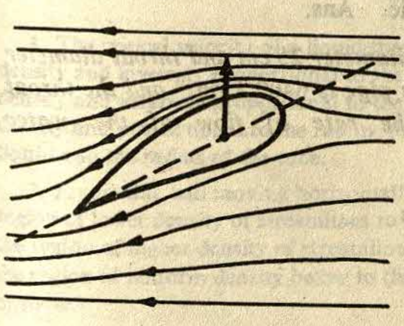


Fig. 15.11

5. *The Sprayer.* The common flit sprayer is yet another example of fall in pressure due to an increase in velocity as predicted by Bernoulli's theorem. Here, air is blown by a pump P across a small opening O which results in a reduction of pressure above O. The atmospheric pressure pushes the liquid up the tube. On reaching the opening it is blown into a fine spray by the airstream.

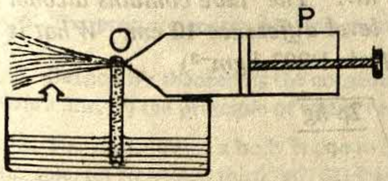


Fig. 15.12

Examples

1. *How much work is done by pressure in forcing* 1·35 *cubic metre of water through a* 1·27 *cm pipe if the difference in pressure is* 9000 $kg\ m^{-2}$.

Sol. $\text{Thrust} = 9000 \times 9{\cdot}8 \times \pi \times \left(\dfrac{1{\cdot}27 \times 10^{-2}}{2}\right)^2$

$= 11{\cdot}17\ N$

$$\text{Displacement} = \frac{1{\cdot}35}{\pi\left(\dfrac{1{\cdot}27 \times 10^{-2}}{2}\right)^2} = 1{\cdot}0655 \times 10^4$$

$\therefore$ Work done $= 11{\cdot}17 \times 1{\cdot}0655 \times 10^4$
$= 11{\cdot}9 \times 10^4$ joule. **Ans.**

2. *A venturimeter has a pipe of diameter* 25 *cm and throat diameter* 10 *cm. If the water pressure in the pipe is* 6400 kgm^{-2} *and the throat pressure* 4800 kgm^{-2}, *determine the rate of flow of the water (volume flow).*

Sol. $\alpha_1 = \pi\left(\dfrac{{\cdot}25}{2}\right)^2 = {\cdot}049\ m^2$

$\alpha_2 = \pi\left(\dfrac{{\cdot}1}{2}\right)^2 = {\cdot}0078\ \text{m}^2$

$p_1 - p_2 = (6400\text{-}4800)\ \text{kgm}^{-2} = 1600 \times 9{\cdot}8\ \text{Nm}^{-2}$

$$V = \alpha_1\alpha_2\sqrt{\frac{2(p_1 - p_2)}{\rho(\alpha_1{}^2 - \alpha_2{}^2)}} = {\cdot}049 \times {\cdot}0078\sqrt{\frac{2 \times 1600 \times 9{\cdot}8}{1000({\cdot}049^2 - {\cdot}0078^2)}}$$

or $V = {\cdot}00038\sqrt{\dfrac{31360}{2{\cdot}34}} = {\cdot}044\ m^3s^{-1}$. **Ans.**

3. *A pilot tube is mounted on an airplane wing to determine the speed of the plane relative to the air. The tube contains alcohol (specific gravity* $= {\cdot}8$) *and indicates a level difference* 10 *cm. What is the plane's speed in kph ? (density of air* 1·293 kgm^{-3})

Sol. We have by Eq. 15.6 $v = \sqrt{\dfrac{2\rho' hg}{\rho}}$

$$\therefore\ v = \sqrt{\frac{2 \times {\cdot}8 \times 1000 \times {\cdot}1 \times 9{\cdot}8}{1{\cdot}293}} = 34{\cdot}8\ ms^{-1} = \frac{34{\cdot}8 \times 3600}{1000}$$

$= 125$ kph. **Ans.**

4. *A garden hose having an internal diameter of 2 cm is connected to a lawn sprinkler that consists of an enclosure with 24 holes, each 125 cm in diameter. If the water in the hose has a speed of ·9 ms⁻¹, at what speed does it leave the sprinkler holes ?*

Sol. From the principle of continuity we have

$$\pi(\cdot01^2)\times\cdot9=24\times\pi\left(\frac{\cdot125\times10^{-2}}{2}\right)^2\times v$$

$$v=\frac{36\times10^5}{24\times125^2}=9\cdot6\ \text{ms}^{-1}.\quad \textbf{Ans.}$$

QUESTIONS

(A)

1. **The critical velocity of a liquid through a tube is (a) proportional to its density and inversely proportional to the radius, (b) inversely proportional to its density and inversely proportional to the radius, (c) inversely proportional to its density and proportional to the radius, (d) proportional to both the density of the liquid and the radius of the tube.**

2. A spinning ball moving horizontally experiences a dynamic lift (a) from the region of lower density of streamlines to the region of higher density, (b) from the region of higher density of streamlines to the region of lower density, (c) from the region of uniform density below to the region of uniform density, (d) none of these.

3. The pressure energy per unit volume is (a) p, (b) $\frac{p}{\rho}$, (c) $p\rho$, (d) none of these.

4. For a flow to be streamlined the velocity should be (a) less than the critical velocity, (b) more than the critical velocity, (c) equal to the critical velocity, (d) $\sqrt{2}$ times the critical velocity.

5. There is a constriction in a water pipe. The velocity of water is (a) maximum at the constriction, (b) minimum at the constriction, (c) the same as at other points, (d) there is an intermediate velocity at the constriction.

6. The pressure energy per unit mass is (a) p, (b) $\frac{p}{\rho}$, (c) $p\rho$, (d) $p\rho^2$.

7. Bernoulli's theorem is the consequence of (a) Newton's laws of motion, (b) Boyle's law, (c) the principle of conservation of energy, (d) Dalton's law.

8. The static lift on a body is due to (a) Bernoulli's theorem, (b) Archimedes' principle, (c) Newton's third law, (d) Pascal's law.

9. The dynamic lift on a body is due to (a) Bernoull's theorem, (b) Archimedes' principle, (c) Newton's third law, (d) Pascal's law.

Ans. : 1. b, 2. a, 3. a, 4. a, 5. a, 6. b, 7. c, 8. b, 9. a.

(B)

1. What are streamlines ? What are their properties ?
2. Explain the action of a pilot tube and venturimeter.
3. Show that the pressure energy per unit volume is numerically equal to the pressure of the liquid.
4. Find the critical velocity through a tube by the method of dimensions.
5. Apply Bernoulli's theorem to find the velocity of efflux of a liquid.
6. Explain how a dynamic lift arises on an airplane.

(C)

1. State and prove Bernoulli's theorem. Describe important applications of the theorem.

2. Distinguish between steady and nonsteady, rotational and irrotational, compressible and incompressible and viscous and nonviscous flow of a liquid.

State and explain Bernoulli's theorem.

3. What do you mean by dynamic lift and how does it differ from static lift. Explain how a dynamic lift arises on a spinning ball and on the wing of an airplane.

(D)

1. Water flows through a horizontal tube of non-uniform cross-section. At a point where the velocity is 15 ms^{-1}, the pressure is 15×10^4 Nm^{-2}. What is the pressure at a point where the velocity is 10 ms^{-1} ?

(Ans. $21 \cdot 25 \times 10^4$ Nm^{-2})

2. Water flows through a tapering tube horizontally. The velocity of water is 3 ms^{-1} where the diameter of the tube is 1 m. What is the velocity where the diameter is ·05 m ? (Ans. 12 ms^{-1})

3. Water falls from a height of 20 m at the rate 20 m^3 per minute and drives a water turbine. What is the maximum power that can be developed by the turbine ? (Ans. $6 \cdot 5 \times 10^4$ watt)

4. In a horizontal oil pipe of constant cross-sectional area the pressure difference between two points 300 m apart is 4000 kgm^{-2}. What is the energy loss per cubic metre of oil per unit distance ? (Ans. 130·6 Jm^{-3} per metre)

5. If the speed of flow past the lower surface of a wing is 120 ms^{-1}. what speed of flow over the upper surface will give a lift of 20 kgm^{-2}. Density of air 1·29 kgm^{-3}. (Ans. 121·3 ms^{-1})

6. Air streams horizontally past an air-plane wing of area 3·24 m^2 and weighing 270 kg. The speed over the top surface is 60 ms^{-1} and 45 ms^{-1} under the bottom surface. What is the lift on the wing ? The net force on it ? Density of air 1.3 kgm^{-3}. [Ans. 338·6 kgwt. 68·6 kgwt. (up)]

(E)

1. In a lecture demonstration a ping-pong ball is kept in mid-air by a vertical jet of air. Is the equilibrium stable, unstable or neutral ? (I. I. T.)

2. Two row boats moving parallel to one another in the same direction are pulled towards one another. Explain.

3. Why does water flow in a continuous stream down a vertical pipe whereas it breaks into drops when falling freely ?

4. The destructive effect of a tornado is greater near the centre of the disturbance than near the edge. Explain.

5. Why does an object falling from a great height reach a steady terminal velocity.

[Ans. 1. Unstable. 2. The velocity of water between boats increases which results in a reduction in pressure. The sideways pressure then pushes the boats together. 3. There is no free surface of water in a tube. Due to surface tension it falls in drops in air. 4. Due to Bernoulli's effect. 5. Due to viscous force on the falling body.]

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CHAPTER 16

PRODUCTION OF LOW PRESSURE AND MEASUREMENT

16.1. Low Pressure

It is the technique of producing high vacuum that helped scientists to have entry into the realm of atomic physics. The very first thing that we need for the production of cathode rays, x-rays, positive rays etc. and study their properties is the production of sufficiently low pressure. A cathode ray tube needs low pressure of the order of ·01 mm of *Hg*, an *X*-ray tube needs vacuum of the order of 10^{-6} mm of *Hg*, a radio valve also needs vacuum of the same order etc. Let us now take up some simple and important types of pumps

16.2. The Common Air Pump (or Exhaust Pump)

The simplest type of exhaust pump was invented by Otto van Guericke. It consists of a stout metal cylinder fitted with an air-tight piston P having a value V_2 which can open upward. At the bottom of the cylinder there is another valve V_1 which can also open upward only. The cylinder is connected by means of a pipe to the plate D on which the vessel to be evacuated is placed in an inverted position.

Action : To start with, suppose the piston is at the bottom and it is pulled up. The piston valve V_2 closes due to pressure from above and the cylinder valve V_1 opens due to a fall of pressure inside the cylinder. So air from the vessel rushes into the cylinder. When the piston is pushed down, the inside air gets compressed. The piston valve opens and the cylinder valve is closed. The air in the cylinder escapes into the atmosphere. The process is repeated a number of times until the air in the vessel is too rarefied to lift the cylinder valve.

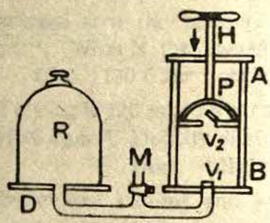


Fig. 16.1

This pump is unable to give a high vacuum, because the pressure of the residual air in the vessel is unable to force open the cylinder valve and get into the cylinder.

Degree of evacuation : Let V = volume of the vessel to be evacuated (receiver) and v = volume of the cylinder. To start with, the mass of air in the vessel (receiver) is $V D$ where D is the density of the air originally present in the vessel. After the first up-stroke the same mass of air occupies the volume $(V+v)$. Hence the density decreases to, say, D_1.

Since the mass remains constant

$$(V+v)D_1 = VD$$

or

$$D_1 = \left(\frac{V}{V+v}\right)D$$

During the down stroke the cylinder valve is closed and so the density of air in the vessel remains fixed at D_1. In the beginning of the second stroke the mass of air, to start with, is VD_1 which expands to $(V+v)$ at density D_2.

$$VD_1 = (V+v)\,D_2$$

$$D_2 = \left(\frac{V}{V+v}\right)D_1 = \frac{V}{V+v}\cdot\frac{V}{V+v}\,D$$

or

$$D_2 = \left(\frac{V}{V+v}\right)^2 D$$

Proceeding in this way the density D_n at the end of n strokes is given by

$$D_n = \left(\frac{V}{V+v}\right)^n D \qquad \text{.. (16.1)}$$

Since at constant temperature the pressure is proportional to the density, the pressure after n strokes is

$$P_n = \left(\frac{V}{V+v}\right)^n P \qquad \text{.. (16.1a)}$$

16.3. Rotary Oil Pumps

The most efficient pumps which can start from the atmospheric pressure and produce vacuum of the order of 10^{-3} mm of Hg are rotary oil pumps. There are two types of rotary pumps : (i) the rotary-vane oil pump or the Gaede rotary oil pump, and (ii) the stationary-vane oil pump or the Hyvac rotary oil pump.

(a) The rotary vane oil pump or the Gaede pump : The pump consists of a stout hollow cylinder and a stout and massive solid cylinder mounted eccentrically inside the hollow cylinder such that it is always in contact with the cylinder called *stator* along a fixed line because the axis of rotation of the rotor is off the axis of the stator. The rotor rotates about its own axis which is off the axis of the stator.

A slot, cut diametrically right across the rotor, carries two vanes *A* and *B* separated by a strong spring which not only keeps them apart from each other, but also presses them well against the walls. The vane divides the space between the stator and the rotor into three separate chambers.

On either side of the line of contact, the stator is provided with an inlet (I) and an outlet part *O*, which is closed by a spring-operated valve. This valve can open outward only. The whole pump is kept immersed in oil which serves a threefold purpose : (i) It provides automatic lubrication, (ii) it prevents leakage of gas or vapour into the high vacuum created and (iii) it works as an efficient cooling agent. The rotor is driven at high speed by means of an electric motor.

To begin with, suppose the position of the vane is as shown in the Fig. 16.2a. As the rotor rotates in the direction shown in the figure by the arrow head, the air is sucked in through the inlet *I* in

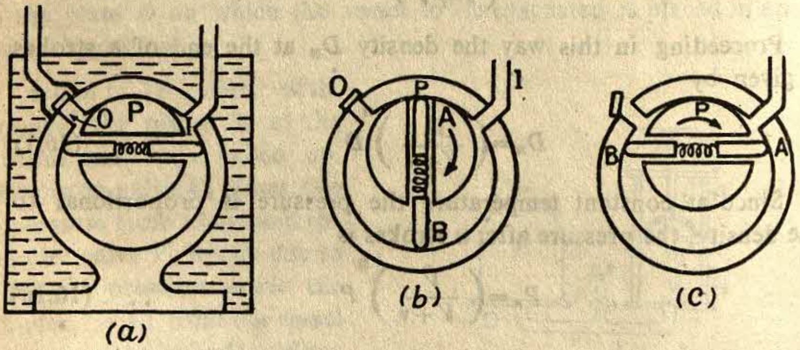


Fig. 16.2

the first right chamber till the vane sets vertically as shown in Fig. 16.2b, when the space between the stator and rotor is momentarily divided into two chambers. As the vane rotates a little, again three chambers appear distinctly. Air is compressed in the left chamber and fresh air is sucked in through *I* in the right chamber. When the

compression becomes sufficiently high, air in compression chamber forces open the valve and escapes through the outlet *O*. The process goes on repeating itself and within a short time produces a pressure as low as 10^{-3} mm of mercury.

(*b*) *The stationary-vane or the Hyvac Rotary oil pump*: This is exactly the same as the Gaede pump, except that here the vane is stationary and the rotors axis of rotation is off its own axis, but it is the same as the axis of the stator. It consists of a stout hollow cylinder, inside which is mounted eccentrically a massive solid cylinder. A rectangular vane in the form of a thick plate is slid into the slot cut in the stator and kept well pressed against the rotor with the help of a powerful spring *S*. The vane divides the space between the stator and the rotor into two compartments 1 and 2. On either side of the vane there are two ports—inlet and outlet. The outlet port is closed by a valve which can open outwards. The whole pump is immersed in oil which serves three-fold purposes as explained in the Gaede pump The rotor of the pump is driven at a very high speed by means of an electric motor.

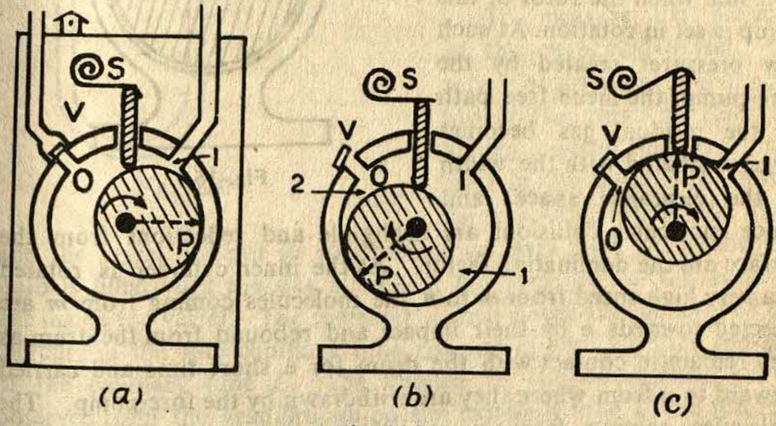


Fig. 16.3

To start with, suppose the position of the rotor is as shown in the Fig. 16.3a. As the rotor moves on in the direction shown in the figures by the arrow head, the air is sucked in through the inlet from the vessel in the chamber 1 and air in chamber 2 is compressed When the compression becomes sufficiently high, air in the chamber 2 forces open the valve of the outlet port and escapes to the atmosphere. When the rotor is in its highest position (Fig. 16. 3c) the two chambers 1 and 2 momentarily coalesce and become one

chamber. As the rotor rotates a little further a fresh cycle starts again. The cycle of operation is repeated in quick succession and evacuation proceeds at a fast rate. It also produces evacuation of about 10^{-3} mm in a very short time.

16.4. Molecular Pumps

These are very efficient pumps which push the vacuum created by rotary oil pumps to about 10^{-6} mm. This pump consists of a drum rotating within an outer cylinder with a very small clearing space between the two, except for a very short length between the ports *m* and *n* where the clearance space is larger. The port *m* is connected to the vessel to be evacuated and *n* to the fore-pump. First the fore-pump is worked and the pressure is reduced to about 10^{-3} mm when the rotor of this pump is set in rotation. At such low pressures created by the fore-pump, the mean free path of the residual gas becomes large compared with the width of the clearance space, and hence molecular collisions are negligible and reflections from the surface are the dominating factor. As the inner cylinder is rotated at a very high speed from *m* to *n*, the molecules coming from *m* are directed towards *n* by their impact and rebound from the drum as they remain in contact with the drum for a short time and carried forward to *n* from where they are withdrawn by the fore-pump. The molecules coming from *n* are almost prevented from moving towards *m*.

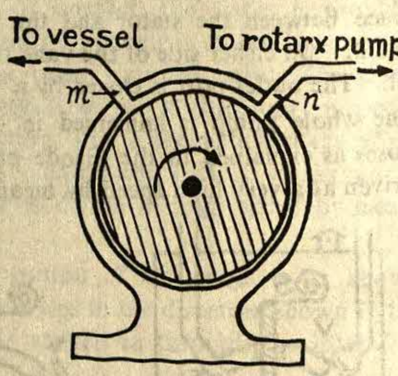


Fig. 16.4

16.5. Measurement of Low Pressure : McLeod Gauge

With the development of high vacuum pumps, very delicate gauges to measure the low pressures produced by them have also been developed. Among these which are on the fore-front are McLeod Gauge, the Pirani Gauge and the Knudsen Gauge. Here we

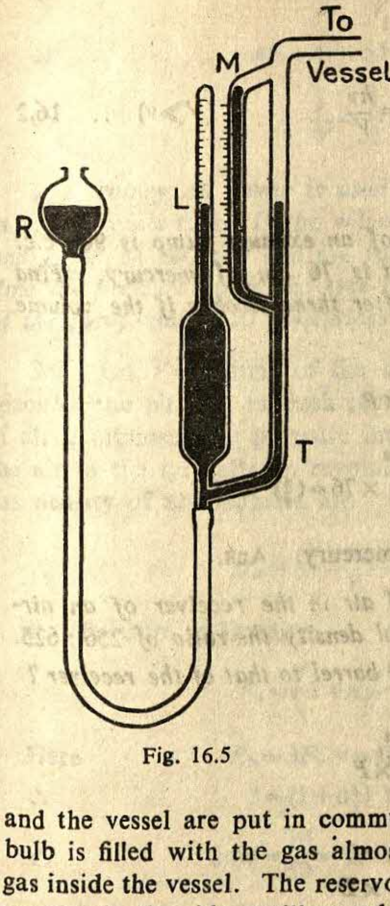


Fig. 16.5

will see only McLeod Gauge in detail. It consists of a cylindrical or spherical bulb of known volume. The bulb is attached to a closed capillary tube L graduated in c c. at its top and at the bottom to a reservoir of mercury R and a side tube T to be connected to the vessel of which pressure is to be measured. Another tube M of the same capillary bore as of L is sealed as a by-path to the side tube T to avoid errors due to capillarity. A millimetre scale allows the measurement of the difference in mercury levels in L and M.

To measure the pressure of the evacuated vessel, the tube T of the gauge is connected to it. The mercury reservoir is then lowered till the mercury level in the bulb comes down the junction, when the bulb and the vessel are put in communication with each other and the bulb is filled with the gas almost at the pressure of the residual gas inside the vessel. The reservoir is then raised till the level of the mercury in the side capillary tube rises up in level with the top-end of the capillary tube L. The volume of the compressed gas is observed from the graduations on the capillary tube L and also the differencc of mercury level is noted from the millimetre scale.

Let the volume of the compressed gas be v, the original volume of the gas contained in the bulb and the capillary tube be V, and h be the difference in mercury levels in the capillary tubes. Then the initial pressure and the volume of the gas in the bulb and the capillary tube are respectively p and V, where p is the pressure to be measured. The final pressure and volume of the same gas after compression are $(p+h)$ and v respectively.

From Boyle's law we have,

$$pV=(p+h)v$$

or $$pV=pv+hv$$

or $$p=\frac{hv}{V-v}=\frac{hv}{V} \qquad (V \gg v) \quad .. \quad 16.2$$

Examples

1. *The volume of the receiver of an exhaust pump is* 900 *c.c. and the pressure of the air inside it is* 76 *cm of mercury. Find the pressure of air in the receiver after three strokes if the volume of the barrel is* 300 *c.c.*

Sol : We have $P_n=\left(\frac{V}{V+v}\right)^n \times P$

$$P_3=\left(\frac{900}{900+300}\right)^3 \times 76=(\tfrac{3}{4})^3 \times 76$$

$$=32{\cdot}1 \text{ cm of mercury.} \quad \textbf{Ans.}$$

2. *After* 4 *strokes the density of air in the receiver of an air-pump is found to bear to its original density the ratio of* 256 : 625. *What is the ratio of the volume of the barrel to that of the receiver ?*

Sol. We have, $P_n=\left(\frac{V}{V+v}\right)^n \times P$

Pressure $\propto$ Density

$$D_n=\left(\frac{V}{V+v}\right)^n \times D$$

$$D_4=\left(\frac{V}{V+v}\right)^4 \times D$$

or $$\frac{D_4}{D}=\left(\frac{V}{V+v}\right)^4$$

$$\frac{D_4}{D}=\frac{256}{625} \text{ (given)}$$

$$\frac{256}{625}=\left(\frac{V}{V+v}\right)^4 \text{ or } \left(\frac{4}{5}\right)^4=\left(\frac{V}{V+v}\right)^4$$

or $$\frac{4}{5}=\frac{V}{V+v}$$

or $$4V+4v=5V$$

or $$\frac{v}{V}=\frac{1}{4}.\quad \textbf{Ans.}$$

3. *A rotary oil pump is used as a compression pump to fill air in a motor cycle tyre. If the volume of the space between the stator and rotor be* 100 *cm*³ *and the volume of tyre be* 500 *cm*³*; in what time the pressure inside the tyre becomes* 3 *atmosphere ? The rotor of the pump makes* 50 *revolutions in one second.*

Sol. Let V=volume of the tyre and v=volume of the inner space of the pump. In each revolution the pump forces v volume of air at atmospheric pressure into the tyre. Therefore, mass of the air in the tyre after n revolutions is $(V+nv).D$, where D is the density of atmospheric air.

$$\therefore \qquad D_n=\frac{(V+nv).D}{V}=\left(1+n\frac{v}{V}\right).D$$

$\therefore$ Pressure $\propto$ Density

$$P_n=\left(1+n\frac{v}{V}\right)P.$$

Here $\qquad P_n=3P;\ v=100\ \text{cm}^3;\ V=500\ \text{cm}^3$

$\therefore \qquad 3=(1+n\tfrac{1}{5})$ or $n=10$

$$\therefore \qquad \text{time}=\frac{10}{50}=\cdot 2s.\quad \textbf{Ans.}$$

QUESTIONS

(A)

1. McLeod Gauge works on (a) Charles' law, (b) Boyle's law, (c) The pressure law, (d) Archimedes' principle.

2. In the Gaede rotary oil pump the axis of rotation of the rotor is (a) off the axis of the stator and at the axis of the rotor, (b) off the axis of the rotor and at the axis of the stator, (c) at the axis of the stator, (d) none of these.

3. In the Hyvac rotary oil pump the axis of rotation of rotor lies (a) off the axis of the stator at the axis of the rotor, (b) off the axis of the rotor at the axis of the stator, (c) at the common axis of the stator and rotor, (d) none of these.

4. The contact line between the stator and the rotor of the Gaede oil pump (a) rotates, (b) does not rotate, (c) oscillates, (d) there is no contact line between the two.

5. The line of contact between the stator and the rotor of the Hyvac rotary oil pump (a) rotates, (b) does not rotate, (c) oscillates, (d) there is no contact line between the two.

Ans : 1.b 2.a 3.b 4.b 5.a

(B)

1. Describe in detail an air-pump with a neat diagram and explain its action. Calculate the degree of evacuation after *n* strokes. Can the apparatus create perfect vacuum ? If not, why ?

2. Describe the working of the Gaede rotary oil pump for producing low pressure. Describe McLeod's gauge method of measuring low pressure.

3. Describe, with a neat sketch, the working of the Hyvac rotary oil pump for producing low pressure.

4. Describe a molecular pump and explain its working principle ? What are the advantages and disadvantages of this pump ?

(C)

1. The pressure inside the receiver of an air pump is reduced to $\frac{1}{4}$th of the atmospheric pressure after 3 strokes. What will be the pressure after 8 strokes ?

[Ans : $\frac{1}{40}$ of the atmospheric pressure]

2. If the pressure inside the receiver of an air pump is reduced to $\frac{1}{4}$ of the atmospheric pressure after 5 strokes, what will be the pressure after 10 strokes ?

[Ans : $\frac{1}{16}$ of the atmospheric pressure]

CHAPTER 17

ERRORS AND THEIR MEASUREMENTS

17 1. Errors and Deviations

All experimental measurements involve uncertainty or error due to known or unknown causes. We cannot make measurements with all certainty and claim that measurements are hundred per cent accurate because neither the instruments used may be supposed to be 'perfect' nor any experimenter may be perfect. Hence errors in measurements are to be taken as a matter of rule and not as an exception.

If X be the true value of a physical quantity and X be the average of n equally reliable measurements of that quantity, then $x_i = X_i - X$ is defined as the error in its ith observation and $\delta_i = X_i - \overline{X}$ is defined as the deviation (or residual) in its ith observation. Obviously, therefore, in actual practice deviations can be computed, but not the errors because the true value of the physical quantity is not known. Though 'error' and deviations (residuals) are two distinctly different quantitites as defined above, in the following discussions *the term error will be reserved to mean only deviations.*

17.2. Kinds of Errors

There are two types of errors :

(i) Systematic, and

(ii) Non-systematic or random.

(i) SYSTEMATIC ERRORS

Systematic errors are those which creep in measurements in a regular way due to imperfection of instruments, improper adjustment, neglect of some factors etc. We can always find the cause of such errors and due attention may be paid to them to eliminate or at least minimise their effects. The error of eccentricity, the error of the magnetic axis, the error of the magnetic centre in a deflection magnetometer are few examples of systematic errors.

(ii) RANDOM ERRORS

These are errors which creep in the measurement of physical quantities due to unknown causes beyond the control of the experimenter. These errors are inherently random in nature and hence are to be treated by statistical methods. When one takes several measurements for the same physical quantity, the error in the individual measurement will be random. Suppose you want to find the focal length of a convex lens by the single pin method. All the readings are expected to be identical. But due to unknown reasons different readings will slightly differ from each other. These are non-systematic or random errors.

17.3. Absolute, Relative and Percentage Errors

Suppose the measure of some physical quantity is X and $\triangle X$ be the error (or uncertainty) then $\triangle X$ is called the absolute error, $\frac{\triangle X}{X}$ is called the relative error and $\frac{\triangle X}{X} \times 100$ is called percentage error.

17.4. Significant Figures

The number of significant figures in a measurement is an indication of the relative accuracy. Suppose the measured length of a bar is reported by an experimenter as 25·7 cm. This means that the experimenter uses a metre scale and reads up to a mm. Here we say that the measure as reported by the experimenter contains three significant figures. Now, if the same measurement is reported by the experimenter as 25 cm, then the measurement contains only 2 significant figures and the experimenter takes readings up to the nearest cm. On the other hand a measurement reported as 25·0 cm means that the experimenter has tried to read up to a mm but it happens to be zero in the place of mm. Thus 25 and 25·0 contain 2 and 3 significant figures respectively, though arithmetically there is hardly any difference between 25 and 25·0

A result obtained from measurements must contain only that number of significant figures as would be consistent with the accuracy in the measurements. Suppose we measure the length of a rod by a slide-calipers and find it as : 2·21, 2·24, 2·23, 2·20, 2·23 cm. The mean must be written as 2·22 and **not** 2·222 because the measurements given by the instrument are of three significant figures. The mean must also be of 3 significant figures.

The following are applicable to arithmetical operations of numbers of different significant figures :

(i) The process of addition of the numbers of different significant figures should be carried up to the least number of columns of the significant figures. The sum of 28·37872 and 0·02 is 28·5 after the usual procedure of rounding off (see below for methods).

(ii) In multiplication or division the number of significant figures in the result is equal to that contained in the factor which has the least number of significant figures.

The mass of a body is 24·325 kg and its volume is 10·2 m³. What is its density ?

$$\text{Density} = \frac{24 \cdot 325}{10 \cdot 2} = 2 \cdot 3848$$

Since the least number of significant figures is three (density has the least number of significant figures), hence we round up the result up to three significant figures and write the result after carrying to the correct number of significant figures as :

Density = 2·38 kgm^{-3}

The length of a rectangle is 11·35 cm and its breadth is 2·3 cm, then

Area = 11·35 × 2·3 = 26·105 = 26 cm^2 (after carrying to the correct number of significant figures).

Remember always that *the number of significant figures is independent of the position of the 'decimal' in the number.*

RULES FOR ROUNDING OFF

(i) If the digit to be discarded is greater than 5, the preceding digit is increased by 1.

(ii) If the discarded digit is 5 itself, the preceding digit is increased by 1 if it is odd but kept unchanged if it is even.

22·387	is	22·4	when rounded up to 3 significant figures.
16·258	is	16·2	
8·375	is	8·38	
118·53	is	118	

17.5. Standard Deviation of Sample and Parent Distribution

A set of infinitely large number of readings is often called the parent distribution and a set of limited number of readings is called

a sample. Naturally, we are interested in the deviations of the parent set rather than that of the sample. But we have the data only of the sample.

If there are n readings in a *sample* and δ_1, δ_2, δ_2, ...δ_n are the errors (deviations) then the root mean square value of δ's is defined as *the standard deviation of the sample.*

$$S.\,(\text{S. D.}) = \sqrt{\frac{\Sigma\,|\,\delta_i\,|^2}{n}}$$

If n is made very large, then the sample becomes the universe (or the parent distribution). The standard deviation of the parent distribution is denoted by σ. In practice it is not possible to obtain thousand and thousand of readings. From theoretical considerations the standard deviation (σ) of the parent distribution bears a definite relations with the standard deviation (S) of a sample. The relation is :

$$\sigma = \sqrt{\frac{\Sigma\,|\,\delta_i\,|^2}{n-1}}$$

A simple theoretical explanation for the replacement of n by $(n-1)$ is that all the deviations are not independent. The deviations in a sample or the universe will be fifty-fifty positive and negative and hence $\Sigma\delta_i = 0$ or $\Sigma(X_i - \overline{X}) = 0$. This leaves only $(n-1)$ independent variables and hence $(n-1)$ in averaging $\delta's$.

A second argument that is put forward is that the relations must be true for all samples including the one of only one reading. The standard deviation of a sample of one reading is zero but not of the parent distribution. If n occurs in the denominator then σ is definitely zero which is not true. In fact σ must be of indeterminate form when a sample of only one reading is considered. To make it of indeterminate form, the denominator is made $(n-1)$ so that when $n = 1$, σ takes the form of zero divided by zero—an indeterminate form.

17.6 Accuracy and Precision

A result is accurate when it comes close to the true value. Since the true value is unknown, we rarely can know if a result is accurate or not. The systematic error is a measure of the accuracy of a method. A method in which there is the least number of systematic errors is an accurate method.

On the other hand, the precision of the experiment is a measure of 'how carefully the observations are taken'. Measurements are **precise** if they agree among themselves. Precision has something to do with the random errors and not of the systematic errors. See the examples below.

·30 mm	·37 mm
·32	·38
·45	·38
·51	·37
·47	·39
(not precise)	(precise)

17.7. Distribution Curve : Histogram

For visual representation of the observations a graph is drawn by dividing the original set of observations into intervals of predetermined magnitude and then finding the number of observations in that interval called the frequency. Suppose we want to visually represent the result of a terminal. Let us divide the 100 marks into 10 ranges 0-10, 10-20....90-100 and then count the number of students in each range. Suppose in the range 0-10 there are only 5 students. Then 5 is the frequency of the range 0-10. A plot of frequency versus range is called a histogram (Fig. 17.1a). If the range is made smaller and smaller, the histogram approaches a continuous curve. This is called distribution curve (Fig. 17.1b).

The value of x at which the peak of the distribution curve occurs is called the *mode* of the distribution.

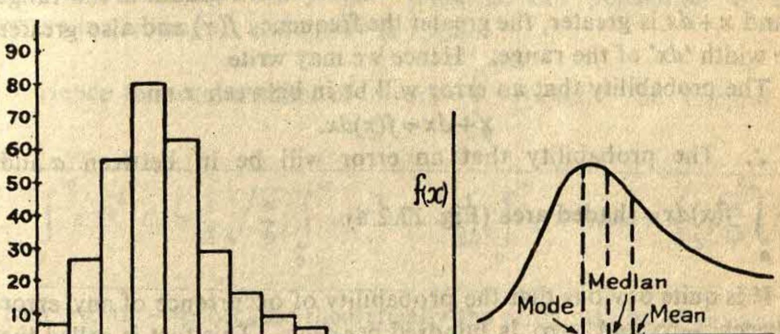


Fig. 17 1.

The value of x which divides the curve into two equal halves so far area is concerned is called the *median*.

The most commonly occurring distribution curve in physical problems is the Gaussian (also called the Normal distribution) distribution. The characteristics of a Gaussian distribution is that it has a central peak and the curve is symmetrical about the peak (Fig. 17.2).

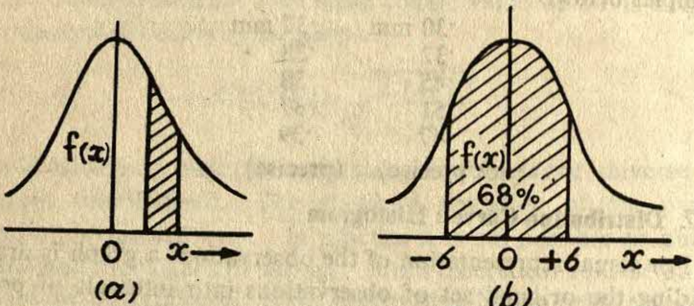


Fig. 17.2.

In our daily measurements of physical quantities there is every chance that there will be as many positive errors as there will be negative errors, the chance of zero error being maximum. Hence if we plot frequency versus errors, the plot will be obviously a Gaussian one. The mathematical expression for the Gaussian distribution is of the form

$$f(x)=A\ e^{-h^2(x-m)^2}$$

where A, h and m are constants and x is the error and $f(x)$ is the frequency of the error x. It is not very difficult to realise that the probability of occurrence of an error selected at random in the range x and $x+dx$ is greater, the greater the frequency $f(x)$ and also greater the width 'dx' of the range. Hence we may write

The probability that an error will be in between x and

$$x+dx=f(x)dx.$$

$\therefore$ The probability that an error will be in between a and $b=\int_a^b f(x)dx=$ shaded area (Fig. 17.2 a).

It is quite obvious that the probability of occurrence of any error between $-\infty$ and $+\infty$ is hundred per cent. This fact is called the 'normalisation condition'. Every distribution curve must satisfy this condition. For the Gaussian distribution we must have then,

$$1=\int_{-\infty}^{+\infty} f(x)dx=\int_{-\infty}^{+\infty} Ae^{-h^2(x-m)^2}dx=2A\int_0^{\infty} e^{-h^2(x-m)^2}dx.$$

Put $$x-m=u$$

$\therefore$ $$dx=du$$

$\therefore$ $$1=2A\int_0^\infty e^{-h^2u^2}du.$$

The task of evaluating this integral is not easy for beginners in Calculus. The list of integrals is given below. Students can use them freely without bothering about their evaluation.

$\therefore$ $$1=2A.\frac{1}{2}\sqrt{\frac{\pi}{h^2}}=\frac{A\sqrt{\pi}}{h}.$$

$\therefore$ $$A=\frac{h}{\sqrt{\pi}}.$$

$\therefore$ $$f(x)=\frac{h}{\sqrt{\pi}}e^{-h^2(x-m)^2}.$$

If x be the notation of the physical quantity, then distribution of x's is given by the same function and its error is $(x-\bar{x})$ where $\bar{x}$ is the average value of the physical quantity.

17.8. The Mean, Standard and Probable Error (deviation)

(i) *The mean deviation (error).* The average of deviations with proper sign is called the mean deviation.

Since $f(x)$ is the frequency of the error x, the number of errors in the range x and $x+dx$ is $f(x)dx$. That is, the '*population*' of the errors in the range x and $x+dx$ is $f(x)dx$.

Hence the total number of errors $=\int_{-\infty}^{+\infty} f(x)dx$. The integral is

---

$$\int_0^\infty e^{-bu^2}du=\frac{1}{2}\sqrt{\frac{\pi}{b}};\int_0^\infty e^{-bu^2}udu=\frac{1}{2b};\int_0^\infty e^{-bu^2}u^2du=\frac{1}{4}\sqrt{\frac{\pi}{b^3}}$$

---

from $+\infty$ to $-\infty$ because theoretically the error may be of any value in between $-\infty$ to $+\infty$.

The sum of x's is the range x and $x+dx$

$=x\times$ population of errors in the range x and dx

$=x\times f(x)dx.$

$$\therefore \quad \bar{x} \text{ (mean error)} = \frac{\text{Sum of } x\text{'s for the entire population}}{\text{the entire population}}$$

$$= \frac{\int_{-\infty}^{+\infty} xf(dx)dx}{\int_{-\infty}^{+\infty} f(x)dx} = \int_{-\infty}^{+\infty} xf(x)dx; \quad \left(\because \int_{-\infty}^{+\infty} f(x)dx = 1 \right)$$

$$\therefore \quad \bar{x} = \int_{-\infty}^{+\infty} xf(x)dx = \int_{-\infty}^{+\infty} \frac{h}{\sqrt{\pi}} e^{-h^2(x-m)^2} xdx$$

Put $$x - m = u \quad \therefore \quad dx = du.$$

Carrying out integrations with the help of standard integrals we have $$\bar{x} = m.$$

Thus the constant m of the distribution law represents the mean value of the errors.

$$\therefore \qquad f(x) = \frac{h}{\sqrt{\pi}} e^{-h^2(x-\bar{x})^2}.$$

The mean value of deviations (errors) in Gaussian distribution is zero as there are as many positive errors as there are negative errors. Hence $$f(x) = \frac{h}{\sqrt{\pi}} e^{-h^2x^2}.$$

If x be the notation of the physical quantity then

$$f(x) = \frac{h}{\sqrt{\pi}} e^{-h^2(x-\bar{x})^2},$$

where $\bar{x}$ is the mean value of the physical quantity.

(ii) *The mean square (standard) deviation.* The average of the squares of deviations is called mean square deviation. If x be the physical quantity, then $(x-\bar{x})$ is its deviation.

The sum of $(x-\bar{x})^2$'s for the population of errors in the range x and $x+dx = (x-\bar{x})^2 \times$ population of errors

$$= (x-\bar{x})^2 f(x)dx.$$

$\therefore$ The sum of $(x-x)^2$'s for the entire population of errors

$$= \int_{-\infty}^{+\infty} (x-\bar{x})^2 f(x)dx$$

By definition, $\sigma^2 = \dfrac{\text{sum of } (x-\bar{x})^2}{\text{the entire population}}$

$$\therefore \quad \sigma^2 = \frac{\int_{-\infty}^{+\infty} (x-\bar{x})^2 f(x\, dx}{\int_{-\infty}^{+\infty} f(x)dx} = \int_{-\infty}^{+\infty} (x-\bar{x})^2 f(x)dx \qquad \left(\because \int_{-\infty}^{+\infty} f(x)dx = 1\right)$$

$$\text{or} \quad \sigma^2 = \frac{h}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-h^2(x-\bar{x})^2} (x-\bar{x})^2 dx$$

Put $\qquad x-\bar{x}=u \quad \therefore \quad dx=du$

$$\therefore \quad \sigma^2 = \frac{h}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-h^2u^2} u^2 du = \frac{h}{\sqrt{\pi}} 2 \int_0^{\infty} e^{-h^2u^2} u^2 du$$

$$= \frac{2h}{\sqrt{\pi}} \frac{1}{4} \sqrt{\frac{\pi}{h^6}} = \frac{1}{2h^2}.$$

$$\therefore \quad h = \frac{1}{\sqrt{2}\sigma}.$$

$$\therefore \quad f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\bar{x})^2}{2\sigma^2}}.$$

If x be the notation of errors then $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$.

(iii) *The mean absolute deviation* α. The average of the absolute value of deviations (value after dropping plus or minus sign) is called the mean absolute deviation α. If x be the physical quantity then $|(x-\bar{x})|$ is the absolute deviation or error.

The sum of $|(x-\bar{x})|$'s for population of errors in the range x and $x+dx = |(x-\bar{x})| f(x)dx$.

$\therefore$ The sum $|(x-\bar{x}|$'s for the entire population

$$= \int_{-\infty}^{+\infty} |(x-\bar{x})| f(x)dx.$$

$$\therefore \quad \alpha=\frac{\int_{-\infty}^{+\infty} |(x-\bar{x})| f(x)dx}{\int_{-\infty}^{+\infty} f(x)dx}=\int_{-\infty}^{+\infty} |(x-x)| f(x)dx$$

$$\left(\because \int_{-\infty}^{+\infty} f(x)dx=1\right)$$

or $$\alpha=\int_{-\infty}^{+\infty} \frac{h}{\sqrt{\pi}} e^{-h^2(x-\bar{x})^2} |(x-\bar{x})| \, dx$$

$$=\frac{2h}{\sqrt{\pi}}\int_{0}^{\infty} e^{-h^2(x-\bar{x})^2} |(x-\bar{x})| \, dx.$$

(since we have to sum up only for absolute values of $|x-\bar{x}|$)

Put $$x-\bar{x}=u$$

$\therefore$ $$dx=du.$$

Then $$\alpha=\frac{2h}{\sqrt{\pi}}\int_{0}^{\infty} e^{-h^2u^2} u du=\frac{2h}{\sqrt{\pi}}\cdot\frac{1}{2h^2}=\frac{1}{\sqrt{\pi}h}$$

$$\therefore \quad \frac{\alpha}{\sigma}=\frac{\frac{1}{\sqrt{\pi}h}}{\frac{1}{\sqrt{2}h}}=\sqrt{\frac{2}{\pi}} \quad \text{or} \quad \sigma=\sqrt{\frac{\pi}{2}}\,\alpha=1{\cdot}25\alpha.$$

Thus $\sigma=1{\cdot}25\alpha$.

(iv) The Probable Error. The particular error (α_p) which has a probability $\frac{1}{2}$ is called the probable error. Its value is given by

$$\frac{1}{2}=\frac{h}{\sqrt{\pi}}e^{-h^2\alpha_p^2} \quad \text{or} \quad e^{-h^2\alpha_p^2}=\frac{\sqrt{\pi}}{2h} \quad \text{or} \quad e^{h^2\alpha_p^2}=\frac{2h}{\sqrt{\pi}}.$$

or $$\alpha_p=\frac{{\cdot}4769}{h} \text{ (after solving graphically)}$$

or $$\alpha_p=\frac{{\cdot}4769}{\frac{1}{\sqrt{2}\sigma}}={\cdot}4769\sqrt{2}\sigma$$

or $$\alpha_p={\cdot}6744\sigma$$

17.9. How to write Experimental Results

Our aim in any experiment is to find an unknown physical quantity. It is true that the true value of a physical quantity is the mean of an infinitely large sample (the universe or parent distribution) It is, however, impossible to find the mean of thousands and thousands of readings, because in practice, only a finite number (ten, twenty etc.) of observations are taken. The mean value X of an actual sample of a limited number of observations is taken as the 'best' value of the physical quantity. Next comes the question of uncertainty (deviation).

There are three conventions in this regard

(i) $X \pm \sigma$ (the standard deviation)

(ii) $X \pm \alpha$ (the absolute mean deviation)

(iii) $X \pm \alpha_p$ (the probable error)

One thing must be clearly stated here that σ (the standard deviation of the 'universe' distribution) can be computed from the standard deviation of a sample. This is, however, not true for the actual value of the physical quantity, that is from the mean value of a sample we cannot compute the mean value of the universe distribution. This is why the mean value of the sample distribution is taken as the 'best' value of the unknown physical quantity.

The probability of error being in between $+\sigma$ and $-\sigma$ is the area of the plot between $+\sigma$ and $-\sigma$. This is about 68% of the total area. The area between $+2\sigma$ and -2σ is 95% of the total area. This means that there is 68% probability of an error being less than $|\sigma|$ and 95% of being less than $|2\sigma|$. The absolute mean deviation represents 54·4% probability of an error being in between $+\alpha$ and $-\alpha$ The probable error represents 45·4% probability of error being between $+\alpha_p$ to $-\alpha_p$. *The most widely accepted convention nowadays is* (1). The final result, thus, is to be reported like this :

The observed physical quantity $= X \pm \sigma$.

17.10. Propagation of Mean Errors

In experiments we measure the basic quantities (length, mass and time etc.) and make calculations of some physical quantity with the help of standard formula connecting the physical quantity with the base quantities. For example, in a simple pendulum we measure the

length and the time period of the pendulum and hence calculate the value of g.

Errors in the measurement of the observed base quantities lead to errors in the calculated result. This is called propagation of errors.

PROPAGATION OF MEAN ERROR IN A SUM

The absolute error (not the relative error) in a sum is the sum of the errors of the observed quantities.

Proof. Suppose $C=A+B$. Let $\triangle C$, $\triangle A$ and $\triangle B$ be the absolute errors in C, A and B respectively. Then we have,

$$\begin{aligned}(C \pm \triangle C) &= (A \pm \triangle A)+(B \pm \triangle B)\\ &= (A+B) \pm (\triangle A+\triangle B)\\ &= C \pm (\triangle A+\triangle B) \quad \because \quad C=A+B.\\ \therefore \qquad \triangle C &= \triangle A+\triangle B.\end{aligned}$$

By Calculus. Suppose $C=A+B$ In calculus the error in a physical quantity is given by its differential.

Differentiating we have

$$dC=dA+dB.$$

PROPAGATION OF M ERRORS IN A DIFFERENCE

The absolute error (and not the percentage error) is the sum of the errors of the observed quantities.

Proof. Suppose $C=A-B$.

Then $C \pm \triangle C=(A \pm \triangle A)-(B \pm \triangle B)=A \pm \triangle A-B \pm \triangle B$

$$=(A-B) \pm (\triangle A+\triangle B).$$

We have not followed the ordinary rule of algebra in removing the bracket so far the error term is concerned because we want to be sure that the true value (unknown always) falls between the upper and the lower limits. We cannot afford that errors cancel out. Hence to get the maximum possible error all errors are treated as positive.

or
$$C \pm \triangle C=C \pm (\triangle A+\triangle B)$$
$$\triangle C=\triangle A+\triangle B.$$

By Calculus. Suppose $C=A-B$.

Differentiating, $dC=dA+dB$ (ignoring the minus sign due to reason given above).

PROPAGATION OF ERROR IN A PRODUCT

The relative error (not the absolute error) in a product is equal to the sum of the individual relative errors of the observed quantities.

Proof. Suppose $C=AB$.

Then $(C \pm \triangle C)=(A \pm \triangle A)(B \pm \triangle B)$

$$=AB \pm B\triangle A \pm A\triangle B \pm \triangle A\triangle B.$$

$\triangle A\ \triangle B$ is negligible with respect to other terms

$$C \pm \triangle C=C \pm B\triangle A \pm A\triangle B$$

or $$\pm \triangle C= \pm B\triangle A \pm A\triangle B$$

or $$\pm \frac{\triangle C}{C}= \pm \frac{B\triangle A}{C} \pm \frac{A\triangle B}{C}$$

or $$\pm \frac{\triangle C}{C}= \pm \frac{\triangle A}{A} \pm \frac{\triangle B}{B}$$

or $$\frac{\triangle C}{C}=\frac{\triangle A}{A}+\frac{\triangle B}{B}$$

By Calculus. Suppose $C=AB$.

Differentiating $dC=BdA+AdB$.

Dividing by C

$$\frac{dC}{C}=\frac{BdA}{C}+\frac{AdB}{C}$$

or $$\frac{dC}{C}=\frac{dA}{A}+\frac{dB}{B} \quad (\because \quad C=AB)$$

PROPAGATION OF MEAN ERROR IN A QUOTIENT

The relative error (and not the absolute error) in a quotient is equal to the sum of the individual relative errors of the observed quantities.

Proof. Suppose $C=A/B$,

Then $$C \pm \triangle C=\frac{A \pm \triangle A}{B \pm \triangle B}=(A \pm \triangle A)(B \pm \triangle B)^{-1}$$

$$=(A \pm \triangle A)B^{-1}\left(1 \pm \frac{\triangle B}{B}\right)^{-1}$$

or $C \pm \Delta C = (A \pm \Delta A)B^{-1}\left(1 \pm \frac{\Delta B}{B}\right)$, expanding and neglecting terms of higher powers as they are negligible.

or $$C \pm \Delta C = AB^{-1} \pm B^{-1}\Delta A \pm \frac{A\Delta B}{B^2} \pm \frac{\Delta A\ \Delta B}{B^2}.$$

$\Delta A\ \Delta B$ is negligible in comparison to other terms.

$$\therefore \quad C \pm \Delta C = AB^{-1} \pm B^{-1}\Delta A \pm \frac{A\Delta B}{B^2}.$$

$$= C \pm \frac{\Delta A}{B} \pm \frac{A\Delta B}{B^2} \quad \left(\because \quad C = \frac{A}{B} = AB^{-1}\right)$$

or $$\pm \Delta C = \pm \frac{\Delta A}{B} \pm \frac{A\Delta B}{B^2}$$

or $$\frac{\Delta C}{C} = \frac{\Delta A}{CB} + \frac{A\Delta B}{CB^2}$$

or $$\frac{\Delta C}{C} = \frac{\Delta A}{A} + \frac{\Delta B}{B}.$$

By Calculus. Suppose $C = \frac{A}{B} = AB^{-1}$

Differentiating we have

$$dC = B^{-1}dA + AB^{-2}dB$$ (ignoring minus sign for the reason given above)

or $$dC = \frac{dA}{B} + \frac{AdB}{B^2}$$

$$\therefore \quad \frac{dC}{C} = \frac{dA}{CB} + \frac{AdB}{CB^2}$$

or $$\frac{dC}{C} = \frac{dA}{A} + \frac{dB}{B}$$

17.11. Propagation Rule For Standard Deviations

PROPAGATION RULE FOR SUM AND DIFFERENCE

Suppose $C = A \pm B$. If ΔC, ΔA and ΔB are the errors of C, A and B respectively then

$$\Delta C = \Delta A + \Delta B$$

Now, $\sigma_C^2 = \frac{1}{N}\Sigma(\triangle C)^2$ where N is the number of observations.

$$= \frac{1}{N}\Sigma(\triangle A + \triangle B)^2$$

$$= \frac{1}{N}\Sigma(\triangle A)^2 + \frac{1}{N}\Sigma(\triangle B)^2 + \frac{1}{N}\Sigma 2 \triangle A \triangle B$$

The errors $\triangle A$ and $\triangle B$ are random; they are as often positive as they are negative. Hence the summation of terms like $\Sigma \triangle A \triangle B$ will be zero.

$\therefore$ $\sigma_C^2 = \sigma_A^2 + \sigma_B^2$ ..... Propagation rule of S. D's in a sum and difference.

PROPAGATION RULE FOR PRODUCT AND QUOTIENT

Suppose in general $C = A^m B^n$

Then by the rule of partial differentiation

$$\triangle C = \frac{\partial C}{\partial A}\triangle A + \frac{\partial C}{\partial B}\triangle B$$

$$\text{Now } \sigma_C^2 = \frac{1}{N}\Sigma(\triangle C)^2 = \frac{1}{N}\left[\left(\frac{\partial C}{\partial A}\right)^2 \Sigma(\triangle A)^2 + \left(\frac{\partial C}{\partial B}\right)^2 \Sigma(\triangle B)^2 + 2\frac{\partial C}{\partial A}.\frac{\partial C}{\partial B}\Sigma\triangle A \triangle B\right]$$

$\Sigma\triangle A.\ \triangle B = 0$ for the reason explained above.

$$\therefore \quad \sigma_C^2 = \left(\frac{\partial C}{\partial A}\right)^2 . \sigma_A^2 + \left(\frac{\partial C}{\partial B}\right)^2 \sigma_B^2$$

Now $\quad C = A^m B^n$

$$\therefore \quad \frac{\partial C}{\partial A} = B^n mA^{m-1} \text{ and } \frac{\partial C}{\partial B} = A^m nB^{n-1}$$

$$\therefore \quad \sigma_C^2 = B^{2n} m^2 A^{2m-2} \sigma_A^2 + A^{2m} n^2 B^{2n-2} \sigma_B^2$$

Dividing throughout by C^2

$$\left(\frac{\sigma_C}{C}\right)^2 = m^2\left(\frac{\sigma_A}{A}\right)^2 + n^2\left(\frac{\sigma_B}{B}\right)^2$$

When $\quad m = n = 1$

$$\left(\frac{\sigma_C}{C}\right)^2 = \left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2 \text{.. rule for propagation of S. D's in a product.}$$

When $m=1, n=-1$

$$\left(\frac{\sigma_C}{C}\right)^2=\left(\frac{\sigma_A}{A}\right)^2+\left(\frac{\sigma_B}{B}\right)^2 \quad \text{.. for propagation of S. D's in a quotient.}$$

Examples

1. *A student makes ten trials for the diameter of a wire by a screw gauge. His readings are* : 2·35, 2·32, 2·39, 2·38, 2·33, 2·31, 2·34, 2·34, 2·36, 2·37 *mm. Calculate the standard deviation.*

Sol.

d	$\bar{d}=\Sigma d/n$	$\Delta d=d-\bar{d}$	$\lvert\Delta d\rvert$	$\lvert\Delta d\rvert^2$
2·35		·00	·00	·0000
2·32		−·03	·03	·0009
2·39		+·04	·04	·0016
2·38	2·35	+·03	·03	·0009
2·33		−·02	·02	·0004
2·31		−·04	·04	·0016
2·34		−·01	·01	·0001
2·34		−·01	·01	·0001
2·36		+·01	·01	·0001
2·37		+·02	·02	·0004

$$\text{S.D.}=\sqrt{\frac{\Sigma\lvert\Delta d\rvert^2}{n-1}}=\sqrt{\frac{\cdot 0061}{10-1}}=\sqrt{\frac{\cdot 0061}{9}}=\cdot 026 \textbf{ Ans.}$$

2. *The measure of the diameter of a cylinder is* (7·20 ± ·1%) *cm and its length* (10·0 ± ·2%) *cm. Calculate the percentage error of in the volume. Also, calculate the absolute error in the volume.*

Sol. We have $V=\pi r^2 l$

$\therefore$ $dV=\pi(r^2 dl+2rdrl)$

or $$\frac{dV}{V}=\frac{dl}{l}+\frac{2dr}{r}$$

$\therefore$ % error in volume = ·2 + 2 × ·1 = ·4 **Ans.**

Volume $=\pi\left(\frac{7\cdot 20}{2}\right)^2\times 10\cdot 0=407\cdot 203=407$ cm³ (the least number of significant figures in the diameter and the length is 3 and so the volume is rounded off up to 3 significant figures.)

$\therefore$ Volume = 407 ± ·4% = (407 ± 2) cm³. **Ans.**

3. *Show that the percentage error in the measurement of* $\frac{m}{B}\left(\frac{\text{magnetic moment}}{\text{magnetic induction}}\right)$ *is minimum when the deflection is near about* 45°.

Sol. In a deflection magnetometer in tan$A$-position

$$\frac{m}{B}=\frac{4\pi}{\mu_0}\frac{(d^2-l^2)^2}{2d}\tan\theta \quad \text{or} \quad m=k\tan\theta$$

where k stands for the constant $\frac{4\pi(d^2-l^2)^2}{\mu_0 2d}.B$.

Now $$\partial m=k\sec^2\theta\partial\theta$$

$$\therefore \quad \frac{\partial m}{m}=\frac{k\sec^2\theta\partial\theta}{k\tan\theta}=\frac{\partial\theta}{\sin\theta\cos\theta}=\frac{2\partial\theta}{\sin 2\theta}$$

$\therefore$ % error in $m=\frac{\partial m}{m}\times 100=\frac{2\partial\theta}{\sin 2\theta}\times 100$

Obviously, the % error is minimum when $\sin 2\theta$ is maximum, that is, $\sin 2\theta=1$ or $\theta=45°$ Proved.

4. *The voltage across a lamp* $6{\cdot}0\pm 0{\cdot}1$ *volt and the current through it is* $4{\cdot}0\pm 0{\cdot}2$ *ampere. What is the power consumed, and what is the error in it? The errors quoted here are the S. D's in the respective measurements.*

Sol. We know $P=VC$

(Power is the product of P. D. and current.)

$$\left(\frac{\sigma_P}{P}\right)^2=\left(\frac{\sigma_V}{V}\right)^2+\left(\frac{\sigma_C}{C}\right)^2$$

Here $$P=6{\cdot}0\times 4{\cdot}0=24$$

$$\therefore \quad \left(\frac{\sigma_P}{24}\right)^2=\left(\frac{0{\cdot}1}{6}\right)^2+\left(\frac{0{\cdot}2}{4}\right)^2=(0{\cdot}052)^2$$

or $$\sigma_P={\cdot}052\times 24=1{\cdot}248=1{\cdot}2. \quad \textbf{Ans.}$$

$$\therefore \quad P=24\pm 1{\cdot}2 \text{ watts.} \quad \textbf{Ans.}$$

5. *The mass of an object is* $24{\cdot}320\pm 0{\cdot}005$ *kg and the volume* $10{\cdot}2\pm 0{\cdot}05$ m^3. *Calculate the density of the object, the errors quoted here are mean errors.*

Sol. We have $$D=\frac{M}{V}$$

$$\therefore \qquad \frac{\Delta D}{D}=\frac{\Delta M}{M}+\frac{\Delta V}{V}$$

Here, $D=\frac{24\cdot320}{10\cdot2}=2\cdot3843=2\cdot38$ (the least significant figure being 3 in volume).

$$\therefore \quad \frac{\Delta D}{2\cdot38}=\frac{0\cdot005}{24\cdot320}+\frac{0\cdot05}{12\cdot2}=0\cdot0002+\cdot00409=\cdot004298.$$

$\Delta D=\cdot01022=\cdot010$ (the least significant figure in the error being 3).

$$\therefore \quad D=(2\cdot38+\cdot010)\text{ kgm}^{-3}. \quad \textbf{Ans.}$$

QUESTIONS

(A)

1. In a 'product' of two quantities (a) the absolute error of the product is equal to the sum of the absolute errors of the two quantities, (b) the absolute error of the product is equal to the sum of the percentage errors of the quantities, (c) the percentage error of the product is equal to the sum of the percentage errors of the two quantities, (d) the percentage error of the product is equal to the difference of the percentage error of the two quantities.

2. In a quotient of two quantities (a) the absolute error of the quotient is equal to the sum of the absolute errors of the two quantities, (b) the absolute error of the quotient is equal to the sum of the percentage of the two quantities, (c) the percentage error of the quotient is equal to the sum of the percentage errors of the two quantities, (d) the percentage error of the quotient is equal to the difference of the percentage errors of the denominator and the numerator.

3. The significant figures in $2\cdot8\times10^4$ is (a) 2, (b) 4, (c) 6, (d) 7.

4. When 7·325 is rounded off up to three significant figures, it is equal to (a) 7·32, (b) 7·33, (c) 7·34, (d) 732.

5. If length is $(4\cdot7\pm\cdot02)$ cm and breadth is $(4\cdot73\pm\cdot005)$ cm, the area should be of (a) 2 significant figures, (b) 3 significant figures, (c) 5 significant figures, (d) 7 significant figures.

6. The standard deviation of the parent distribution is (a) $\sqrt{\frac{\Sigma|\delta_i|^2}{n}}$, (b) $\sqrt{\frac{\Sigma|\delta_i|^2}{n+1}}$, (c) $\sqrt{\frac{\Sigma|\delta_i|^2}{n-1}}$, (d) $\sqrt{\frac{\Sigma|\delta_i|^2}{n(n-1)}}$, where n is a finite number of observations in a sample distribution.

7. If $C=A+B$ and $\triangle C$, $\triangle A$ and $\triangle B$ are the mean errors in C, A and B respectively then (a) $\frac{\triangle C}{C}=\frac{\triangle A}{A}+\frac{\triangle B}{B}$, (b) $\triangle C=\frac{\triangle A}{A}+\frac{\triangle B}{B}$, (c) $\triangle C=\triangle A+\triangle B$, (d) $\triangle C=\triangle A-\triangle B$.

8. If $C=A-B$ and $\triangle C$, $\triangle A$ and $\triangle B$ are the errors in C, A and B respectively then (a) $\frac{\triangle C}{C}=\frac{\triangle A}{A}+\frac{\triangle B}{B}$, (b) $\triangle C=\frac{\triangle A}{A}+\frac{\triangle B}{B}$, (c) $\triangle C=\triangle A+\triangle B$, (d) $\triangle C=\triangle A-\triangle B$.

9. If σ_A, σ_B and σ_C are the S. D's in A, B, C respectively and $A=B\times C$, then (a) $\sigma_A=\sigma_B+\sigma_C$, (b) $\sigma_A=\sqrt{\sigma_B^2+\sigma_C^2}$, (c) $\left(\frac{\sigma_A}{A}\right)^2=\left(\frac{\sigma_B}{B}\right)^2+\left(\frac{\sigma_C}{C}\right)^2$ (d) $\sigma_A=\sqrt{\sigma_B\sigma_C}$.

Ans. 1. c, 2. c, 3. a, 4. a, 5. a, 6. c, 7. c, 8. c, 9. c.

(B)

1. Distinguish carefully between 'errors' and 'deviations' (or residuals). Which one can be computed in practice and why ?

2. Show that in a sum it is the absolute error that propagates and not the percentage error.

3. Show that in a product it is the percentage error that propagates.

(C)

1. Define the 'average', 'r. m. s.' and 'probable' value of errors.

2. Explain carefully why large number of readings should be taken to obtain the best possible value of a given physical quantity.

3. What do you mean by the least square fit to a straight line. Explain how to obtain 'm' and 'c' of the best 'fit' line.

(D)

1. An ammeter and a voltmeter connected in series and parallel with a resistor respectively read as follows

Voltmeter	2·00	1·80	1·60	1·00	0·60	volt
Ammster	0·98	0·91	0·82	0·49	0·31	ampere

Write the report about the resistance of the resistor in terms of S. D.

(Ans. 1·89±·045)

2. The measures of the three sides of a triangle are (5±·10) cm, (3·5±·12) cm and (6·5±·11) cm. Calculate the percentage error in the perimeter of the triangle. **(Ans. 2·2)**

3. The velocity of sound through a gaseous medium is given by

$$C=\sqrt{\frac{E}{D}}$$

Find the formula giving the propagation of error in C due to errors in E and D.

$$\left(\text{Ans. } \frac{\partial C}{C}=\frac{1}{2}\frac{\partial E}{E}+\frac{1}{2}\frac{\partial D}{D}\right)$$

4. The acceleration due to gravity is given by

$$t=2\pi\sqrt{\frac{l}{g}}$$

Establish the formula giving the propagation of the error in g due to errors in t and l. $\left(\text{Ans. } \frac{\partial g}{g}=2\frac{\partial t}{t}+\frac{\partial l}{l}\right)$

5. The flow of a liquid through a capillary tube is given by

$$V=\frac{\pi P r^4}{8\eta l}.$$

Establish the formula giving the propagation of the error in η due to errors in V, l, P and r^4. Which one of these quantities should be measured carefully and why ?

$\left(\text{Ans. } \frac{\partial \eta}{\eta}=4\frac{\partial r}{r}+\frac{\partial p}{p}+\frac{\partial l}{l}+\frac{\partial V}{V};\right.$ the radius because it contributes maximum to the error in η)

(E)

1. In a sum of two quantities,..................(percentage or absolute error) propagates.

2. In a product of two quantities,..................(percentage or absolute error) propagates.

3. Can you compute directly from observations the average value of 'errors' (absolute) ?

4. Can you compute directly from observations the average value of 'deviations' ?

(Ans. 1. absolute error, 2. percentage error, 3. No, 4. Yes,)

CHAPTER 18

THE UNIVERSE**

18.1. The Universe : Stars and Galaxies : Quasars : Pulsars

The entire system of heavenly bodies is called the universe. The small part of the universe which we live in is called the solar system. The nucleus of the solar system is the sun. It has nine nonluminous bodies revolving round it in different orbits called its planets. These are Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune and Pluto. The farthest planet Pluto is at a distance of 6×10^{12} m from the earth. The twinkling luminous small bodies are called stars. Many of them are much bigger in size and luminosity than the sun. They appear small because they are far off. The nearest star 'Alpha centauri' is at a distance of 4 light-years. We can see many stars on a clear night and many more with the help of telescopes. The space between the stars is not completely empty. It is filled thinly with interstellar matter consisting of gas and cosmic dust (particles of sizes approximately $\frac{1}{256}$ mm are dust particles). The 'milky way' or the Akashganga that we see as a band of white light in the sky is nothing but a cluster of stars. A telescope resolves the band into millions of faint stars. The solar system and the visible stars are also constituent units of this milky way system called 'a galaxy'. There are billions of galaxies like our own galaxy in this universe. So vast is the universe !! On the top of this staggering vastness of the universe, it is ever expanding as is evidenced by the 'redshift' of some of the luminous heavenly bodies (Doppler's effect).

Hubble Law

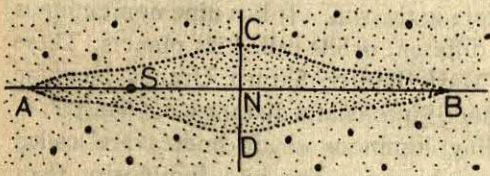
Edwin Hubble, a great astronomer, gave a relation between the distance and velocity of recession of a luminous heavenly body. The velocity of recession is calculated from the 'redshift' by applying Doppler's principle. According to Hubble the speed of recession (v) of a heavenly body increases proportionately with its distance 'r'. That is,

$$v = Hr \qquad \text{.. (1)}$$

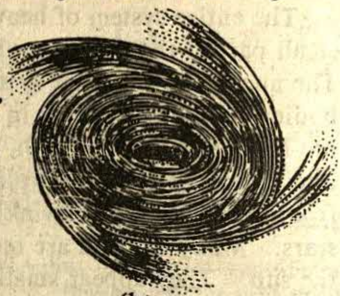
where H is a constant called Hubble's constant. This is known as Hubble's law. The value of the constant is approximately $1{\cdot}6 \times 10^{-2}$ metre per second per light-year, corresponding to

$1{\cdot}7 \times 10^{-18}\text{s}^{-1}$ (1 light-year $= 9{\cdot}4605 \times 10^{15}$ m).

All galaxies do not look alike. Some galaxies are irregular in shape, some look like a whirlpool called spiral galaxies (Fig. 18.1) and some are elliptical in shape. Our own galaxy (milky way) and the Andromeda galaxy are examples of spiral galaxies. The edge-on



(a)



(b)

Fig. 18.1

view of a galaxy is a flat lens-shaped disc which is thicker near the centre and thins out towards the edges (Fig. 18.a). Fig. 18.a shows the egde-on view of our own galaxy (Milky Way). The line AB represents the main plane of the galaxy. S represents the position of the sun. Note that the sun lies well away from the centre N, out toward one edge. Surrounding the main system are globular clusters (large dots) and scattered stars (small dots). Normally all galaxies emit radio radiations which are very small in comparison to their light output. But there are certain galaxies which are found to emit million times more energy in the radio region compared to normal galaxies also called 'quiet' galaxies so far radio emission is concerned. These are called radio galaxies. They are identified by their peculiar structure that they consist of a central optical galaxy G and two radio sources R_1 and R_2 occurring on either side of the optical part G, like the two ears on the two sides of the face of a man (Fig. 18.2).

The source of energy of all stars is the thermonuclear process, i.e. the principle of hydrogen bomb. When all the fuel burns out, the star becomes 'dead'

Fig. 18.2

Astronomers have classified stars into types, designated by initials, forming the sequence $O, B, A, F, G, K, M, R, N, S$ in which temperature decreases from O to S. Since according to

Wien's displacement law, λT=constant, the colour of stars (colour is determined by the radiation that the star emits dominantly) changes from blue to red in the above sequence. We owe the explanation of spectra of stars to Dr. M. N. Saha and his theory of ionisation enables us to determine the surface temperature of stars from their spectra.

From the consideration of brightness of stars, they are divided into different classes called their magnitudes. The brightest stars are arbitrarily put in one group and they were called the stars of the first magnitude.* The stars whose brightness is 2·512 times fainter are called stars of the second magnitude. The stars 2·512 times fainter than the second magnitude or $2{\cdot}512^2$ times fainter than the first magnitude stars are called stars of the third magnitude. In general the stars of the *n*th magnitude is $(2{\cdot}512)^{n-1}$ times fainter than the first magnitude stars. Apart from these normal stars we have many other heavenly objects having starlike appearance but differing from the normal stars in respect of mass, luminosity, stability, variability, radio emission etc. These are :

Meteors : Shooting stars are called meteors. They do not occur occasionally but a patient observer may observe about half a dozen in an hour which may increase to more than hundred in an hour. This phenomenon of increased occurence of meteors is called 'meteor shower'. Meteors are not stars. Their origin—whether they belong to the solar system or other heavenly bodies—is still not obvious but definitely they are detached parts of heavenly bodies shooting into our atmosphere. They burn up in the atmosphere due to excessive friction and appear as shooting stars. Some of the largest meteors survive their passage through the air and reach the earth. The largest of all weigh about some hundred of tons. In all, about ten tons of material are added to our earth every day, by the impact of meteors. It is found that there are two distinct types of meteors, the stone and iron varieties. The stone ones usually contain some iron, but on the whole their composition corresponds closely with the outer rocky parts of the earth; the iron meteors contain up to 99% nickel-iron alloy.

Comets : They are luminous objects that move rapidly across the the sky, coming into sight as a faint diffuse point and developing

*Absolute magnitude of a star corresponds to the brightness if it were taken to a standard distance (32·6 light-year from the sun).

the characteristic tail as they approach the sun. As they recede, after only a few days, the tail swings round so that it always points away from the sun, as though the sun were continually blowing it away from itself.

Nova and *Supernova :* A nova, sometimes called a new or temporary star, is a kind of variable star. They blaze suddenly into view in the sky and rise quickly from extreme faintness to great brightness. They then subside to their former faintness. One of the most brilliant novae ever seen was discovered by Tycho Brahe in 1572 in the constellation* Cassiopeia. This was so bright that it was visible even in broad daylight. It gradually decreased in brightness and after 16 months was seen no more. Supernovae are rare guest stars like novae but after disappearance they leave benind a vast glowing and expanding mass to be seen even after hundreds of years after disappearance. Uptill now only three supernovae have been observed respectively in 1054, 1572 and 1604 A.D. The 1054 supernova, watched by Chinese and Japanese astronomers, has left an expanding mass glowing with light even after 900 years of its disappearance; it is known as the Crab nebula in the constellation of Taurus. It is believed that supernova is due to release of energy due to nuclear fusion process going on inside an extinct star, until the energy released was so fast that the star exploded with the violence of a colossal thermonuclear bomb. This is called 'supernova explosion'.

Dwarfs : Very dense stars shining very feebly are called dwarfs. Kuiper's star is a famous dwarf. It is as massive as the sun but as small as the earth in size. Because of their small size they are justifiably called dwarfs. A man will weigh 'fifty million tons' on the surface of Kuiper's star. It is believed that the reason for high density of dwarfs is that all the constituent fundamental particles of matter, namely electrons, protons and neutrons are packed tightly together; there is no waste of space, whereas an ordinary atom is mostly empty space.

Quasars : Quasars are powerful radio sources having starlike appearance. They differ from other radio sources in respect of their abnormally high 'redshift' and staggering brilliancy. They are receding at tremendous speed ranging from ·5*c* to ·8*c* where *c* is the speed of light as observed from their 'redshift' Their brilliancy is equivalent

*In early times the people who watched the sky a great deal thought that the stars formed figures resembling all sorts of objects, particularly man and animals. These are called constellations. There are 88 constellations.

to million million suns, so that a single quasar is more luminous than a galaxy. They appear like a star because they are million and million light-years off.

Pulsars : Pulsars are also radio sources but their intensity of radiation varies with time and hence the name 'pulsars'. They are observed in Crab Nebula. It is believed that pulsars are 'neutron stars' of very small size and credible density, which are themselves supposed to be burn-out or last stage of an ordinary star.

18.2. Astronomical Instruments : Optical and Radio Telescopes

(a) *Optical Telescopes :* See chapter 7, Light, article 7.3 (2).

(b) *Radio Telescopes :* Karl Jansky of America while carrying out the noise level in a short wave radio receiver with directional aerial system found that the signal always occurred when the aerial pointed out in a certain direction in space. This direction turned out to be the centre of our galaxy and Jansky was able to say, without any doubt, that he was listening to a signal 'broadcast' from the Milky Way. Jansky's discovery opened up a new branch of astronomy, namely, radio astronomy and many countries developed 'radio telescopes' of high precision.

A radio telescope essentially consists of a steerable paraboidal reflector *R*. In the 'altitude-azimuth' mounting, steering is achieved by an elevation movement about an axis between towers *A* and *B*, and an azimuthal movement by a movement of the towers round a circular railway track *T*.

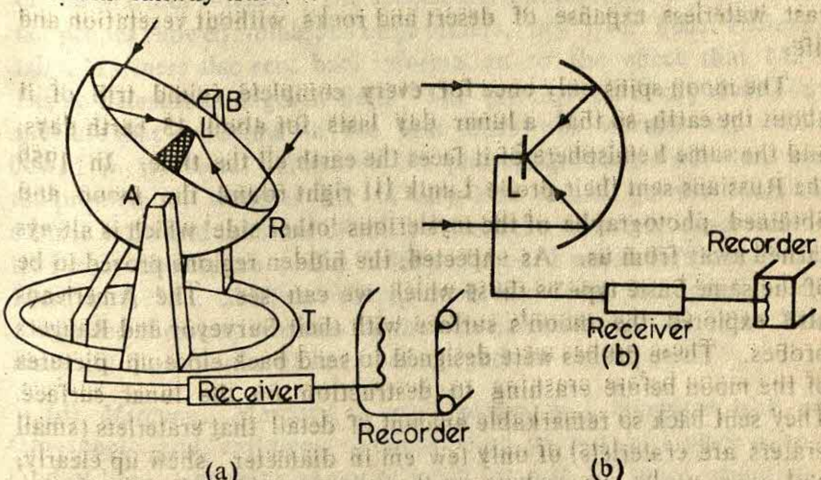


Fig 18.3

The reflector collects and focusses radio waves just as an optical telescope collects and focusses light. At the focus the radio signals are received on an aerial *L*. The signals received on the aerial are delivered to a receiver. The ouput from the receiver drives a mechanical recorder which traces waves on moving paper.

18.3. Exploration of the Solar System by Probes

Outer space exploration by probes (bodies sent for various scientific operations) started with the launching of the first artificial satellite—Russia's Sputnik I in 1957. The main aim of space scientists has been to gain the first-hand knowledge about the objects in the solar system and to search for life on them. Their first target was obviously our own atmosphere. In 1958 the first American artificial satellite Explorer I discovered Van Allen radiation belts of charged particles trapped in earth's magnetic field. Scientists studied thoroughly well about the density, temperature and composition of the earth's atmosphere. Thereafter they turned toward the moon and the planets.

(a) *Moon :* The first probe sent to the moon was Russian space probe Lunik I which went past the moon at a distance of about 6400 km. The next probe Lunik II, sent up in the same year after few months, actually hit the lunar surface. These probes confirmed that the moon has no detectable atmosphere, and neither are there any radiation zones of the Van Allen type, which indicates a virtual lack of any magnetic field of the moon. The surface of the moon is a vast waterless expanse of desert and rocks without vegetation and life.

The moon spins only once for every complete round trip of it about the earth, so that a lunar day lasts for about 15 earth days, and the same hemisphere of it faces the earth all the time. In 1959 the Russians sent their probe Lunik III right round the moon and obtained photographs of the mysterious 'other side' which is always turned away from us. As expected, the hidden regions proved to be of the same basic type as those which we can see. The Americans also explored the moon's surface with their Surveyor and Rangers probes. These probes were designed to send back close-up pictures of the moon before crashing to destruction on the lunar surface. They sent back so remarkable amount of detail that craterlets (small craters are craterlets) of only few cm in diameter shew up clearly, and seen to be everywhere on the moon's surface. In July 1969 in

the historic event of space exploration Neil Armstrong set his little step but a giant step for mankind on the soil of the moon. They brought back lunar soil and rock only to be examined in our laboratories. Analysis of lunar soil and rock has shown that it is similar to those in the earth's crust. The age of the lunar rocks agrees with the age of the earth (about 4½ billion years), confirming the belief that all objects in the solar system were formed more or less simultaneously.

(b) *Venus :* The next world to be attacked by space scientists was Venus, which has often been described as the earth's twin in respect of radius, mass and density. Naturally it was thought in the beginning that it could have water and life similar to the earth. But Vanera probes sent by Russians showed Venus had a dense atmosphere containing 95% carbon dioxide and its atmospheric pressure is 100 times our atmospheric pressure and its surface temperature is 480°C. Under these circumstances there is no possibility of life on Venus.

(c) *Mars :* Mars, with its thin atmosphere, chilly climate, red-soil like colour, white polar caps and 24 hours duration of day and night was supposed to support some primitive forms of life, such as lichens, if not man-like beings. The Americans started exploring Martian surface with their Mariner probe programme and Russians with their Zond probe programme. They were designed to pass within few thousand kilometres of Mars and send back close-up photographs of the Martian surface. The spectacular photographs sent by Mariner probes showed numerous large craters, dry river beds. Incidentally, Mariners also sent back information to the effect that Mars has no detectable magnetic field. The photograph of dry river beds indicate that sometime in the past Mars was warmer and water flowed in its rivers. However, the photographs could not clear up the problem of whether life of any kind exists on the Martian surface. To find solution of this problem the American space scientists sent two space crafts Viking 1 and 2 which soft landed on Mars in 1976. The first Viking took excellent photographs and conducted three experiments on the Martian soil to detect life. The results of the experiment have been negative.

(d) *Mercury :* Mercury, a small planet with complete lack of atmosphere and extremes of temperature, cannot be a place of any kind of life. Hence it was not of much interest to space scientists. The only probe sent to it by the American scientists is Mariner-10

which passed within a few kilometres of Mercury in 1973. Photographs taken at that time show that mercury is pockmarked by craters of all sizes.

18.4. Theories of the Origin of the Universe

There are three theories of origin of the universe : (i) The "Big Bang" theory, (ii) The "Steady-state" theory, (iii) The theory of pulsating universe.

(i) *The 'Big Bang' Theory :* According to this theory, due largely to Abbe' Lemaître, all matter in the universe was created suddenly as a dense consolidated mass consisting of "primaeval atom" (atom which is simply a mere jumble of electrons, protons and neutrons, leaving no empty space), so dense that 10^{17} kgm^{-3} would be a moderate estimate. An explosion occurred at about 20 billion years ago and since then the matter is moving away in all directions. Expansion went on for thousands of millions of years, until the universe had a diameter of about one billion light-years. Meanwhile the original density had been much reduced, so that instead of a mere jumble of fundamental particles, different types of normal atoms (hydrogen in particular) started to form. The expanding concept of the universe is in direct clash with Newton's law of gravitation. The theory conceives a kind of repulsion called cosmical repulsion which increases with some power of distance so that over short distances (here even several million light-years also are to be treated as short distances) cosmical repulsion is negligible, but at greater distances it supercedes the Newtonian gravitational attraction. This explains why individual galaxies, and group of galaxies, are relatively stable but the velocities of the distant bodies increases due to unbalanced cosmic repulsive force.

According to this theory the universe has beginning and end. When the velocity of expansion equals the velocity of light (according to Hubble's law this velocity is acquired at a distance of 20 billion light-years), light from galaxies moving with this speed will never reach us. The distance of such galaxies is called the boundary of the 'observable universe'. On account of continuous expansion more and more galaxies will go beyond this boundary and they will be lost.

(ii) *The steady-state theory :* The universe had no beginning, and will have no end. It has always existed; as old stars and galaxies die, fresh material is created out of 'nothingness' to form new

systems. The rate of creation of matter is so slow that it is quite undetectable, but the result is the universe in the steady state. A time-traveller coming back in, say, billions of years would have the same broad impression as we have to-day. He, too, would see the same approximate number of stars and galaxies, even though they would not be the same stars and galaxies as ours.

(iii) *The theory of pulsating universe :* This theory begins in the same way as the 'big bang' theory but it denies the end of the universe by assuming that expansion may be stopped by gravitational pull and the universe may contract again. Thus we may have alternate expansion and contraction giving rise to a pulsating universe.

None of the theories is perfect. The 'big bang' theory and the pulsating universe theory' require us to assume that creation occurred at some moment suddenly; therefore, what happened before that ? The steady state theory is no better, since it asks us to imagine creation of matter out of 'nothingness'. Once we have our material, we can trace its development, but material must be there to begin with. If one suggests that to begin with we had energy and matter was created by conversion of energy; this may be true but that also does not help us. Then question arises which came first ? And how ?

Examples

1. *The redshift of a quasar is as much as* 37%. *Assuming Hubble's law, calculate its distance in metres and light-years. Hubble's constant* $=1{\cdot}7\times10^{-18}\ s^{-1}$. *1 light-year* $=9{\cdot}46\times10^{15}\ m$.

Sol. We have by Doppler's principle,

$$\nu'=\frac{c}{c+v}.\nu \quad \text{when the source is receding.}$$

Hence, $$\frac{\Delta\nu}{\nu}=\frac{\nu-\nu'}{\nu}=\frac{v}{c+v}.$$

Here $$\frac{\Delta\nu}{\nu}={\cdot}37.$$

$\therefore$ $${\cdot}37=\frac{v}{c+v}, \quad \text{or} \quad v={\cdot}59c.$$

By Hubble's law we have, $v=Hr$

or $$r=\frac{v}{H}=\frac{59\times 3\times 10^8}{1\cdot 7\times 10^{-18}}=1\cdot 04\times 10^{26}\text{ m. Ans.}$$

Since 1 light-year $=9\cdot 46\times 10^{15}$ m,

$$\therefore \quad r=\frac{1\cdot 04\times 10^{26}}{9\cdot 46\times 10^{15}}=1\cdot 1\times 10^{10}\text{ light-years. Ans.}$$

2. *Calculate the boundary of the observable universe assuming Hubble's law. Given that Hubble's constant* $=1\cdot 7\times 10^{-18}\ s^{-1}$.

Sol. The boundary of the 'observable universe' is the distance of the star moving with the velocity of light.

We have by Hubble's Law, $v=Hr$.

Here $v=c$. $\therefore\ c=Hr$

or $$r=\frac{c}{H}=\frac{3\times 10^8}{1\cdot 7\times 10^{-18}}=1\cdot 76\times 10^{26}\text{ m}$$

or $$r=\frac{1\cdot 76\times 10^{26}}{9\cdot 46\times 10^{15}}=2\cdot 0\times 10^{10}\text{ light-years}$$

$$=20\text{ billion light-years. Ans.}$$

QUESTIONS

(A)

1. The nearest star to our solar system is (a) Procyon, (b) Sirius, (c) Alpha centauri, (d) Spica.

2. A star of the sixth magnitude is (a) 5 times fainter, (b) 5 times stronger, (c) 10 times stronger, (d) 10 times fainter, than the star of the first magnitude.

3. Venus has (a) very dense atmosphere, (b) thin atmosphere, (c) no atmosphere, (d) same atmosphere as ours.

4. The moon spins once about its axis (a) in 24 earth hours, (b) 15 earth hours, (c) 30 earth days, (d) 12 earth months.

5. The brilliancy of a quasar is equal to (a) the brilliancy of the sun, (b) brilliancy of million million suns, (c) zero, (d) the brilliancy of 10 suns.

6. Which one of the following is the twin of the earth ? (a) Mars, (b) Venus, (c) Mercury, (d) Saturn.

7. The boundary of the observable universe is at a distance (a) 10 million light-years, (b) 10 billion light-years, (c) 20 million light-years, (d) 20 billion light-years.

Ans. 1. c, 2. d, 3. a, 4. c, 5. b, 6. b, 7. d.

(B+C)

1. Give a list of various constituents of the universe.

2. Describe and compare the various kinds of telescopes used for astronomical considerations.

3. Describe the structure and contents of the Milky Way.

4. What are quasars ? What are their characteristics ?

5. Give an account of the exploration of the solar system by probes.

6. Briefly discuss the three main cosmological theories of the universe.

7. State and explain Hubble's Law.

(D)

1. The lens of our eye has a diameter of 8 mm. How much fainter objects can be seen through a telescope of 120 cm aperture as compared to the faintest naked eye stars ?

(**Ans.** $2 \cdot 25 \times 10^4$ times fainter)

2. Calculate the distance of the star whose redshift is only 1%. Assume Hubble's law. Hubble's constant $= 1 \cdot 7 \times 10^{-18}\ s^{-1}$ (**Ans.** 2 billion light-years)

3. The planet Mars has two satellites Phobos and Deimos. When they are in opposition relative to Mars their distances are 25″ and 62″ respectively as observed from the earth. Calculate the distances of the two satellites from Mars in astronomical units. Given that the distance of Mars from the eartl is 0·524 A. U.

(**Ans.** 6×10^{-5} A. U; 16×10^{-5} A. U.)

CHAPTER 19

THE THEORY OF RELATIVITY**

19.1. Space and Time

All physical phenomena take place in space and time. Every physical law in every physical domain contain explicitly or otherwise space-time relationships.

We know from our experience that space and time possess certain symmetry properties. Among these properties, the most important is their space-time 'homogeneity'. Because of time homogeneity a physical phenomenon is always the same whenever it is observed. More than two thousand years ago Archemedes discovered laws of buoyancy, yet today they can be readily reproduced, provided the conditions of observations are the same. The physical equivalence of different instants of time so far physical phenomena is concerned is called the principle of homogeneity of time. Because of space homogeneity a physical phenomenon is always the same, wherever it is observed. The same physical experiment conducted in Moscow or New York gives the same result. The physical equivalence of different points of space so far physical phenomenon is concerned is called the principle of homogeneity of space.

19.2. Frame of Reference : Event and Point

It is impossible to investigate physical phenomena without introducing a frame of reference, reference bodies or landmark relative to which observations are carried out. Imagine a road across a vast maidan. A car travelling along it in the distance at first appears to stand still. But when one finds a landmark—a telegraph pole, a mile-post or a tree and continues to observe it for some time one detects a change in the position of the car and declares that the car is moving. Therefore, the concept of the frame of reference is a fundamental one in physics. *Definition* : 'A frame of reference is a body which is regarded fixed for a given problem and with respect to which the positions of all other bodies are specified'.

In principle any body can serve as a reference frame but all reference frames do not turn out to be equally convenient. The motion of the car, for instance, can be more conveniently

dealt with in a frame fixed to the earth and not the sun or the moon. The motion of the planets, on the contrary, is most conveniently represented in a frame linked precisely to the sun (heliocentre) and not the earth.

A frame of reference is usually associated with three mutually perpendicular straight lines called the coordinate axes. The path along which a particle moves in a frame is called its trajectory. It is to be noted carefully that the shape of a trajectory depends on the chosen reference frame. For example, the trajectory of a body dropped in a moving train is a straight line in a frame fixed to the train but it is a parabolic curve with respect to the frame fixed to the earth. Similarly in a frame fixed to an aeroplane the tip of the propeller describes a circle but in the one linked to the earth the same point describes a helix.

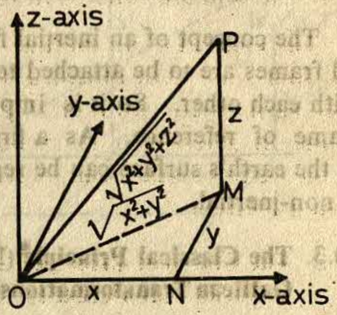


Fig. 19.1

Point and Event

When only the three coordinates are specified we call it a 'point' in space and when we specify the place of occurrence of a phenomenon as well as the time of occurrence we call it an 'event'.

Inertial and noninertial frames

If a body does not interact with surrounding bodies, i.e. no force acts on it, the velocity of the body does not change either in magnitude or direction, i.e. it continues to move in a straight line. This is called law of inertia. This statement is certainly incomplete. The motion of the bodies is mentioned here but nothing is said about the reference frame in which the motion takes place. Perhaps subconsciously we take the frame of reference fixed to the earth. But this is also not to be done because as mentioned above the path that is rectilinear in a frame of reference may turn out to be curvilinear in some other frame. Thus the law of inertia leads to the concept of two types of reference frames. If no force acts on a body, it has no acceleration, i.e. the body is either at rest or in uniform motion in a straight line. Now if the body is observed from a frame of reference which is at rest or in uniform motion in a

straight line then, also the body will appear to be at rest or in motion in a straight line. Thus in this frame also 'no force, no acceleration', i.e. law of intertia holds good. Such frames are called inertial frames. Frames of reference in which 'no force, no acceleration' does not hold are called non-inertial frames.

The concept of an inertial frame is an hypothetical one because all frames are to be attached to a body. In nature all bodies interact with each other. So it is impossible to point out strictly inertial frame of reference. As a first approximation reference frame fixed to the earth's surface can be regarded inertial. Strictly speaking it is non-inertial.

19.3. The Classical Principle (Newtonian Principle) of Relativity and Galilean Transformations)

Experience tells us that on a ship travelling at a constant velocity in a straight line with respect to the earth, if we shoot a revolver, the bullet requires the same time to fly from the bow to the stern as it does from the stern to the bow. A body dropped from a height falls vertically downward whether the ship is at rest or in uniform motion in a straight line. Water poured into a vessel has a horizontal surface whether the ship is at rest or moving at a constant velocity in a straight line. It follows from these examples and many other that by conducting any **mechanical experiment** it is not possible to single out any inertial frame out of the many available. This basic fact of nature and the concept of absoluteness of space and time, i.e. space and time measured from any inertial frame are the same form the basis of the classical principle of relativity. Thus the classical principle of relativity is :

(i) *All inertial frames of reference are equivalent with regard to the description of the laws of mechanics.*

(ii) *Time and space are absolute.*

Galilean transformation

Suppose that S and S' are two inertial frames, S is at rest and S' is moving with constant velocity along the common x-axis. To simplify the problem we assume that the x-axis of both the frames is along the direction of travel of the primed frame (S') and take their origins O and O' to coincide in the beginning (i.e. at $t=0$). Let us fix up two observers in the two frames and provide them with two identical clocks and ask them to observe an event, say the

flight of an aeroplane. For simplicity sake let us assume that the plane takes off at $t=0$ from O when the primed frame (S') begins to move out from the unprimed frame S and let the two observers start their clocks at that very instant. They will report the event (take-off of the plane) as (0, 0, 0, 0) and (0, 0, 0, 0). Let them report the event (flight of plane at K) when S' has moved out through

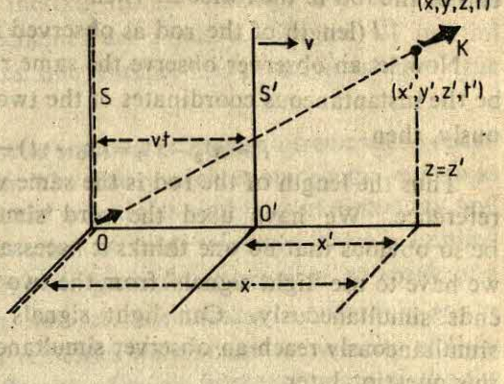


Fig. 19.2

OO'. The event is, say (x, y, z, t) in frame S and (x', y', z', t') in the frame S'. The first observer will report the distance described by the plane in time t (time recorded by his clock) as $\sqrt{x^2+y^2+z^2}$ using the formula $r=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$ for the distance between two points in the same frame. Similarly the distance described by the plane in time t' (time recorded by the observer in frame S' by his own clock) in S' as $\sqrt{(x'+vt)^2+y'^2+z'^2}$.

Since time is absolute according to the principle of relativity, $t=t'$ and since space is absolute

$$\sqrt{x^2+y^2+z^2}=\sqrt{(x'+vt)^2+y'^2+z'^2}$$

This is possible only when $x=x'+vt$

$$y=y'$$

$$z=z'.$$

Thus
$$\left.\begin{aligned} x&=x'+vt\\ y&=y'\\ z&=z'\\ t&=t' \end{aligned}\right\} \quad \ldots (1).$$
and

These equations are called Galilean transformation equations.

Now we will examine whether these transformation equations agree with the first relativity principle, i.e. laws of mechanics are equally valid in all inertial frames of reference.

(i) Let a rod be held in the stationary frame S along the x-axis. Let x_2 and x_1 be the instantaneous coordinates of the far and near end of the rod in the frame S. Then

$$l \text{ (length of the rod as observed in } S) = x_2 - x_1.$$

Now let an observer observe the same rod from S'. If x_2' and x_1 be the instantaneous coordinates of the two ends observed simultaneously, then

$$l' = x'_2 - x'_1 = (x_2 - vt) - (x_1 - vt) = x_2 - x_1 = l.$$

Thus the length of the rod is the same whatever be the frame of reference. We have used the word 'simultaneous', which seems to be so obvious that no one thinks it necessary to question it. After all we have to use 'light signals' from the two ends to observe the two ends simultaneously. Can light signals emitted by the two ends simultaneously reach an observer simultaneously ? We will return to this question later.

(ii) We have from transformation equations,

$$x = x' + vt.$$

$$\therefore \quad \frac{dx}{dt} = \frac{dx'}{dt} + v \quad \text{or} \quad \frac{dx}{dt} = \frac{dx'}{dt'} \cdot \frac{dt'}{dt} + v = \frac{dx'}{dt'} + v \qquad (\because \ t' = t).$$

Differentiating once more with respect to t,

$$\frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt}\left(\frac{dx'}{dt'}\right) = \frac{d}{dt'}\left(\frac{dx'}{dt'}\right)\left(\because \ v \text{ is a constant, } \frac{dv}{dt} = 0\right)$$

or

$$\frac{d^2x}{dt^2} = \frac{d^2x'}{dt'^2}.$$

Thus, acceleration in frame S = acceleration in frame S'.

Since masses are constant according to classical concept,

$$F = m\frac{d^2x}{dt^2} \quad \text{and} \quad F' = m'\frac{d^2x'}{dt'^2}.$$

Thus the second law of force (force = mass × acceleration) is the same in all inertial frames.

(iii) Classical law of addition of velocities : Suppose a particle travels uniformly along the x-axis at the speed $u = \frac{dx}{dt}$. We want to find the velocity of the same particle from another inertial frame moving with velocity v relative to the first.

We have $$x = x' + vt.$$

$$\therefore \quad \frac{dx}{dt} = \frac{dx'}{dt} + v \quad \text{or} \quad \frac{dx}{dt} = \frac{dx'}{dt'} \cdot \frac{dt'}{dt} + v$$

or

$$u = u' + v \quad (\because \ t = t') \qquad \ldots (2).$$

19.4. The Special Theory of Relativity

The classical theory of relativity says that it is impossible to distinguish between two inertial frames through any mechanical experiments, i.e. law of mechanics are equally valid in all inertial frames. Now we wish to examine whether this principle is also true in electrodynamics.

Light is an electromagnetic wave travelling at a tremendous speed $c=3\times10^8$ ms^{-1} in a vacuum. The theory of electromagnetic waves was developed by Maxwell on the principles and laws of electricity and magnetism. A question that arises is: what reference frame is implied for the velocity of light ? It is, in general, meaningless to speak of a velocity without specifying the reference frame. According to the classical law of addition of velocities we see that the velocity of light should be different in different reference frames. Consequently, the given velocity of light holds only for the particular frame containing the source of light. This reasoning can be seen in the following 'thought experiment'.

Assume that an instrument (rectangular box in the figure) for measuring velocity of light is at rest in the frame of reference containing the source of light (L). The instrument will register a velocity of light

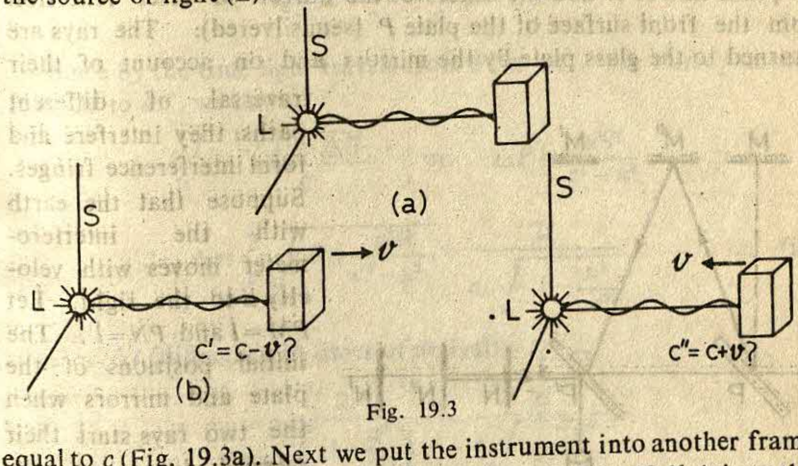


Fig. 19.3

equal to c (Fig. 19.3a). Next we put the instrument into another frame which moves with respect to S with velocity v, in Fig. 19.3b it is to the right and in Fig. 19.3c it is to the left. On the basis of the classical law of addition of velocities, one would expect $c-v$ and $c+v$ respectively. This means that through optical experiments uniform rectilinear motion between two inertial frames could be detected. This also means that principle of classical relativity is true in mechanics and

not in electro-dynamics. But it is known that laws of electricity and magnetism implicitly involve mechanical effect (force connected with charges and poles) and hence laws of electro-dynamics must be identical in all inertial frames. Moreover it would appear quite illogical if we had some principle for laws of mechanics and some other principles for laws of electricity and magnetism. Thus there is a direct clash between the principle of classical relativity and physical laws. Since experimental observation is the supreme arbiter of any controversy, a number of experimental attempts were made to do away with these difficulties.

19.5. Michelson and Morley's Experiment

The aim of this experiment was to decide whether by an optical experiment it was possible to distinguish between inertial frames, i.e. to decide whether velocity of light was really different in different inertial frames. The earth moves in its orbit round the sun with an average speed 3×10^4 ms^{-1}. For the short duration of experiment this speed may be taken to be uniform and rectilinear.

In the Michelson interferometer a ray of light (monochromatic) is split into two parts by a glass slab P—one travelling straight to the plane mirror N and the other to the mirror M after reflection from the front surface of the plate P (semisilvered). The rays are returned to the glass plate by the mirrors and on account of their traversal of different paths, they interfere and form interference fringes. Suppose that the earth with the interferometer moves with velocity u to the right. Let $PM = l$ and $PN = l'$. The initial positions of the plate and mirrors when the two rays start their onward journey to the mirrors from the plate are P, M and N respectively. As the earth is moving with velocity u, the interferometer will

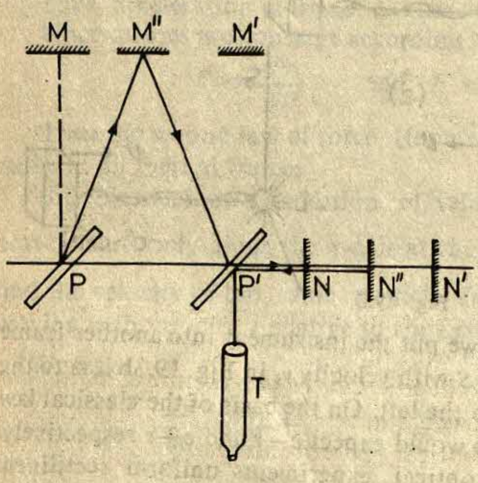


Fig. 19.4

also move in the same direction with velocity u. Let the interfering beams meet when the plate and mirrors are in the positions P', M' and N' respectively. The reflection of rays will take place from their respective intermediate positions M'' and N''. In the frame of reference linked to the interferometer light will have velocity $(c-u)$ in the direction from P to N and $(c+u)$ in the direction N to P according to the classical principle of relativity. Hence to calculate the time of travel of light from P to N'' and back to P', we may assume interferometer stationary and calculate the time of travel of light from P to N and then from N to P. The two timings will be the same.

$\therefore$ t_1 (time of travel from P to N'' and back to P')

$$=\frac{l'}{c-u}+\frac{l'}{c+u}=l'.\frac{2c}{c^2-u^2}=\frac{2l'}{c\left(1-\frac{u^2}{c^2}\right)} \quad \ldots \text{(i)}.$$

Let t_2 be the time of travel of the second interfering ray from P to P' after reflection from the mirror M in its intermediate position M''. By the time light reaches the mirror M in the position M'', the whole apparatus moves through a distance $MM''=\triangle l$ (say).

$$\therefore \quad t_2=2.\frac{PM''}{c}=\frac{2}{c}\sqrt{l^2+\triangle l^2}.$$

Since by the time light travels from P to M'', the mirror moves from M to M'', we have

$$\frac{\triangle l}{u}=\frac{\sqrt{l^2+\triangle l^2}}{c} \quad \text{or} \quad \triangle l^2=\frac{u^2l^2}{c^2-u^2}.$$

$$\therefore \quad t_2=\frac{2}{c}\sqrt{l^2+\frac{u^2l^2}{c^2-u^2}}=\frac{2l}{c\sqrt{1-\frac{u^2}{c^2}}} \quad \ldots \text{(ii)}.$$

$\therefore$ $\triangle t$ (difference in times of arrival) $=t_1-t_2$

$$=\frac{2l'}{c\left(1-\frac{u^2}{c^2}\right)}-\frac{2l}{c\sqrt{1-\frac{u^2}{c^2}}}.$$

In the experimental arrangement l is set equal to l'.

$$\therefore \quad \triangle t=\frac{2l}{c}\left(\frac{1}{1-\frac{u^2}{c^2}}-\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\right)$$

$$=\frac{2l}{c}\left[\left(1-\frac{u^2}{c^2}\right)^{-1}-\left(1-\frac{u^2}{c^2}\right)^{-\frac{1}{2}}\right]$$

$$=\frac{2l}{c}\left[\left(1+\frac{u^2}{c^2}\right)-\left(1+\frac{u^2}{2c^2}\right)\right]=\frac{lu^2}{c^3}$$

$$(\because\ u^2 \ll c^2).$$

To find the corresponding path difference we multiply $\triangle t$ by c.

$$\triangle \text{ (path difference)}=c\left(\frac{lu^2}{c^3}\right)=l.\frac{u^2}{c^2} \quad \ldots \quad \text{(iii)}.$$

If now the interferometer is turned through 90°, the direction of u remains unchanged, but the two paths in the interferometer will be interchanged. This would introduce a path difference $\triangle$ in the opposite sense. Hence we expect a fringe shift corresponding to a change of path difference $2\triangle$.

Michelson and Morley made the distance l large by reflecting the light back and forth between sixteen mirrors. The interferometer was mounted on a large concrete block floating in mercury and observations were made as it was rotated slowly and continuously about a vertical axis. In one experiment l was 11 metres, and so if we take $u=3\times10^4$ ms^{-1} and $c=3\times10^8$ ms^{-1}, we find a path difference of 22×10^{-8} m. For light of wavelength 6000Å, this corresponds to a change of ·4λ, so that the fringes should be displaced by ·4 of a fringe width. No fringe shift was observed although the instrument was able to detect much smaller shift by ·01 of a fringe width. Thus it was proved beyond doubt that by optical experiments also uniform rectilinear motion between two inertial frames could not be detected.

19.6. Einstein's Special Theory of Relativity

Since the invariance of velocity of light in all inertial frames was a firmly established fact supported by the negative result of Michelson and Morley experiment, need arose for critical revision of the absoluteness of space and time. This was done by Einstein in 1905. According to Einstein time and space are not absolute (a strong contradiction of classical theory of relativity). They are also relative, i.e. clocks in different inertial frames will record different time and length of a rod measured from different inertial frames will be different. According to Einstein it is the velocity of light which is really absolute and not space and time. Another obvious conclusion from Michelson and Morley experiment is that not only laws of mechanics

but laws of all phenomena of nature are equally valid in all inertial frames. This way we arrived at Einstein's special theory of relativity. The theory comprises of two basic principles :

1. Principle of relativity : *All inertial frames are equally valid with regard to the description of not only laws of mechanics but also laws of all phenomena of nature.*

2. Principle of constancy of the velocity of light : *The velocity of light in vacuum is the same and is equal to c in all inertial frames of reference.*

Now we shall discuss the censequences from the theory.

(a) *Simultaneity of events* : The concept of simultaneity is relative, and the course of time differs, i.e. clocks run differently in different inertial frames. To make the point clear let us consider the following experiment. Let one frame of reference be fixed to the earth and another to a train moving at a constant speed. Two events appear simultaneous when flashes of light emitted simultaneously by the two events reach an observer simultaneously. Let us mark off the points A, B and M on the earth and similar three points A', B' and M' in a train car such that $AM=BM$, $A'M'=B'M'$ and

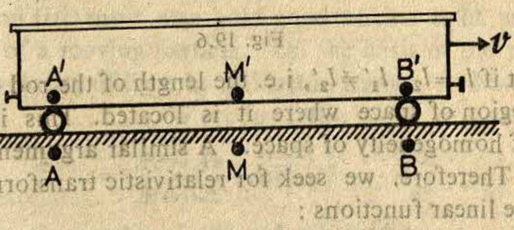


Fig. 19.5

$AB=A'B'$. Suppose that in course of motion of the train when $A'B'$ is exactly above AB, two events occur, say two thunderbolts, strike these points. To an observer fixed at M, the two events will appear to be simultaneous but to an observer fixed in the car at M' the events will not appear simultaneous. Thus simultaneity is not absolute, it differs from one inertial frame to another.

(b) *Lorentz's transformation equations* : Let us derive now transformation equations for passing over from one inertial frame to another for the same simplified case which we considered earlier

(Fig. 19.2). Here also as there is no motion along y-axis and z-axis we have $y=y'$ and $z=z'$ but x must transform linearly into x' and t' and t must transform linearly into x' and t'. This is needed for homogeneity of time and space. When the transformation is linear, the length of a rod will not depend what region of space it is located. We measure the length or distance by comparing with a 'measuring rod'. So it is necessary that if $l_1=l_2$ where l_1 is the length of the rod and l_2 is the reading on the metre scale in one frame (unprimed frame), in the primed frame also $l_1'=l_2'$ so that the rod and scale contract or elongate equally and an observer gets the same reading on the scale. If the transformation is nonlinear (Fig. 19.6b) then it

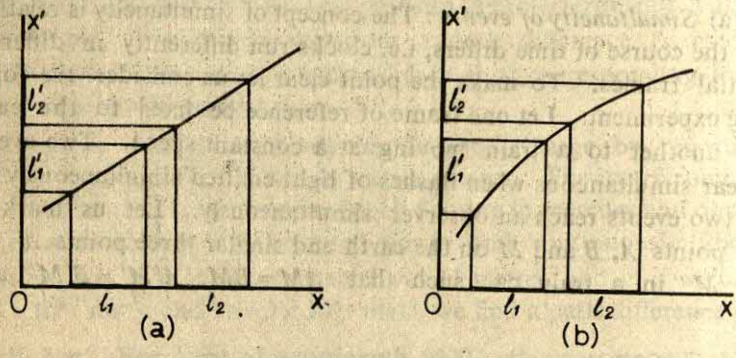


Fig. 19.6

follows that if $l_1=l_2$, $l_1'\neq l_2'$, i.e. the length of the rod would depend upon the region of space where it is located. This is against the principle of homogeneity of space. A similar argument can be made for time. Therefore, we seek for relativistic transformations in the forms of the linear functions :

$$x'=Ax+Bt \qquad \text{.. (i),}$$

$$t'=Mx+Nt \qquad \text{.. (ii)}$$

where A, B, M, N are constants.

$$\therefore \quad \triangle x'=A\,\triangle x+B\,\triangle t$$

$$\text{and } \triangle t'=M\,\triangle x+N\,\triangle t.$$

Dividing, we have

$$\frac{\triangle x'}{\triangle t'}=\frac{A.\triangle x+B.\triangle t}{M.\triangle x+N.\triangle t}=\frac{A\dfrac{\triangle x}{\triangle t}+B}{M\dfrac{\triangle x}{\triangle t}+N}.$$

Now $\frac{\Delta x'}{\Delta t'}=u'$ is the velocity of a particle in the primed frame

and $\frac{\Delta x}{\Delta t}=u$ is the velocity of the same particle in the unprimed frame.

$$\therefore \quad u'=\frac{Au+B}{Mu+N} \quad \ldots \text{(iii)}.$$

This relation must hold good for any velocity of the particle in the primed frame and hence must hold for $u'=0$ as well. But when $u'=0$, the particle is at rest in the prime frame and has velocity v relative to the unprimed frame (S). Here v is the velocity of the primed frame. Thus when $u'=0$, $u=v$.

$$\therefore \quad 0=\frac{Av+B}{Mv+N}$$

or $$Av+B=0 \quad \text{or} \quad B=-Av.$$

The relation (iii) must also hold good when the particle is at rest in the frame S. Then $u=0$ and $u'=-v$.

$$\therefore \quad -v=\frac{B}{N} \quad \text{or} \quad B=-Nv.$$

$$\therefore \quad A=N.$$

The relation (iii) must also hold good when a light wave propagates instead of a moving particle. On the basis of the principle of constancy of velocity of light in all inertial frames, we have

$$u=u'=c \text{ for light.}$$

$$\therefore \quad c=\frac{Ac-Av}{Mc+A} \quad \text{or} \quad M=-A\frac{v}{c^2}.$$

Substituting these values in (i) and (ii), we have

$$\left.\begin{aligned} x'&=Ax-Avt=A(x-vt) \\ \text{and} \quad t'&=-\frac{Av}{c^2}x+At=A\left(t-\frac{vx}{c^2}\right) \end{aligned}\right\} \quad \ldots \text{(iv)}.$$

According to the first postulate of the theory of relativity, i.e. equivalence of all inertial frames, the above relations given by Eq. (iv) must hold good if S' is at rest and S moves with velocity $-v$. Thus we must have

$$\left.\begin{aligned} x&=A(x'+vt') \\ \text{and} \quad t&=A\left(t'+\frac{vx'}{c^2}\right) \end{aligned}\right\} \quad \ldots \text{(v)}.$$

Substituting values of x and t from (v) into equations (iv), we have

$$x'=A^2\left\{x'+vt'-v\left(t'+\frac{vx'}{c^2}\right)\right\}$$

or
$$x'=A^2\left(x'-\frac{v^2x'}{c^2}\right) \quad \text{or} \quad A=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}.$$

Thus finally we have transformation equations as

$$\left.\begin{aligned} x'&=\frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}}, \quad y=y', \quad z=z' \\ t'&=\frac{t-\frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \end{aligned}\right\} \quad \ldots (3)$$

These are called Lorentz transformation equations.

Replacing primed quantities by the corresponding unprimed quantities and unprimed quantities by the cosrresponding primed one and v by $-v$, we obtain

$$\left.\begin{aligned} x&=\frac{x'+vt'}{\sqrt{1-\frac{v^2}{c^2}}}, \quad y=y', \quad z=z' \\ t&=\frac{t'+\frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \end{aligned}\right\} \quad \ldots (3a).$$

These are called reciprocal Lorentz transformation equations. Substituting the values of B, M and N in terms of A in (iii) we obtain

$$u'=\frac{Au-Av}{\left(-\frac{Av}{c^2}\right)u+A}$$

or
$$u'=\frac{u-v}{1-\frac{uv}{c^2}} \quad \ldots (4).$$

Replacing u by u', u' by u and v by $-v$, we have

$$u=\frac{u'+v}{1+\frac{u'v}{c^2}} \quad (4a).$$

These Eqs. (4) and (4a) are called the relativistic law of addition of velocities.

(c) *Limiting velocity of any material particle* : Suppose that the velocity of a particle in an inertial frame S is u. We can always find another inertial frame S' moving with velocity $v<c$ relative to S in which velocity of the particle is also less than velocity of light in vacuum, i.e. $u'<c$. This is neither difficult nor impossible because there is an infinitely great number of inertial frames relative to the given inertial frame. Any frame moving with uniform velocity with respect to a certain inertial frame is also inertial.

$$\text{Now,}\qquad c-u=c-\frac{u'+v}{1+\frac{u'v}{c^2}}=\frac{c+\frac{u'v}{c}-u'-v}{1+\frac{u'v}{c^2}}.$$

$$\text{or}\qquad c-u=\frac{(c-v)(c-u')}{c\left(1+\frac{u'v}{c^2}\right)}.$$

Since $c>v$ and $c>u'$, $c-u=+\text{ve}$ or $c\geqslant u$.

Thus the velocity of light in vacuum is the highest possible velocity of a particle in nature.

19.7. Consequences of Lorentz Transformations

(a) *Contraction of length* : Let a rod fixed in frame S' be of length l_0 as measured by an observer in S' and l as observed by an observer in S. The velocity of the rod is the same as the velocity of the frame S', namely, v. Obviously, $l_0=x'_2-x'_1$ and $l=x_2-x_1$.

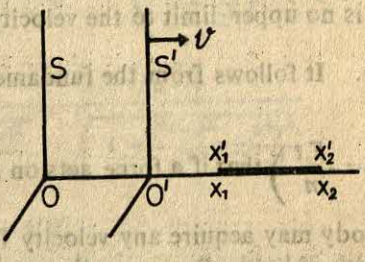


Fig. 19.7

$$\text{Now,}\qquad x'_2=\frac{x_2-vt}{\sqrt{1-\frac{v^2}{c^2}}}\text{ and }x'_1=\frac{x_1-vt}{\sqrt{1-\frac{v^2}{c^2}}}.$$

$$\therefore\quad l_0=x'_2-x'_1=\frac{x_2-x_1}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{l}{\sqrt{1-\frac{v^2}{c^2}}}$$

or $$l=l_0\sqrt{1-\frac{v^2}{c^2}} \qquad (5).$$

Thus the length of the rod is shorter when it is moving with velocity v. This cannot, however, be observed because to see the real contraction light must be emitted simultaneously and must reach also simultaneously the observer which is not possible.

(b) *Time dilation* : Let the instants of occurrence of an event in the inertial frame S be t_1 and t_2 and that in S' moving with velocity v be t'_1 and t'_2. Then

$$t_1=\frac{t'_1+\frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{and} \quad t_2=\frac{t'_2+\frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}.$$

$$\therefore \quad t_2-t_1=\frac{t'_2-t'_1}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{or} \quad \tau=\frac{\tau_0}{\sqrt{1-\frac{v^2}{c^2}}} \quad .. \quad (6).$$

Here $\tau=t_2-t_1$ is the interval measured by a clock in S

and $\tau_0=t'_2-t'_1$ is the interval measured by a clock in S'.

Obviously $\tau<\tau_0$. This is called the principle of dilation of time.

(c) *Variation of mass with velocity* (*Relativistic mass*) : In classical mechanics mass of a body is the same in all inertial frames and there is no upper limit to the velocity with which material bodies can move. It follows from the fundamental law of dynamics $\left(F=\frac{mu}{t}\right.$ or $\left.u=\frac{F.t}{m}\right)$ that if a force acts on a body for sufficiently long time, the body may acquire any velocity from rest—even more than the velocity of light. But according to the special theory of relativity no material body is permitted to move with velocity exceeding that of light in vacuum. It is this 'upper limit' of velocity of material bodies which forces us to the conclusion that mass of a body must vary with velocity and become infinite when its velocity becomes equal to c. So we may write

$$m=m_0 f(u)$$

where f is a function to be determined and m_0 is the rest mass of the body. It must satisfy the conditions

$$f(u) \to 1 \text{ when } u \to 0$$

and $$f(u) \to \infty \text{ when } u \to c,$$

Let u and u' be respectively the velocities of a particle in inertial frames S (at rest) and S' moving with velocity v relative to S. According to the relativistic law of addition of velocities,

$$u' = \frac{u-v}{1-\dfrac{uv}{c^2}}.$$

Let us use the following notations :

$$\alpha = \frac{1}{\sqrt{1-\dfrac{u^2}{c^2}}},\ \alpha' = \frac{1}{\sqrt{1-\dfrac{u'^2}{c^2}}} \text{ and } \beta = \frac{1}{\sqrt{1-\dfrac{v^2}{c^2}}}.$$

Let us first show the important result $\alpha\beta(u-v) = \alpha' u'$.

$$\alpha\beta(u-v) = \frac{1}{\sqrt{1-\dfrac{u^2}{c^2}}} \cdot \frac{1}{\sqrt{1-\dfrac{v^2}{c^2}}} \cdot (u-v) = \frac{u-v}{\sqrt{1-\dfrac{u^2+v^2}{c^2}+\dfrac{u^2v^2}{c^4}}}.$$

$$\alpha' u' = \frac{1}{\sqrt{1-\dfrac{u'^2}{c^2}}} \cdot \frac{u-v}{\left(1-\dfrac{uv}{c^2}\right)} = \frac{u-v}{\sqrt{1-\left(\dfrac{u-v}{1-\dfrac{uv}{c^2}}\right)^2 \dfrac{1}{c^2}}\left(1-\dfrac{uv}{c^2}\right)}$$

$$= \frac{u-v}{\sqrt{\left(1-\dfrac{uv}{c^2}\right)^2 - \dfrac{(u-v)^2}{c^2}}} = \frac{u-v}{\sqrt{1-\dfrac{u^2+v^2}{c^2}+\dfrac{u^2v^2}{c^4}}}.$$

$$\therefore \qquad \alpha\beta(u-v) = \alpha' u' \qquad \text{.. (i).}$$

All intertial frames are identical so far the description of physical laws are concerned. In the matter of collisions of a system of particles, mass and momentum remain conserved in any frame. This fact must be the same in all inertial frames.

$$\therefore \qquad \Sigma m_i = \text{a constant} \qquad \text{.. (ii),}$$

$$\Sigma m_i u_i = \text{a constant} \qquad \text{.. (iii).}$$

and $$\Sigma m_i' = \text{a constant} \qquad \text{.. (iv),}$$

$$\Sigma m_i' u_i' = \text{a constant} \qquad \text{.. (v).}$$

β and v are the same for all particles as they are constants of the frame S'.

$$\therefore \qquad \Sigma m_i \beta v = \text{a constant} \quad .. \quad \text{(vi)},$$

$$\Sigma m_i u_i \beta = \text{a constant} \quad .. \quad \text{(vii)}.$$

Subtracting (vi) from (vii), we have

$$\Sigma m_i \beta(u_i - v) = \text{a constant} \quad .. \quad \text{(viii)}.$$

By the relation (i) already established above,

$$\alpha_i \beta(u_i - v) = \alpha_i' u_i' \qquad \text{or} \quad \beta(u_i - v) = \frac{\alpha_i' u_i'}{\alpha_i}.$$

Multiplying by m_i on both sides we have

$$m_i\, \beta(u_i - v) = \frac{m_i\, \alpha_i'\, u_i'}{\alpha_i} \quad ..$$

$$\text{or} \qquad \sum \frac{m_i \alpha_i' u_i'}{\alpha_i} = \Sigma m_i \beta(u_i - v) = \text{a constant by (viii)}.$$

But $\Sigma m_i' u_i' = \text{a constant by (v)}$.

These two relations agree only if

$$\frac{\alpha_i' m_i}{\alpha_i} = m_i' \qquad \text{or} \qquad \frac{m_i}{\alpha_i} = \frac{m_i'}{\alpha_i'}.$$

Dropping the suffix i, $\qquad \dfrac{m}{\alpha} = \dfrac{m'}{\alpha'}$,

$$\text{or} \qquad m\sqrt{1 - \frac{u^2}{c^2}} = m'\sqrt{1 - \frac{u'^2}{c^2}}$$

$$\text{or} \qquad m\sqrt{1 - \frac{u^2}{c^2}} = \text{a constant } (k).$$

Let $m = m_0$ called the rest mass when $u = 0$, i.e. when the particle is at rest. Then $m_0 = k$

$$\therefore \qquad m_0 = m\sqrt{1 - \frac{u^2}{c^2}}$$

$$\text{or} \qquad m = \frac{m_0}{\sqrt{1 - \dfrac{u^2}{c^2}}} \quad .. \quad (7).$$

(d) *Relativistic kinetic energy* : Laws of mechanics and definitions of dynamic quantities are the same in relativistic mechanics as in Newtonian mechanics, i.e. force is the rate of change of momentum and power is the dot product of force and velocity.

$$\therefore \quad F=\frac{dp}{dt}=\frac{d}{dt}(mu)=\frac{d}{dt}\;\frac{m_0u}{\sqrt{1-\frac{u^2}{c^2}}}=\frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}\cdot\frac{du}{dt}$$

$$+\frac{m_0\,\frac{u^2}{c^2}}{\left(1-\frac{u^2}{c^2}\right)^{\frac{3}{2}}}\cdot\frac{du}{dt}$$

$$=\frac{m_0}{\left(1-\frac{u^2}{c^2}\right)^{\frac{3}{2}}}\;\frac{du}{dt} \qquad \text{(i).}$$

In vector notation, $\vec{F}=\dfrac{m_0}{\left(1-\frac{u^2}{c^2}\right)^{\frac{3}{2}}}\;\dfrac{d\vec{u}}{dt}$ .. (ii).

Now, $\dfrac{dW}{dt}$ (rate of doing work) $=\vec{F}\,.\,\vec{u}$

$$=\frac{m_0}{\left(1-\frac{u^2}{c^2}\right)^{\frac{3}{2}}}\;\frac{d\vec{u}}{dt}\,.\,\vec{u}$$

$$=\frac{m_0}{\left(1-\frac{u^2}{c^2}\right)^{\frac{3}{2}}}\,u\,\frac{du}{dt}\;\left(\because\;\frac{d\vec{u}}{dt}\,.\,\vec{u}=\frac{du}{dt}u\cos 0^\circ=u\;\frac{du}{dt}\right)$$

or $\quad \dfrac{dW}{dt}=\dfrac{d}{dt}\;\dfrac{m_0c^2}{\sqrt{1-\frac{u^2}{c^2}}}$ or $dW=d\left(\dfrac{m_0c^2}{\sqrt{1-\frac{u^2}{c^2}}}\right)$.

$\therefore \quad W$ (total work done in time τ) $=\displaystyle\int_{t=0}^{t=\tau} dW$

$$=\int_{t=0}^{t=\tau} d\left(\frac{m_0c^2}{\sqrt{1-\frac{u^2}{c^2}}}\right)=\left[\frac{m_0c^2}{\sqrt{1-\frac{u^2}{c^2}}}\right]_{t=0}^{t=\tau}.$$

When $t=0$, $u=0$ and $t=\tau$, $u=v$ (say).

Then $W=\left[\dfrac{m_0c^2}{\sqrt{1-\dfrac{u^2}{c^2}}}\right]_{u=0}^{u=v}=\dfrac{m_0c^2}{\sqrt{1-\dfrac{v^2}{c^2}}}-m_0c^2$

$$=mc^2-m_0c^2=(m-m_0)c^2.$$

By the 'work-energy' theorem, this is the kinetic energy (T) of the body.

$$\therefore \quad T=(m-m_0)c^2 \quad .. \quad (8).$$

(e) *Mass-energy Equivalence* : Eq. (8) shows that mass is equivalent to energy because it shows that the kinetic energy is the difference of two terms which are themselves of the dimension of energy. The mass-energy equivalence is thus given by

$$E=mc^2 \quad .. \quad (9).$$

Examples

1. *Calculate the relativistic kinetic energy of a particle of rest mass 1 gm moving at a speed of half the speed of light in vaccum. Also calculate the classical kinetic energy of the particle.*

Sol. We have $T=mc^2-m_0c^2=\dfrac{m_0c^2}{\sqrt{1-\dfrac{v^2}{c^2}}}-m_0c^2$

or $$T=\frac{m_0c^2}{\sqrt{1-\dfrac{c^2}{4c^2}}}-m_0c^2=m_0c^2\left(\frac{2}{\sqrt{3}}-1\right)$$

or $$T=\frac{1}{1000}\times(3\times10^8)^2\times\left(\frac{2}{\sqrt{3}}-1\right)=1{\cdot}39\times10^{13} \text{ joule.} \quad \textbf{Ans.}$$

$$T_{classical}=\tfrac{1}{2}mv^2=\tfrac{1}{2}\times\frac{1}{1000}(\tfrac{1}{2}\times3\times10^8)^2=1{\cdot}13\times10^{13} \text{ joule. } \textbf{Ans.}$$

2. *A distant star is found to send 'radio signals' at the frequency 20 Mhz. What will be the frequency from a spaceship moving at the speed of ·1c ?*

Sol. We have $$\tau=\frac{\tau_0}{\sqrt{1-\dfrac{v^2}{c^2}}}$$

or $$\nu=\nu_0\sqrt{1-\frac{v^2}{c^2}} \qquad \left(\because\ \nu=\frac{1}{\tau}\right)$$

$$=20\times10^6\sqrt{1-\frac{\cdot1^2c^2}{c^2}}=20\times10^6\times\sqrt{\cdot99}=19\cdot9\text{ Mhz. Ans.}$$

3. *The average longevity of a man is 70 years. What is the longevity of a man on the earth as observed from a star moving with speed ·9c ?*

Sol. We have $\tau=\dfrac{\tau_0}{\sqrt{1-\dfrac{v^2}{c^2}}}=\dfrac{70}{\sqrt{1-\dfrac{\cdot9^2c^2}{c^2}}}=160$ years. **Ans.**

4. *A red hot iron sphere of mass 1000 kg and temperature 1200°C cools to 30°C. What is the energy lost by it and what mass ? Specific heat capacity of iron = 460 J kg^{-1} K^{-1}.*

Sol. Energy lost $=1000\times460\times(1200-30)=5\cdot38\times10^8$ joule. **Ans.**

Mass lost $(m)=\dfrac{E}{c^2}=\dfrac{5\cdot38\times10^8}{(3\times10^8)^2}=5\cdot98\times10^{-7}$ kg. **Ans.**

QUESTIONS

(A)

1. If l_0 is the rest length of a rod, then its length l when it moves with velocity v is

(a) $l=\dfrac{l_0}{\sqrt{1-\dfrac{v^2}{c^2}}}$. (b) $l_0=\dfrac{l}{\sqrt{1-\dfrac{v^2}{c^2}}}$, (c) $l=l_0\sqrt{1-\dfrac{c^2}{v^2}}$,

(d) $l_0=l\sqrt{1-\dfrac{c^2}{v^2}}$.

2. If τ_0 is the interval between two events in a frame at rest, then the interval between the same two events as observed from a frame moving with velocity v is

(a) $\tau=\dfrac{\tau_0}{\sqrt{1-\dfrac{v^2}{c^2}}}$, (b) $\tau=\tau_0\sqrt{1-\dfrac{c^2}{v^2}}$, (c) $\tau_0=\dfrac{\tau}{\sqrt{1-\dfrac{v^2}{c^2}}}$,

(d) $\tau_0=\tau\sqrt{1-\dfrac{c^2}{v^2}}$.

3. If m_0 is the rest mass of a body then its mass at velocity v is

(a) $m=\dfrac{m_0}{\sqrt{1-\dfrac{v^2}{c^2}}}$, (b) $m_0=\dfrac{m}{\sqrt{1-\dfrac{v^2}{c^2}}}$, (c) $m=m_0\sqrt{1-\dfrac{c^2}{v^2}}$,

(d) $m_0 = m\sqrt{1-\frac{c^2}{v^2}}$.

4. If u is the velocity of a body along the x-axis in a frame at rest, u' that in a frame moving with velocity v, then

(a) $u = \dfrac{u' - v}{\sqrt{1-\frac{v^2}{c^2}}}$, (b) $u = \dfrac{u'+v}{\sqrt{1+\frac{v^2}{c^2}}}$, (c) $u = \dfrac{u'+v}{1+\frac{u'v}{c^2}}$,

(d) $u = \dfrac{u'+v}{\sqrt{1-\frac{u'v}{c^2}}}$.

Ans. 1. b, **2.** a, **3.** a, **4.** c.

(B+C)

1. State the classical principle of relativity. Obtain Galilean transformation equations and hence obtain classical law of addition of velocities.

2. Describe Michelson and Morley experiment and show how the negative result of this experiment led to the special theory of relativity.

3. State Einstein's postulates of special theory of relativity. Obtain Lorentz transformation equations and discuss two important consequences.

4. Deduce the following relations :

(i) $l = l_0\sqrt{1-\frac{v^2}{c^2}}$, (ii) $\tau = \dfrac{\tau_0}{\sqrt{1-\frac{v^2}{c^2}}}$, (iii) $m = \dfrac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$

where all the notations have their usual significance.

(D)

1. A rod 1 m long is placed in a fast moving train of speed ·6c relative to the ground. Calculate Lorentz contraction of the rod. **(Ans.** ·2 m**)**

2. How many times does the mass of an electron increase when it moves with velocity one-tenth of the velocity of light in vacuum ? **(Ans.** 1·005 times**)**

3. Find the velocity of a particle at which the mass of the particle is double its rest mass. **(Ans.** $2{\cdot}6\times10^{8}$ ms^{-1}**)**

4. Calculate the velocity of an electron moving with kinetic energy 5 Mev. (rest mass of electron $= 9\times10^{-31}$ kg.) **(Ans.** $2{\cdot}99\times10^{8}$ ms^{-1}**)**

5. A block of iron of 1 kg is heated from 0°C to 100°C. By how much will its mass increase ? Specific heat capacity of iron $= 420$ Jkg^{-1} K^{-1}.

(Ans. $4{\cdot}67\times10^{-13}$ kg**)**

CHAPTER 1

THERMOMETRY

1.1. Concept of Temperature

The sense of touch is the simplest way to distinguish hot bodies from cold bodies. Our judgement regarding hotness or coldness of a body is our sense of temperature. A hot body is said to possess a higher temperature than a cold one. But our judgement about hotness or coldness of a body by the sense of touch is never absolute. If we immerse our two hands, one in hot water and the other in cold water and then lift them and again immerse them in water of intermediate hotness, the two hands will give different judgement about the degree of hotness of the same bath. Thus our judgement of temperature is not only unreliable but also misleading. Therefore, what we need is an objective, absolute and reproducible measure of temperature. In other words we need a scientific basis for defining and measuring temperature. Let us see how we achieve this from the so called 'zeroth law of thermodynamics'.

When two bodies of different hotness are brought in contact, they exchange certain energy on account of their different degree of hotness. This energy is called heat. Due to the exchange of heat it is found that, in general, there is a change in their properties, such as volume, pressure etc. Finally, a state is attained after which there is no further change. The two bodies are then said to be in '*thermal equilibrium*' with each other.

Now consider two systems A and B separated from each other by *adiabatic* wall (wall that does not allow flow of heat), but each is in contact with a third system C through a diathermic wall (wall that allows heat to pass).

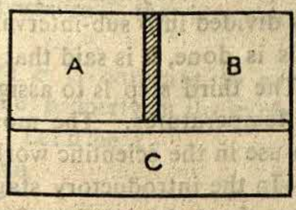


Fig. 1.1.

A and B will reach thermal equilibrium with C with appropriate changes in their properties. Now, if the adiabatic wall between A and B is replaced by a diathermic wall, it is found that

no further change occurs in A and B which means that A and B are also in thermal equilibrium. Thus it is found as a fact that "if two systems are in thermal equilibrium with a third system, then the two systems are in thermal equilibrium with each other." This is often called the Zeroth law of thermodynamics.

This law of thermal equilibrium is the basis for defining and measuring temperature. In the above example the system C may be an ordinary thermometer with which A and B are in thermal equilibrium. All those three bodies may be said to possess a property that ensures their being in thermal equilibrium with one another. This property of a body is called temperature. *The temperature of a body is thus a property which determines whether or not the body is in thermal equilibrium with other bodies.* Obviously, *temperature can also be measured by observing the change of some such property of any one of these bodies.* Thus the zeroth law of thermodynamics forms the basis for measuring and defining the 'temperature' of a body.

1.2. Measurement of Temperature

When heat is added or subtracted from a, body there are many changes in the physical properties of the body such as expansion, change in electrical resistance, development of emf at the junction of two dissimilar metals etc. These changes in the properties of matter are explored to measure temperature. The earliest and commonest thermometers utilise the property of expansion, e.g., ordinary mercury-in-glass thermometer, the standard hydrogen thermometer etc. The platinum resistance thermometer utilises the property of change of electrical resistance and the thermocouple thermometer makes use of the development of emf due to the difference in temperature between the junctions of two dissimilar metals.

To fix up the numerical value of a temperature, first '*a standard temperature interval*' between two easily '*reproducible fixed temperatures*' is selected and then this interval is divided into sub-intervals called '*degree of temperature*'. When this is done, it is said that a scale of temperature has been defined. The third step is to assign some arbitrary value to one of the fixed temperatures. The most common scale of temperature which is in use in the scientific world is the Celsius scale (old centigrade scale). In the introductory stage the two fixed temperatures on this scale were the melting point of ice and the boiling point of water at one atmospheric pressure. The

lower point was fixed arbitrarily at 0 and the interval was divided into 100 degrees. At the Tenth General Conference on Weight and Measures in Paris in 1954 this has been slightly changed to fit with the most scientific scale developed by Lord Kelvin on the thermodynamic property of a reversible Carnot engine. The efficiency of an ideal Carnot reversible engine depends only on the temperature of the source and the sink between which it works and it does not depend on the working substance of the engine. This fact was utilised by Kelvin to define a scale of temperature called the Kelvin or thermodynamic scale of temperature. Any engine at most may have hundred per cent efficiency. Hence on the thermodynamical scale of temperature absolute zero is that temperature at which a reversible Carnot engine has hundred per cent efficiency. Assigning a zero numerical value to this temperature and the triple point of water, i.e., the temperature at which ice, liquid water, and water vapour coexist in equilibrium as the other fixed point, the interval was divided into 273·16 divisions. This was necessitated to keep the size of a 'degree' of this scale same as that of the Celsius scale. The temperature of a body on this scale is called Kelvin or absolute temperature denoted by 'K'. By the convention of the Tenth Conference on Weight and Measures, it was decided to shift the zero of the Celsius scale to the melting point of ice which is 273·15°K on the thermodynamic scale. Thus the temperature of a body on the thermodynamic scale and the Celsius scale will differ only by 273·15. That is,

$$T = 273{\cdot}15 + t \qquad \text{.. (1.1).}$$

where T is the temperature on the thermodynamic scale and t is the same temperature on the Celsius scale.

Henceforth by '°C' we will mean Celsius Scale and read it as Celsius and K as kelvin. In SI the temperature of material bodies are taken on this scale.

General method after 1954 :

Suppose we select a particular property to define a temperature scale. Let us denote it by X and enforce a linear relation of it with temperature T, that is, we assume

$$T = aX$$

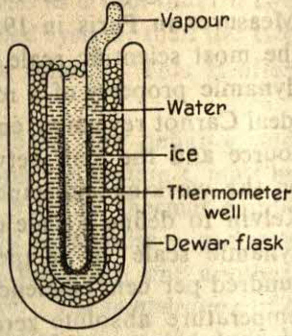
where a is constant to be determined. To determine a, the value of the property at the triple point (T_{tr}) of water is determined (Fig. 1.2). Let it be X_{tr}.

Then

$$T_{tr} = a\,X_{tr}.$$

$$\therefore \quad T = \frac{X}{X_{tr}} \times T_{tr} = \frac{X}{X_{tr}} \times 273{\cdot}16 \text{ K}.$$

This is the relation of the scale on this property with the kelvin scale. However, this temperature requires correction as the calculation is based on an enforced relation. To find the correct temperatures a calibration graph is obtained which gives the correct temperature on the kelvin scale against the temperatures on the scale of the selected property. For calibration many standard temperatures such as freezing point of mercury (−38·87°C), melting point (M. P.) of ice (0·00°C), boiling point (B. P.) of water (100°C), freezing point of lead (F. Z.) (327·3°C), B. P. of sulphur (444·60°C), M. P. of silver (960·8°C) etc. are available.



A triple point cell

Fig. 1.2

Perfect gas scale. As pointed out above a scale of temperature is defined by some property of matter. The scale of temperature defined by the 'expansion property of a perfect gas' on adding heat to it is called the perfect gas scale.

For a perfect gas we have, $V_t = V_0(1+\alpha t)$ where α is the coefficient of expansion of a perfect gas at constant pressure and t is the temperature on the Celsius scale. At $t = -\frac{1}{\alpha}\,°C$, $V_t = 0$. Thus $\left(-\frac{1}{\alpha}\right)°C$ is the lowest possible temperature that we can imagine on the property of a gas, because at most we can think of a gas having zero volume, but we can never think of the negative volume of a gas. Thus $\left(-\frac{1}{\alpha}\right)°C$ is the lowest possible temperature on the gas scale. Assigning zero numerical value to this temperature a scale is designed. This scale is called the *perfect gas scale.*

Thus $\left(-\frac{1}{\alpha}\right)°C = 0°A$ where A (absolute) is the temperature on perfect gas scale.

or $$0°C = \frac{1}{\alpha}\,°A$$

or $$t°C = \left(t + \frac{1}{\alpha}\right)°A$$

By experiments, $$\alpha = \frac{1}{273{\cdot}15}$$

$$\therefore \quad t°C = (t + 273{\cdot}15)°A \qquad \text{.. (1.1a).}$$

It is to be noted carefully here that the temperature of a body on the thermodynamical scale coincides with the temperature of the body on the perfect gas scale. The thermodynamical scale has a quite solid theoretical background, but it cannot be directly put to practical application, because the Carnot engine does not exist in real practice. It is realised in practice by measuring temperatures on the perfect gas scale because it can be shown that whatever is the temperature of a body on the perfect gas scale, that is also the temperature of the body on the thermodynamic scale. Henceforth, therefore, we will make no distinction between the perfect gas scale and the thermodynamic scale, and the Celsius scale, except that the numerical value of the temperature on the Celsius scale differs from the numerical value of the same temperature on the thermodynamic scale by 273·15.

1.3. Mercury-in-Glass Thermometer

The thermometer that is in common use is the mercury-in-glass thermometer. It consists of a glass bulb containing mercury to which a long capillary stem is attached. The stem is graduated later. The freezing point of water is marked 0°C and the boiling point 100°C and the interval divided into 100 equal parts when it is converted into a thermometer giving temperatures on Celsius scale. If the same two points are given values 32 and 212 respectively and the interval is divided into 180 equal parts, it is called the Fahrenheit scale. This scale has no scientific status and is used only in clinical thermometers. When the same two points are given values 0 and 80 respectively and the interval is divided into 80 equal parts, it gives temperatures on Reaumur scale. This scale also has no scientific status and is mostly used in Russia. Mercury as thermometric substance has many advantages. It does not wet glass, can easily be obtained pure, remains liquid over a fairly wide range, has low specific heat capacity and high conductivity, it is opaque and its expansion is quite uniform and regular.

1.4. Gas Thermometers

Gases have an edge over liquids as a thermometeric substance. Gases expand more than liquids and so gas thermometers are more sensitive and the expansion of the containing vessel will require only a very small correction. They can be obtained pure and remain gaseous over a wide range of temperatures. They expand and shrink quite uniformly and regularly with change of temperature. The value of temperature given by gas thermometers is so reliable and

accurate that they are used as standards for checking the readings of other thermometers.

In its simplest form a constant volume air thermometer consists of a glass bulb B to which a long capillary tube bent twice at right angles, is attached. The capillary tube is clamped on to a vertical wooden stand and is connected by rubber tubing to a reservoir of mercury. A metre scale is placed between the capillary tube and the reservoir to read the positions of the mercury levels in them. The whole instrument is supported on a heavy base provided with levelling screws. To keep the volume of air enclosed in the bulb constant, an index mark 'I' is made on the capillary tube. The level of mercury is constantly kept at the index by adjusting the position of the reservoir.

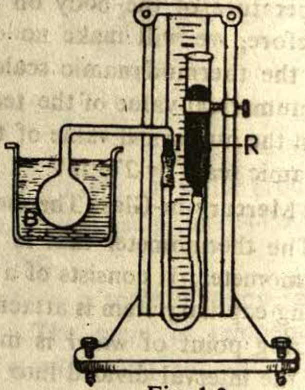


Fig. 1.3

To use it as a thermometer first the bulb of the thermometer is immersed in a bath containing ice cold water and the level of mercury in the closed tube is set at the index mark by raising or lowering the reservoir. When the level remains stationary for a pretty long time (say for 5 minutes), the levels of mercury are read on the scale. The difference added or subtracted from the barometer reading according as the level of mercury in the reservoir stands above or below the index mark gives the pressure of the gas at 0°C. Next heat water of the bath to its boiling point and again note down the difference of the mercury levels after setting the level in the closed tube at the index mark. This difference added to the barometer-reading at the time of the experiment gives the pressure of the gas at 100°C, provided the pressure of the atmosphere is the standard one, namely, 76 cm of mercury. If the pressure of the air is not 76 cm of mercury, then the corresponding temperature of boiling water is calculated from the standard relation

$t = 100 - \cdot 37(76\text{-H})$ °C where H is the barometer-reading in cm at the time of the experiment.

After fixing up its lower and upper fixed points the thermometer is ready for use to determine any unknown temperature, say, the melting point of wax. A little wax is taken in a capillary tube and is held close to the bulb immersed in a water bath. The bath is heated slowly and when wax melts, the difference in mercury levels is noted. The bath is then allowed to cool and when the wax freezes again, the difference in mercury levels is noted. The mean difference added or substracted from the barometer-reading, according as the level of mercury in the open tube stands above or below the index mark, gives the pressure of the enclosed gas at the melting point of wax.

Let P_0 be the pressure of the enclosed gas at 0°C, P_{100} pressure of the gas at 100°C and P_t is the pressure at the melting point of wax which is t°C, say.

Then by Charles' law, we have

$P_t = P_0(1+\alpha t)$ where α is the coefficient of expansion of the enclosed gas at constant volume.

$$\therefore \qquad P_{100} = P_0(1+\alpha 100).$$

Eliminating α we have,

$$t = \frac{P_t - P_0}{P_{100} - P_0} \times 100 \qquad \text{.. (1.2).}$$

To estimate the same temperature graphically, plot two points corresponding to P_0 and P_{100} at 0°C and 100°C respectively and join them by a straight line. This is the 'Calibration Curve'. Now use this calibration curve to find the temperature of melting wax corresponding to the pressure of the enclosed gas at which the wax melts.

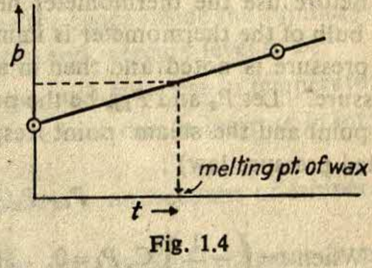


Fig. 1.4

To carry out very accurate measurements of temperature specially for standardising other thermometers the Bureau International des Poids et Mesures recommended the constant-volume hydrogen thermometer of the following description as a standard.

It consists of a platinum-iridium tube C one litre in capacity, 1 metre in length and 36 mm in diameter. It is filled with hydrogen at a pressure of 1 m of mercury at the temperature of melting ice

It is connected to the pressure measuring unit by a capillary tube of platinum 1 metre in length. The pressure measuring unit consists of two tubes A and B and a barometer tube dipped into mercury in A. The barometer is bent so that the upper surface of mercury in it is exactly above B so that the levels of mercury to be read lie on the same vertical line. Since the barometer is incorporated into the thermometer itself, the process of measuring the pressure is much simplified and the number of observations to be taken is reduced to two. B is divided into two compartments of mercury by a strong steel plug H and both these compartments communicate with A. By raising or lowering the mercury reservoir M, the mercury surface in the lower part of B is just made to touch a fine platinum point P projecting from the steel plug H, and thus the volume of the enclosed gas is kept constant.

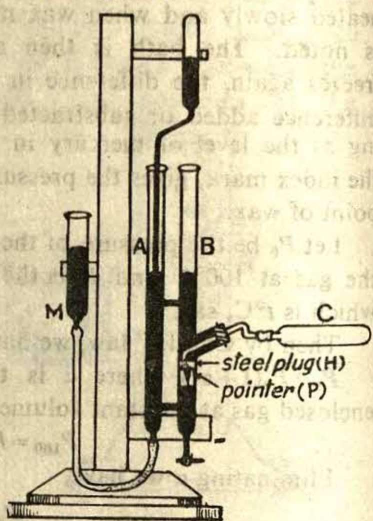


Fig. 1.5

Before use the thermometer needs 'calibration'. To calibrate it the bulb of the thermometer is immersed in a bath of melting ice and its pressure is noted and then in a bath of boiling water at standard pressure. Let P_0 and P_{100} be the pressures of the enclosed gas at the ice point and the steam point respectively. By Charles' law (also called pressure law) :

$$P_t = P_0(1 + \alpha t).$$

When $t = \left(-\frac{1}{\alpha}\right)$°C, $P_t = 0$. Thus $\left(-\frac{1}{\alpha}\right)$°C is the absolute zero on this property (pressure) of a perfect gas.

$$\therefore \qquad \left(-\frac{1}{\alpha}\right)^\circ\text{C} = 0^\circ\text{K}.$$

$$\text{or} \qquad 0^\circ\text{C} = \frac{1}{\alpha}\,^\circ\text{K}.$$

$\therefore$ T_0 (temperature of melting ice on the absolute scale $= \frac{1}{\alpha}$.

Introducing T_0 in the above relation

$$P_t = P_0\left(1+\frac{t}{T_0}\right) = P_0\,\frac{T_0+t}{T_0}$$

or
$$\frac{P_t}{P_0} = \frac{T_0+t}{T_0} = \frac{T}{T_0} \qquad \text{.. (i)}$$

$$\therefore\ \frac{P_{100}}{P_0} = \frac{T_0+100}{T_0} \quad \text{or} \quad \frac{1}{T_0} = \frac{P_{100}-P_0}{100P_0}.$$

If T is the temperature on the absolute scale corresponding to any observed pressure P then from (i)

$$\frac{P}{P_0} = \frac{T}{T_0}$$

or
$$T = \frac{P}{P_0}.\,T_0 = \frac{P}{P_0}\cdot\frac{100P_0}{P_{100}-P_0}$$

or
$$T = \frac{P}{P_{100}-P_0}\times 100 \qquad \text{.. (1.3).}$$

In developing the working formula Eq. 1.3 for a standard hydrogen thermometer it is assumed : (i) that the whole gas attained the temperature of the bath, (ii) that the gas obeyed the perfect gas law, and (iii) that its volume remained constant.

Actually, however, none of these assumptions is true and consequently corrections are necessary. Thus corrections are necessary for the following :

(i) The gas in the 'dead space' is not raised to the temperature of the bulb. The 'dead space' means the space inside the capillary tube and the space between the steel plug and mercury level in B.

(ii) Increase in the volume of the bulb with rise in temperature.

(iii) Increase in the volume of the bulb due to increase in internal pressure.

(iv) Change in density of mercury on account of temperature changes.

(v) Correction are necessary for deviations of real gases from the behaviour of perfect gases.

1.5. Callendar's Compensated Constant Pressure Thermometer

In this thermometer the error due to the 'dead space' is avoided by an ingenious compensating device. The bulb of the thermometer is a silica bulb A connected to an identical graduated bulb B con-

taining mercury and fitted with a tap at the bottom. To compensate for the volume of the connecting tube *bb* and that of the bulb *B*, an exactly identical tube *dd* and a bulb *C* (similar to *A* or *B*) are taken and *bb* and *dd* are connected by a sulphuric acid manometer *M*. At the time of construction the same mass of a gas is introduced into the two parts on either side of the manometer *M*. To measure the temperature of a bath, *A* is immersed up to *b* into it and the bulbs *B* and *C* are fully immersed in a bath of melting ice. After levelling up the levels of acid in *M* by letting out mercury from *B*, the volume of the gas in *B* is read off from

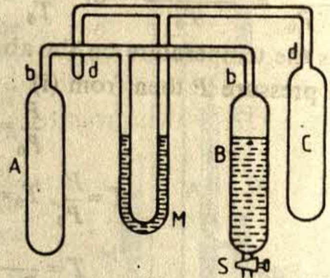


Fig. 1.6

the scale. Let T, T_0 and T' be the temperatures of the test bath, melting ice bath and the room respectively. Let V be the volume of A or C, v be the volume of gas in B and v' be the volume of bb or dd. Then since mass of the gas is the same on the two sides and the mass of a gas is given by $pV = mrT$ we have

$$\frac{pV}{rT}+\frac{pv'}{rT'}+\frac{pv}{rT_0}=\frac{pV}{rT_0}+\frac{pv'}{rT'}$$

or

$$T=\frac{V}{V-v}T_0$$

From this formula unknown temperature T is calculated. Temperatures within the range 0° to 600°C can be accurately measured by this instrument.

1.6 Platinum Resistance Thermometer

Method before 1954.

The basis of construction of this thermometer is that the resistance of a metal increases quite uniformly with temperature according to the equation

$$R_t=R_0(1+\alpha t+\beta t^2) \quad .. \quad \text{(i).}$$

where R_0 and R_t are the resistances of the wire at 0°C and t°C and α and β are constants of the material of the wire called the first and second temperature coefficients of resistance. To calibrate the thermometer, that is, to know the unknowns α and β the resistance of the wire is measured at the ice point (0°C), boiling point of water

(100°C) at the normal pressure and boiling point of sulphur (444·4°C) at normal pressure.

To simplify the process of calculating unknown temperature from equation (i), Callendar introduced a new scale of temperature called platinum scale on the assumption of the simplified formula $R_t = R_0(1+et)$ where e is the mean temperature coefficient between 0° and 100°C.

The unknown temperature on this scale is given by

$$t_p = \frac{R_t - R_0}{R_{100} - R_0} \times 100 \qquad .. \quad (1.4).$$

The correction required is given by

$$\text{Correction} = t - t_p = t - \frac{R_t - R_0}{R_{100} - R_0} \times 100$$

$$= t - \frac{R_0(1+\alpha t+\beta t^2) - R_0}{R_0(1+100\alpha+100^2\beta) - R_0} \times 100$$

$$= t - \frac{\alpha t + \beta t^2}{\alpha + 100\beta}$$

$$= \frac{\alpha t + 100\beta t - \alpha t - \beta t^2}{\alpha + 100\beta}$$

$$= \frac{100\beta t - \beta t^2}{\alpha + 100\beta}$$

$$= \frac{\beta}{\alpha + 100\beta}(100t - t^2)$$

or $$\text{Correction} = -\frac{\beta.100^2}{\alpha + 100\beta}\left\{\left(\frac{t}{100}\right)^2 - \frac{t}{100}\right\}$$

or $$\text{Correction} = \delta\left\{\left(\frac{t}{100}\right)^2 - \frac{t}{100}\right\} \qquad .. \quad (1.5)$$

where $$\delta = -\frac{\beta.100^2}{\alpha + 100\beta} \qquad .. \quad (1.5a).$$

For pure platinum $\alpha = 3{\cdot}94 \times 10^{-3}\text{K}^{-1}$, $\beta = -5{\cdot}8 \times 10^{-7}\text{K}^{-2}$ and $\delta = 1{\cdot}5$ from the Eq. 1 5a.

$\therefore$ For pure platinum

$$\text{Correction} = 1{\cdot}5\left\{\left(\frac{t}{100}\right)^2 - \frac{t}{100}\right\} \qquad .. \quad (1.5b).$$

It is seen that the calculation of the necessary correction requires the correct value of the unknown temperature and hence the calculation of the correction seems to be impossible. But the fact is that it

is quite possible by the so called '*method of successive approximation*. In the first step the value of the platinum scale temperature is substituted in place of t in Eq. 1.5 (b) and the correct temperature is calculated by adding the correction to the platinum scale temperature. This is the correct temperature after a first approximation. In the second step, the correct temperature obtained after the first approximation is substituted in Eq. 1.5 (b) and the correction is calculated. This correction added to the platinum scale temperature gives correct temperature after the second approximation. The process is continued till a fairly steady value of the unknown temperature is obtained. In practice two or three approximations will give the correct temperature. To save time in actual practice, every platinum thermometer is provided with a table giving corrections calculated by the method of 'successive approximation' leisurely in one column and the corresponding platinum scale temperatures in the adjoining columns. In actual practice, the platinum scale temperature is calculated by using Eq. 1.4 and then the correction table is consulted for correction. The platinum scale temperature plus the correction found in the table gives immediately the correct value of the unknown temperature.

Construction. Pure platinum wire free from silicon, carbon, tin and other impurities is non-inductively wound on a thin mica plate. The ends of this wire are attached to platinum leads which pass through holes in mica sheets closely fitting the upper part of the protecting hard glass or glazed porcelain tube and terminate at the terminals P, P at the top of the instrument. The function of the mica sheets is to keep the lead wire well separated and prevent convection currents of air up and down the tube. To compensate for the resistance of the leads, an exactly similar pair of wires, with their lower ends joined together is placed close to the leads of the coil of platinum wire.

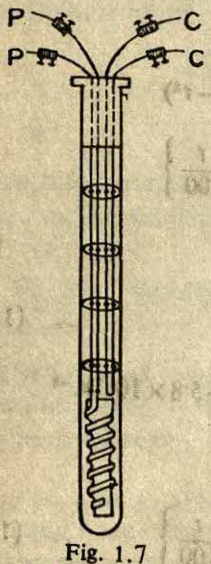


Fig. 1.7

Measurement of resistance. Various methods due to Siemens, Callendar and Thompson are available

for connecting the compensating leads and measuring the resistance of the thermometer at different temperatures. The method due to Callendar and Griffifths is the common method, which is described below. The bridge employed to measure resistance is the usual wheatstone bridge in which the ratio arms P and Q are kept equal so that the balance condition

$\frac{P}{Q}=\frac{R}{S}$ is simplified to $R=S$. This bridge is due to Callendar and Griffifths and after their name it is known as Callendar and Griffith's bridge.

The rheostat arm R of the bridge consists of a set of resistances 1, 2, 4, 8, 16, 32, 64 units. The usual plug contacts are here replaced by mercury cup contacts. To obtain fractions of an 'ohm', a long resistance wire of uniform cross-section is connected at the D-point of the usual wheatstone bridge.

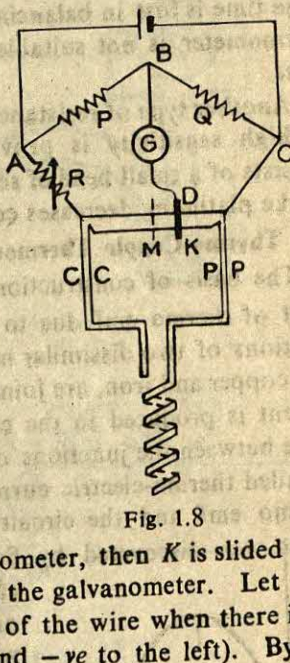


Fig. 1.8

To eliminate the thermoelectric effect another wire of the same material as the first wire is arranged parallel to the latter. The contact maker K touches both the wires simultaneously. The compensating leads are inserted in the rheostat arm and the main coil together with its leads is inserted in the unknown arm of the bridge. The position of the contact-maker on the wire constitutes the D-point of the bridge. Keeping it at the mid-point of the wire R is adjusted within 1Ω till there is no deflection or minimum deflection in the galvanometer. If there is some deflection in the galvanometer, then K is slided over the wire till there is 'no' deflection in the galvanometer. Let x be the distance of K from the mid-point of the wire when there is no deflection (x is $+ve$ to the right of M and $-ve$ to the left). By the principle of this bridge, when there is 'no' deflection

$$R=S.$$

i.e., Resistance between A and D = Resistance between D and C

or $$R+R_c+(l+x)\rho=R_t+R_c+(l-x)\rho$$

where R_t = resistance of the main coil of the thermometer
R_c = resistance of either compensating leads or the primary leads
ρ = resistance per unit length of the wire
$2l$ = length of the wire

or $$R_t = R + 2x\rho. \quad .. \quad (1.6).$$

The great advantage of platinum thermometers lies in their wide range $-200°C$ to $1200°C$) and the reliability of the result given by them. The value of temperature given by a platinum thermometer is so precise and consistent that once standardised by comparison with a gas thermometer, they may serve as secondary standards. There are, however, some drawbacks also. The resistance thermometer has a large thermal capacity and the protecting tube has low thermal conductivity and therefore the thermometer does not quickly attain the temperature of the bath in which it is immersed. In short, we may say that this thermometer has an excessive *'time lag'*. Further some time is lost in balancing the bridge. For these reasons this thermometer is not suitable for measuring rapidly varying temperatures.

Another type of resistance thermometer which has the advantage of high sensitivity is provided by the *thermistor*. A thermistor consists of a small bead of semi-conducting material whose resistance, unlike platinum, decreases considerably with increase of temperature.

1.7. Thermo-Couple Thermometer

The basis of construction of this thermometer is the development of thermo emf due to difference of temperatures between the junctions of two dissimilar metals. When two dissimilar metal wires, say, copper and iron, are joined to form a closed circuit, an electric current is produced in the circuit on setting up a temperature difference between the junctions of the wires. The current so produced is called thermo-electric current and the corresponding emf is called thermo emf and the circuit itself is called a thermo-couple. This effect was discovered by Seebeck and is known as the Seebeck effect. The thermo-emf developed depends on the temperature of the hot junction when the other junction is kept in melting ice. The plot of the thermo emf with the temperature of the hot junction is parabolic. In the beginning the thermo emf increases linearly with the

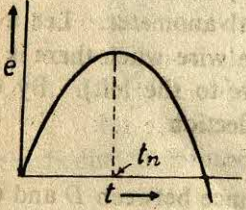


Fig. 1 9

temperature of the hot junction, but later it becomes flat, and then decreases linearly with the increase of the temperature. The temperature at which the curve becomes almost parallel to the temperature axis is called the neutral temperature and the one at which the thermo emf becomes zero is called the temperature of inversion.

If the temperature of the hot junction is well below the neutral temperature the thermoemf may be taken to be proportional to the temperature of the hot junction This fact is utilised in the construction of the thermo-couple thermometer.

Construction. To construct a thermo-couple thermometer the elements of the couple are to be selected properly. The choice of the elements is determined by the range of temperature and the thermo emf developed. The elements should be such that they do not get

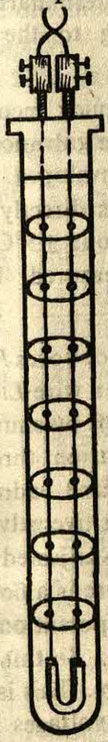


Fig. 1.10

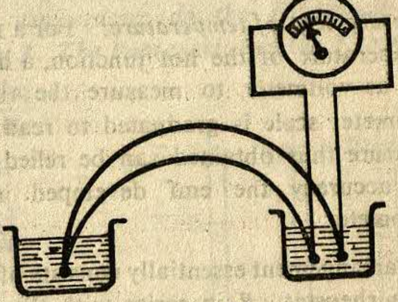


Fig. 1.11

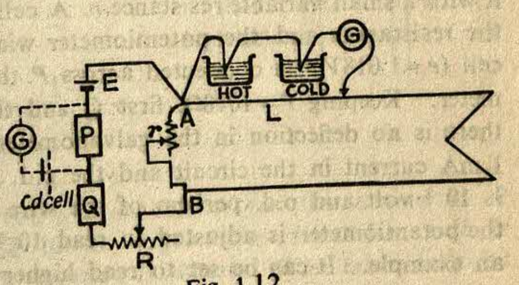


Fig. 1.12

oxidised at such high temperature and that they develop maximum thermoemf. One more point should be considered in the selection and that is the neutral temperature. The range of temperature for

which it is constructed must be well below the neutral temperature. For low temperatures up to 300°C iron-constantan and copper constantan couples are satisfactory, upto 600°C a nickel-iron couple is suitable and for very high temperatures a platinum and platinum-iridium couple must be used.

After proper selection of the elements one end of each element is welded together electrically. This end forms the hot junction. The portions of the wire near the hot junction are kept insulated by a short capillary tube of fire-clay or hard glass for low temperatures. The wires are threaded through mica discs closed in the protecting tube of porcelain, quartz or hard glass depending upon the temperature for which the thermometer is meant. For ordinary works in laboratories the mica discs and the protecting tubes can be dispensed with. The wires are connected to two terminals. To these terminals are connected extension leads leading to the cold junction which lies at a convenient distant place.

Measurement of temperature. For a rough, but quick record of the temperature of the hot junction, a high resistance galvanometer is used as voltmeter to measure the thermoemf developed. The galvanometer scale is graduated to read temperatures directly. The temperature thus obtained can be relied upon to about ±5°C. For higher accuracy the emf developed must be measured by a potentiometer.

This arrangement essentially consists of two resistance boxes P and Q and a rheostat R in series with the potentiometer wire L. The resistance of the potentiometer wire is reduced to 10Ω by shunting it with a small variable resistance r. A cell E drives current through the resistances and the potentiometer wire. A standard cadmium cell ($e = 1{\cdot}018\,V$) is connected across P through a sensitive galvanometer. Keeping $P = 1018\Omega$, first Q and then R are so adjusted that there is no deflection in the galvanometer. Thus there is a flow of 1 mA current in the circuit and the p.d. across the potentiometer is 10^{-2} volt and p.d. per cm of the wire is 10^{-5} volt. In this way the potentiometer is adjusted to read 10^{-5} volt per cm. This is just an example. It can be set to read higher or smaller voltages than this. Now the thermo-couple emf is balanced against the wire of the potentiometer. If it is balanced by l cm of the wire, the thermo emf developed is directly $l \times 10^{-5}$ volt.

To estimate the unknown temperature, first the thermo-couple thermometer is to be calibrated. Keeping the hot junction in two standard baths, namely, boiling water (100°C) at normal pressure and melting lead (327·3°C) the thermoemfs developed are measured and a calibration curve is plotted. This calibration curve is subsequently used to find any unknown temperature.

The advantages of a thermo-couple thermometer are : (i) it is cheap and easy to construct, (ii) it has practically no 'lag', (iii) it has an extensive range of temperatures from—200°C to 1600°C, (iv) it measures temperature at a single point as it itself is a point (hot junction) and (v) it can measure rapidly varying temperatures. The only serious disadvantage of this thermometer is that there is no single theoretical formula which can be utilised over the whole range of temperatures.

Examples

1. *What is the value of the absolute zero on Fahrenheit and the Reaumur scale ? At what temperature do the Fahrenheit and the Celsius give the same reading ? The Fahrenheit and the Kelvin ?*

Sol. $\dfrac{373{\cdot}15-0}{273{\cdot}15-0}=\dfrac{212-x}{32-x}$

or $x=-459{\cdot}69$. Ans.

$\dfrac{373{\cdot}15-0}{273{\cdot}15-0}=\dfrac{80-x}{0-x}$

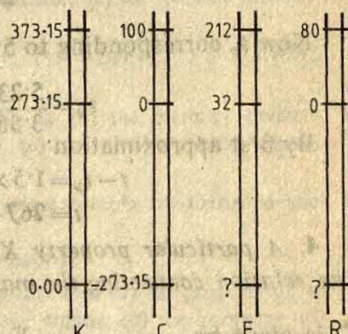
or $x=-218{\cdot}52$. Ans.

$\dfrac{x-32}{212-32}=\dfrac{x-0}{100-0}$

or $x=-40°\text{C}$. Ans.

$\dfrac{x-(-459{\cdot}69)}{32-(-459{\cdot}69)}=\dfrac{x-0}{273{\cdot}15-0}$

or $x=574{\cdot}56$. Ans.



2. *The pressure of air in a constant volume gas thermometer is* 80 *cm and* 109·3 *cm at* 0° *and* 100°C *respectively. When the bulb is placed in some hot bath, the pressure is* 100 *cm. What is the ice point as given by this thermometer and what is the temperature of the hot bath on the absolute scale ?*

Sol. $T_0=\dfrac{100P_0}{P_{100}-P_0}=\dfrac{100\times 80}{109{\cdot}3-80}=\dfrac{100\times 80}{29{\cdot}3}=273{\cdot}037°\text{K}$. Ans.

$$T=\frac{P}{P_{100}-P_0}\times 100=\frac{100\times 100}{109{\cdot}3-80}=\frac{100\times 100}{29{\cdot}3}=341{\cdot}3^\circ\text{K}.$$

Ans.

3. *The resistances of a platinum resistance thermometer are 2·56, 3·56 and 6·78 ohm at 0°C, 100°C and 444·6°C respectively. Calculate the temperature of the bath in which it has 5·23Ω. Make only one approximation.*

Sol. We have $t_p=\frac{R_t-R_0}{R_{100}-R_0}\times 100.$

Corresponding to correct temperature 444·6, the platinum temperature is

$$t_p=\frac{6{\cdot}78-2{\cdot}56}{3{\cdot}56-2{\cdot}56}\times 100=422.$$

Now
$$t-t_p=\delta\left[\left(\frac{t}{100}\right)^2-\left(\frac{t}{100}\right)\right]$$

$\therefore$
$$444{\cdot}6-422=\delta\,\frac{444{\cdot}6}{100}\left(\frac{444{\cdot}6}{100}-1\right)$$

or
$$\delta=\frac{22{\cdot}6\times 100\times 100}{444{\cdot}6\times 344{\cdot}6}=1{\cdot}5.$$

Now t_p corresponding to 5·23Ω is

$$t_p=\frac{5{\cdot}23-2{\cdot}56}{3{\cdot}56-2{\cdot}56}\times 100=267.$$

By first approximation

$$t-t_p=1{\cdot}5\times\tfrac{267}{100}\times\left(\tfrac{267}{100}-1\right)=6{\cdot}69.$$

$$t=267+6{\cdot}69=273{\cdot}69^\circ\text{C}.$$ **Ans.**

4. *A particular property X of a substance changes on heating. The relation connecting the magnitude of X and absolute temperature T is given by,*

$$X=\frac{k}{T-223}$$

when $T>223$ and k is a constant. Derive an expression for celcius temperature t based on this scale and establish the relation of t and T What is the value of t corresponding to $T=423$ K ?

Sol. We have, in general, $t=\frac{X-X_0}{X_{100}-X_0}\times 100.$

$\therefore$ Here,
$$t=\frac{\dfrac{k}{T-223}-\dfrac{k}{273-223}}{\dfrac{k}{373-223}-\dfrac{k}{273-223}}\times 100 \quad \text{(taking } T_0=273 \text{ instead of } 273{\cdot}15)$$

or $$t=\frac{T-273}{T-223}\times 150 \quad °C \quad \textbf{Ans.}$$

When $T=423$°K, $t=\dfrac{423-273}{423-223}\times 150=112\cdot5$°C **Ans.**

QUESTIONS

(A)

1. If α is the coefficient of expansion of a gas, the absolute zero on perfect gas scale is (a) $-\alpha$°C, (b) $+\alpha$°C, (c) $+\frac{1}{\alpha}$ °C, (d) $\left(-\frac{1}{\alpha}\right)$°C.

2. For platinum (a) α and β are both positive, (b) α and β are both negative, (c) α positive, β negative, (d) α negative, β positive.

3. If P_0 and P_{100} are the pressures indicated by a constant volume thermometer at 0°C and 100°C respectively, the absolute temperature of melting ice is

(a) $\frac{P_{100}-P_0}{100P_0}$, (b) $\frac{P_{100}-P_0}{P_0}$, (c) $\frac{100P_0}{P_{100}-P_0}$, (d) $\frac{P_0}{P_{100}-P_0}$.

4. The lower fixed point of the Celsius scale is (a) the melting point of ice, (b) the triple point of water, (c) boiling point of water, (d) none of these.

5. The SI unit of temperature is (a) degree centigrade, (b) degree Celsius, (c) Kelvin, (d) Rankine.

6. To measure accurately a temperature of -150°C one may use (a) a mercury-in-glass thermometer, (b) a gas thermometer, (c) an alcohol thermometer, (d) none of these.

7. The absolute zero of temperature is (a) 0°C, (b) -260°C, (c) $-273\cdot16$°C, (d) -10°C.

8. The thermometer used for standard reference is (a) the mercury thermometer, (b) the platinum resistance thermometer, (c) the alcohol thermometer, (d) the gas thermometer.

9 Which of the following gases would be most suitable as a thermometric substance for low temperature measurement ?

(a) oxygen, (b) helium, (c) hydrogen, (d) nitrogen ?

10 The thermodynamic scale of temperature is based on (a) the property of a perfect gas, (b) the property of a reversible heat engine, (c) the property of alcohol, (d) the property of steam.

(Ans. 1. d. 2. c. 3. c. 4. a. 5. c. 6. b. 7. c. 8. d. 9. b. 10. b)

(B)

1. What do you mean by the perfect gas scale ? How does it differ from the thermodynamical scale ?

2. What do you mean by thermodynamical scale of temperature ? In what respect is it superior to the perfect gas scale ?

3. What is the 'Celsius Scale' ? How is it related to the thermodynamical scale ?

4. What do you mean by the 'absolute zero' of a scale ? Explain the 'absolute zero' of a perfect gas scale and a thermodynamical scale.

5. Explain the action of a thermister as a thermometer.

(C)

1. Carefully bring out the concept of temperature from the Zeroth law of thermodynamics. Describe the constant volume air thermometer and explain how you would use it to determine the melting point of wax.

2. Describe a constant volume air thermometer and explain how with its help you will determine the value of the absolute zero on the centigrade scale

3. Distinguish between heat and temperature. Describe the standard hydrogen thermometer and deduce the working formula which would give the temperature of a bath directly on the absolute scale. What are the corrections necessary ?

4. Describe, with theory, a platinum resistance thermometer. What are the advantages and disadvantages of this thermometer ?

5. Describe, with theory, a thermocouple thermometer. What are the merits and demerits of this thermometer ?

(D)

1. The pressure of air in a constant volume air thermometer are 80 cm and 109·4 cm of mercury at 0°C and 100°C respectively. The pressure is 94·7 cm when the bulb is placed in a bath of hot water. Find the temperature of the hot water.

(Ans. 50°C)

2. The pressure inside the bulb of a constant volume hydrogen thermometer is 73 cm of mercury at 0°C, 100·3 cm at 100°C and 23·7 cm in liquid air. Calculate the temperature of liquid air.

(Ans. $-180{\cdot}6$°C)

3. The resistances of a platinum resistance thermometer at the ice point, the steam point and sulphur point are respectively 10·000, 13·946 and 24·817. Find α and β. (Ans. $\alpha = 4{\cdot}124 \times 10^{-3} K^{-1}$; $\beta = -1{\cdot}780 \times 10^{-6} K^{-2}$)

4. The resistance of a platinum thermometer are 2·57Ω at 0°C, 3·57Ω at 100°C and 2·89Ω at some unknown temperature. Calculate the temperature on the platinum scale and on the celsius scale

(Ans. 32° and 31·67°C by only one approximation)

5. When the bulb of a constant volume air thermometer is immersed in melting ice, the level of mercury in the reservoir 5 cm below the index mark; when the bulb is at 273°C, the level is 65 cm above the index mark. Calcutate the atmospheric pressure at the time of these observations. (Ans. 75 cm of mercury)

(E)

1. Is temperature a microscopic or macroscopic concept ?

2. Why does the column of mercury first descend and then rise, when a mercury-in-glass thermometer is put in a flame ?

3. Is the temperature of a body on the thermodynamical scale the same as that on the perfect gas scale ?

4. What is the unit of α, the first temperature coefficient of resistance ?

5. What is the unit of the second temperature coefficient of resistance ?

6. If α is the pressure coefficient of a gas, the absolute zero on centigrade scale is............

7. Why hydrogen is added in mercury in-glass thermometers ?

(Ans. 1. macroscopic, 2. Bulb expands first and then mercury, 3. Yes, 4. K^{-1}, 5. K^{-2}, 6. $\left(-\frac{1}{\alpha}\right)$°C, 7. Hydrogen elevates boiling point and prevents distilling of mercury to the upper part of the stem.)

CHAPTER 2

THERMAL EXPANSION

2.1. Expansion of Solids

The size of all material bodies changes on being heated. The change may be linear, superficial and cubical. The change in any linear dimension of a solid body such as its length, width or thickness, is called a linear expansion. If the length of the linear dimension is l, the change in length, arising from a change in temperature by $\triangle T$, is $\triangle l$, then experiments show that $\triangle l$ is proportional to the temperature change $\triangle T$ and to the original length l. Hence, we can write

$$\triangle l = \alpha l \triangle T \qquad .. \qquad (2.1)$$

where α is a constant called the *coefficient of linear expansion* or linear expansivity. It has different values for different materials. Rewriting this formula we have

$$\alpha = \frac{1}{l} \frac{\triangle l}{\triangle T} \qquad .. \qquad (2.1a)$$

Obviously then α may be defined as the *fractional change in length per degree temperature change*. Its unit is K^{-1} (per kelvin) or $°C^{-1}$ (per degree celsius). Strictly speaking α is not a constant. It depends on temperature. However, its variation is usually negligible and we can safely take it as a constant for a given material, independent of its temperature.

Exactly in the same way if $\triangle S$ is the change in area arising from a change in temperature $\triangle T$, then the coefficient of superficial expansion (β) or superficial expansivity is given by

$$\beta = \frac{1}{S} \frac{\triangle S}{\triangle T} \qquad .. \qquad (2.2)$$

and the coefficient of cubical expansion or cubic expansivity is given by

$$\gamma = \frac{1}{V} \frac{\triangle V}{\triangle T} \qquad .. \qquad (2.3)$$

where V is the original volume and $\triangle V$ is the change in volume, arising from a change in temperature by $\triangle T$.

It can be shown (see example 1) that $\beta = 2\alpha$ and $\gamma = 3\alpha$.

2.2. Measurement of Linear Expansion

There are many methods of determining the coefficient of linear expansion. We describe here two methods: one is the Pullinger's method, a method for the students and the other is the 'Comparator Method', a standard method used by the International Bureau of Weight and Measures.

(*a*) *By Pullinger's apparatus.* In this method the increase in length of a metal rod is measured by a spherometer or an optical lever. The rod which is about a metre long is placed inside a vertical steam jacket. The rod rests with its lower end on a glass plate fixed on the base board of the jacket *J* so that the rod is forced to expand upward only. At the top end it passes through a circular hole in a glass plate. The three legs of a spherometer or the two hind legs of an optical lever rest on the glass plate. The tip of the screw of the spherometer or the front leg of the optical lever rests on the top of the rod. Before passing steam, the screw of the spherometer is turned downward and made to touch the top of the rod and the reading is noted. Then it is turned up to allow space for expansion of the rod. Pass steam through the steam jacket after noting the temperature of the rod by means of the thermometer inserted into the jacket through the side tube. The rod is heated for a pretty long time until the temperature becomes steady. The screw of the spherometer is turned down till its tip just touches the top of the rod. The difference of the two spherometer readings gives $\triangle l$. The difference of the thermometer readings gives $\triangle T$. Knowing $\triangle l$ and $\triangle T$, l being known already, α is calculated from the formula

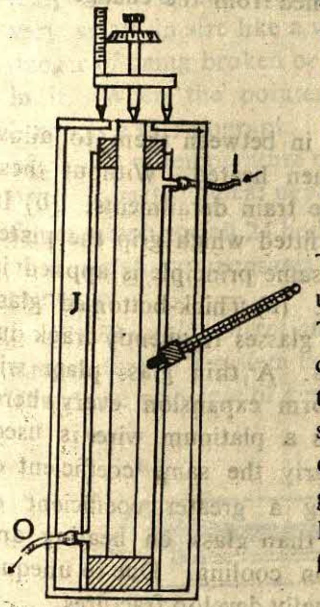


Fig. 2.1

$$\alpha = \frac{1}{l}\,\frac{\triangle l}{\triangle T}.$$

(*b*) *The Comparator Method.* In this method a metre bar A (shown by thick line) having two fixed marks exactly one metre apart is placed on a trolley inside a trough containing water at 0°C.

The experimental bar B is also placed on separate trolleys and there are two scratches on it about one metre apart. Two microscopes provided with micrometer eye-pieces are fixed vertically in rigid horizontal supports projecting out from stout pillars, the distance between the microscopes being about one metre apart. First the fine marks on the bar B are viewed through the two microscopes and their positions are noted in the micrometer eye-pieces. The standard metre bar is then brought below the microscopes and the two marks on it are viewed through the micrometer eye-pieces. From the change in the micrometer readings the length of the bar is obtained. A is then removed and B is again brought in the field of view of the microscopes. The temperature of the bath is then increased and maintained constant at some suitable value. The marks are again viewed through the micrometer eye-piece. The increase in length is determined from the change in the micrometer readings.

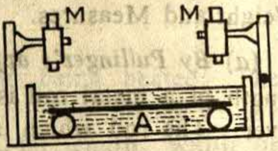


Fig. 2.2

2.3. Some Practical Examples

(a) In laying rails a small gap is left in between them to allow space for the expansion of the rails when heated. Without these gaps the rails would bend and may cause train derailments. (b) In rivetting boiler plates, red hot rivets are fitted which grip the plates tightly on contraction by cooling. The same principle is applied in putting on iron tyres on cart wheels. (c) Thick-bottomed glass wares such as glass chimney, drinking glasses frequently crack due to unequal expansion of different layers. A thin glass plate will conduct heat and there will be uniform expansion everywhere. (d) In sealing metallic wires into glass a platinum wire is used, because glass and platinum have nearly the same coefficient of expansion. If a metal, say *Cu*, having a greater coefficient of expansion is used, *Cu* will expand more than glass on heating and will also contract more than glass on cooling. Due to unequal contraction on cooling the joint will usually develop fractures.

2.4. Some Practical Applications of Expansion of Solids

(*i*) *Bimetallic thermometer*. When two bars of different metals, say iron and copper, are rivetted together, the composite body is called a bimetallic strip. On heating a bimetallic strip the bar bends

with copper on the outside while on cooling it bends on the opposite

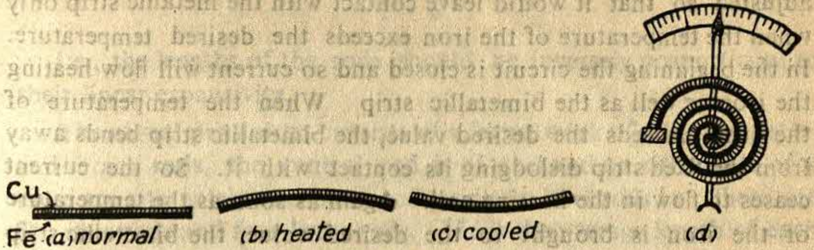


Fig. 2.3

way round. This fact is utilised in the construction of a thermometer.

In the construction of a bimetallic thermometer, two thin strips of iron and copper are rivetted together and the bimetallic strip formed is bent in the form of a spiral at the room temperature. One end of the spiral is fixed and the other end is free and carries a pointer which moves over a dial directly graduated in degrees of temperature by comparison with a standard thermometer. Being very small in size like a watch it can be carried easily and it has no danger of being broken or damaged, as there is no mercury or glass in it. When the pointer is replaced by a lever carrying a pen, it becomes a thermograph. A thermograph keeps a continuous record of change of temperature on a white paper moving slowly below the pen. In metrological departments a temperature record is made by thermograph for all 24 hours. The hair-spring of the balance wheel of a watch compensates for the change of temperatures on this principle.

(*ii*) *Bimetallic thermostat* Thermostats are devices by which the temperature of a system is maintained at a constant temperature. The above principle of bending of a bimetallic strip is utilised in the

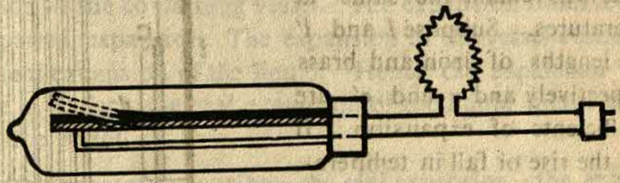


Fig. 2.4

construction of a thermostat. The regulator consists of a thick metallic strip in contact with a bimetallic strip The two strips are enclosed in a glass bulb. The instrument, say an electric iron, which is to be maintained at a particular temperature is connected in series

with the thermostat. The size of the bimetallic strip is previously adjusted so that it would leave contact with the metallic strip only when the temperature of the iron exceeds the desired temperature. In the beginning the circuit is closed and so current will flow heating the iron as well as the bimetallic strip When the temperature of the iron exceeds the desired value, the bimetallic strip bends away from the fixed strip dislodging its contact with it. So the current ceases to flow in the heating coil. Again as soon as the temperature of the iron is brought to the desired value, the bimetallic strip resumes its original position and re-establishes the circuit.

The same principle is applied in the **master bulb** which controls alternate 'off' and 'on' of a series of tiny bulbs used for decoration of houses and shops.

2.5. The Compensated Pendulum

In a pendulum clock the time keeping quality depends upon its length. In summer its effective length increases and so it goes slow and in winter the clock will go fast for thermal contraction of its length. In order to nullify the effects of thermal expansion or contraction, pendulums are so constructed that their lengths remain unaltered inspite of variations of temperatures. Such pendulums are called *Compensated Pendulums.*

Brass expands more than iron. So if a brass rod and an iron rod are arranged in such a way that the iron rod can expand downward only and the brass rod upward, and their lengths are so adjusted that the increase in length of the iron rod (downward), is exactly equal to the increase in length of the brass rod (upward) the 'tip to tip' distance of the iron and brass rods will remain the same at all temperatures. Suppose l and l' are the lengths of iron and brass rods respectively and α and α' are their coefficients of expansion. If $\triangle T$ be the rise or fall in temperature then

$$\triangle l = l\alpha\triangle T \text{ and } \triangle l' = l'\alpha'\triangle T.$$

For no change in 'tip to tip' distance we must have

$$l\alpha\triangle T = l'\alpha'\triangle T$$

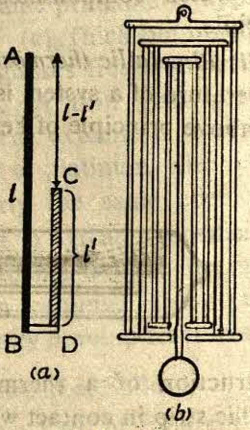


Fig. 2.5

or $$\frac{l}{l'}=\frac{\alpha'}{\alpha} \quad .. \quad (2.4)$$

i.e., the lengths of the rods should be inversely proportional to their linear expansivity.

The actual pendulum consists of a frame work of alternate steel and brass rods, the central rod is of steel and on either side of it there are alternately two brass rods and two steel rods i.e., there are five steel rods and four brass rods. If l is the average length of each steel rod and l' is that of each brass rod, the total length of the brass rods is $2l'$ and that of three steel rods is $3l$ because here expansion (downward) of three steel rods is nullified by the expansion (upward) of the two brass rods.

$$\therefore \quad 3l\times\alpha\times\Delta T=2l'\times\alpha'\times\Delta T$$

or $$\frac{l}{l'}=\frac{\alpha'}{\alpha}\,\frac{2}{3} \quad .. \quad (2.4a)$$

By experiment, $\alpha=12\times10^{-6}$ and $\alpha'=19\times10^{-6}$

$$\therefore \quad \frac{l}{l'}=\frac{19}{12}\times\frac{2}{3}=1.$$

or $$l=l'$$

i.e. the average length of the steel rods = the average length of the brass rods.

2.6. Expansion of Liquids

In case of liquids we have to consider only the cubical expansion. Liquids expand more than solids. A liquid needs a solid vessel to contain it and hence its expansion is always accompanied by the expansion of the containing vessel. The expansion observed is called the apparent expansion. The expansion of the vessel conceals a part of the real expansion of the liquid. Hence the expansion which we observe is, in fact, the real expansion of the liquid minus the expansion of the vessel.

or the apparent expansion = the real expansion − the expansion of the vessel.

Since there are two cubical expansions, real and apparent, of a liquid, there are two cubical expansivities : the real cubical expansivity and apparent cubical expansivity.

By the Eq. 2.3

$$\gamma_a = \frac{1}{V} \frac{(\triangle V)_{apparent}}{\triangle T} \text{ and } \gamma_r = \frac{1}{V} \frac{(\triangle V)_{real}}{\triangle T}$$

where γ_a is the apparent cubic expansivity and γ_r is the real cubic expansivity.

or $$(\triangle V)_{apparent} = \gamma_a V \triangle T \text{ and } (\triangle V)_{real} = \gamma_r V \triangle T$$

If $(\triangle V)_{vessel}$ is the expansion of the containing vessel then

$$(\triangle V)_{vessel} = \gamma_g V \triangle T$$

$$\therefore \quad \gamma_a V \triangle T = \gamma_r V \triangle T - \gamma_g V \triangle T$$

or $$\gamma_r = \gamma_a + \gamma_g \qquad .. \quad (2.5)$$

2.7. Determination of the Cubic Expansivity of Liquids

There are three well-known methods for determining the cubic expansivities of liquids : (a) The Weight Thermometer method, (b) The Volume Thermometer Method and (c) The Hydrostatic Balance Method (Dulong and Petit's method).

(*a*) *The Weight thermometer Method*. The weight thermometer is simply a glass-bulb (spherical or cylindrical) having a bent capillary stem drawn out to a nozzle.

It is first weighed and then completely filled with the liquid by alternate heating and cooling with the open end dipping in a cup of the liquid. With the nozzle still inside the liquid in the cup the bulb of the thermometer is immersed in a bath of water at the room temperature (t_1°C). The bulb is then taken out, wiped dry and weighed again. The difference in weights gives the mass of the liquid filling the thermometer at t_1°C. The bulb is again put in the bath and it is heated slowly to any desired steady temperature t_2°C, stirring water of the bath thoroughly well to maintain uniformity of temperature throughout the mass of water of the bath. As the temperature of the bath is raised, the liquid inside the thermometer expands and some of it is forced out through the nozzle. The thermometer is removed from the bath, cooled to the room temperature and weighed again after wiping it dry. The

Fig. 2.6

difference between this weight and the first weight gives the mass of the liquid filling the thermometer at t_2°C.

CALCULATION. Let m_1 and m_2 represent the masses of the liquid filling the thermometer at temperature t_1°C and t_2°C respectively. If V_1 and V_2 are the volumes of the thermometer at the two temperatures and ρ_1, ρ_2 be the corresponding densities of the liquid, then

$$m_1 = V_1\rho_1,\ m_2 = V_2\rho_2 \qquad \text{.. (i)}$$

If γ_r and γ_g denote cubic expansivity of the liquid and the vessel respectively, then

$$\frac{V_2}{V_1} = \frac{V_0(1+\gamma_g t_2)}{V_0(1+\gamma_g t_1)} = (1+\gamma_g t_2)(1+\gamma_g t_1)^{-1} = (1+\gamma_g t_2)(1-\gamma_g t_1)$$

(expanding and neglecting higher terms)

or $$\frac{V_2}{V_1} = (1+\gamma_g t_2 - \gamma_g t_1) = 1+\gamma_g(t_2-t_1) = 1+\gamma_g t$$

where $$t_2 - t_1 = t$$

and $$\frac{\rho_1}{\rho_2} = \frac{1+\gamma_r t_2}{1+\gamma_r t_1} \qquad (\because\ \rho_0 = \rho_1(1+\gamma_r t_1) = \rho_2(1+\gamma_r t_2)).$$

or $$\rho_1/\rho_2 = (1+\gamma_r t_2)(1+\gamma_r t_1)^{-1} = (1+\gamma_r t_2)(1-\gamma_r t_1)$$
$$= 1+\gamma_r(t_2-t_1) = 1+\gamma_r t$$

Now, $$\frac{m_1}{m_2} = \frac{V_1\rho_1}{V_2\rho_2} = \frac{V_1\rho_2(1+\gamma_r t)}{V_1(1+\gamma_g t)\rho_2} = \frac{(1+\gamma_r t)}{(1+\gamma_g t)} = (1+\gamma_r t)(1-\gamma_g t)$$

or $$\frac{m_1}{m_2} = 1+\gamma_r t - \gamma_g t$$

or $$\gamma_r - \gamma_g = \frac{m_1 - m_2}{m_2(t_2-t_1)} \qquad (\because\ t = t_2 - t_1)$$

or, $$\gamma_r = \frac{m_1-m_2}{m_2(t_2-t_1)} + \gamma_g \qquad \text{.. (2.6)}$$

or $$\gamma_a = \frac{m_1-m_2}{m_2(t_2-t_1)} \qquad (\because\ \gamma_r = \gamma_a + \gamma_g)$$

i.e. apparent cubic expansivity

$$= \frac{\text{mass expelled}}{\text{mass remaining} \times \text{rise in temperature}} \qquad \text{.. (2.6a)}$$

It is clear from Eq. 2.6a that if the apparent cubic expansivity be known, the temperature rise can be calculated by finding the three weights of the thermometer and hence, the name *Weight-thermometer*.

(*b*) *The Volume Thermometer* (*or the Dilatometer*). The volume thermometer consists of a bulb to which a graduated stem is attached. It is nearly filled with the experimental liquid. The bulb is immersed in a bath of melting ice and its level on the stem is read and then the temperature of the bath is raised to any desired temperature t°C and level is again read. Let the liquid stand up to mark x_0 and x_1 at 0°C and t°C respectively. If V_0 is the volume of the bulb at 0°C and each division of the stem is v_0, then the volume of the liquid at 0°C and t°C are

Fig. 2.7

$$V_0+x_0v_0 \text{ and } (V_0+x_1v_0)(1+\gamma_g t)$$

γ_g being the expansivity of the containing vessel. The volume of the liquid at t°C is also

$$(V_0+x_0v_0)(1+\gamma_r t)$$

$$\therefore \quad (V_0+x_0v_0)(1+\gamma_r t)=(V_0+x_1v_0)(1+\gamma_g t)$$

$$\text{or} \quad V_0+x_0v_0=(V_0+x_1v_0)(1+\gamma_g t)(1-\gamma_r t)$$

$$=(V_0+x_1v_0)(1-\gamma_r t+\gamma_g t)$$

$$\text{or} \quad V_0+x_0v_0=V_0+x_1v_0-(\gamma_r-\gamma_g)(V_0+x_1v_0)t$$

$$\text{or} \quad \gamma_r-\gamma_g=\frac{(x_1-x_0)v_0}{V_0 t}$$

$$\text{or} \quad \gamma_r=\frac{(x_1-x_0)v_0}{V_0 t}+\gamma_g$$

$$\text{and} \quad \gamma_a=\gamma_r-\gamma_g=\frac{(x_1-x_0)v_0}{V_0 t}$$

$$\text{i.e. Apparent cubic expansivity}=\frac{\text{increase in volume}}{\text{Original volume}\times\text{rise in temp.}}$$

The volume V_0 and v_0 are determined by filling the thermometer with mercury and noting the weight of mercury filling the bulb and then certain length of the stem. The mass divided by the density of the mercury gives the necessary volume.

(*c*) *The Hydrostatic Balance Method* (*Dulong and Petits Method*). This is the only direct method of determining the absolute cubic expansivity of a liquid and is due to Dulong and Petit. If works on

the hydrostatic balancing of two columns of the same liquid at different temperatures. The pressure due to a column of liquid of density ρ and height h is ρgh. So the pressure is independent of the cross-section of the containing vessel and therefore the pressure is not effected by the expansion of the tube containing the liquid.

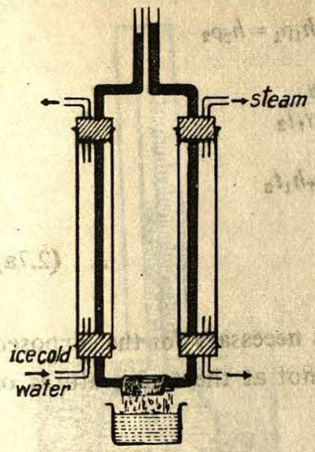


Fig. 2 8

The apparatus consists of a U-tube filled with the experimental liquid. The two limbs of the tube are enclosed in glass jackets Ice cold water is circulated in one jacket and steam in the other. A piece of blotting paper is placed on the horizontal portion of the tube and is constantly soaked with cold water to prevent the flow of the liquid from one limb to another. For convenience of reading the upper ends of the tube outside the jackets are bent twice at right angles and are brought side by side. Since the accuracy of the result depends on how accurately the difference in liquid levels in the two limbs is measured, a cathetometer telescope should preferably be used instead of a metre scale. When the steady state is attained i.e., the levels stand at contant heights for a long time, the heights of columns of liquid above the horizontal portion are measured.

Let h_t and h_0 be the heights of the liquid columns at t°C and 0°C. The pressure on the horizontal portion due to the cold column $=P+h_0\rho_0 g$, and that due to the hot column $=P+h_t\rho_t g$, where $P=$ atmospheric pressure. Since the liquid is at rest, we have by the law of fluid mechanics

$$P+h_0\rho_0 g=P+h_t\rho_t g \text{ or } h_0\rho_0=h_t\rho_t$$

But, $$\rho_0=\rho_t(1+\gamma_r t)$$

where γ_r is the real cubic expansivity of the liquid.

$$\therefore \qquad h_0\rho_t(1+\gamma_r t)=h_t\rho_t$$

or $$\gamma_r=\frac{h_t-h_0}{h_0\times t} \qquad (2.7)$$

If ice-cold water is not available, tap water may be circulated through the cold jacket. Let h_1 and h_2 be the heights of the cold and

hot columns at temperatures t_1°C and t_2°C respectively. Then we have

$$P+h_1\rho_1 g=P+h_2\rho_2 g \text{ or } h_1\rho_1=h_2\rho_2$$

or
$$h_1\frac{\rho_0}{1+\gamma_r t_1}=h_2\frac{\rho_0}{1+\gamma_r t_2}$$

or
$$h_2+\gamma_r h_2 t_1=h_1+\gamma_r h_1 t_2$$

or
$$\gamma_r=\frac{h_2-h_1}{h_1 t_2-h_2 t_1} \quad \text{.. (2.7a).}$$

SOURCES OF ERROR. (i) A correction is necessary for the exposed portion of the liquid columns which are not at the temperatures of the cold and hot jackets.

(ii) The blotting paper does not prevent convection currents completely, and so the hot liquid mixes up with the cold one to some extent.

(iii) The temperatures of the liquid heads in the two limbs are different and this introduces a difference in levels due to the inequality of the surface tension.

2.8. Anomalous Expansion of Water and Its Importance in Nature

The most common liquid, water, does not behave like other liquids. If a mass of water at any temperature, say at 10°C, be taken and allowed to cool, its density will gradually increase, until it reaches the temperature 4°C, when with further cooling the density will decrease instead of increasing. The density of water is maximum at 4°C. This behaviour is peculiar to water. This is studied by two methods: (a) The constant Volume Dilatometer and (b) the Hope's experiment.

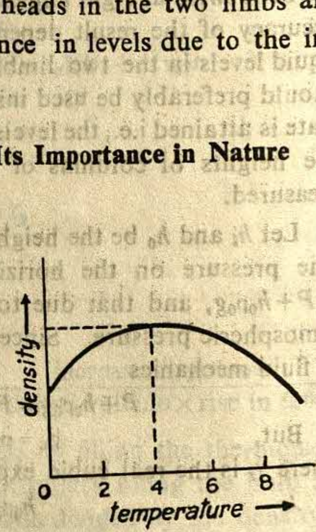


Fig. 2.9

(*a*) *The constant volume dilatometer*. The constant volume dilatometer consists of a large glass bulb to which a graduated stem is attached. Some mercury is put at the bottom of the bulb such that the increase in volume of mercury is equal to the increase in volume of the glass bulb. Then the volume of the bulb above the mercury will remain constant at all temperatures. Let V and V' be the volumes of the bulb and mercury and γ_r and γ_g are the cubic expansivity of the mercury and glass respectively.

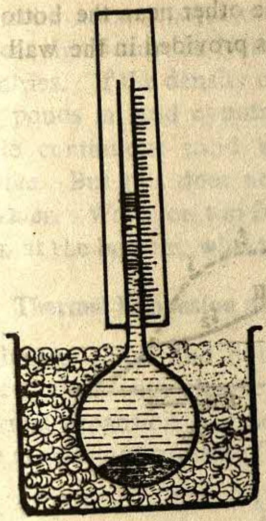


Fig. 2.10

Then $(\triangle V)_{bulb}=\gamma_g V \triangle T$

and $(\triangle V)_{mercury}=\gamma_r V' \triangle T$.

In order that the space above the mercury may remain constant at all temperatures.

$$\gamma_g V \triangle T=\gamma_r V' \triangle T$$

or
$$\frac{V'}{V}=\frac{\gamma_g}{\gamma_r} \quad \cdots \quad (2.8)$$

We know $\gamma_g=27\times10^{-6}K^{-1}$ and $\gamma_r=18\times10^{-5}K^{-1}$

$$\therefore \quad \frac{V'}{V}=\frac{27\times10^{-6}}{18\times10^{-5}}=\frac{3}{20}\cong\frac{1}{7}.$$

Thus if mercury, one seventh of the volume of the bulb, is poured into it, the volume above the mercury will remain constant at all temperatures. The expansion of a liquid in such a vessel will be its real expansion.

To study the anomalous behaviour of water with the help of this instrument, it is filled with water upto the middle of the stem and the bulb is immersed in a bath containing melting ice. The volume of water is noted after some time, when the position of the water level in the stem becomes steady. The temperature of the bath is gradually raised and the volumes of water at the corresponding temperatures are noted from the positions of water level in the stem. It is found that as the temperature is gradually increased the level of water goes down the stem up to 4°C and then rises up the stem from 4°C onwards.

(*b*) *Hope's experiment.* The apparatus consists of a tall metallic cylindrical vessel having a circular trough fixed around its middle. Two thermometers, one near the top and the other near the bottom, are inserted into the vessel through openings provided in the wall of

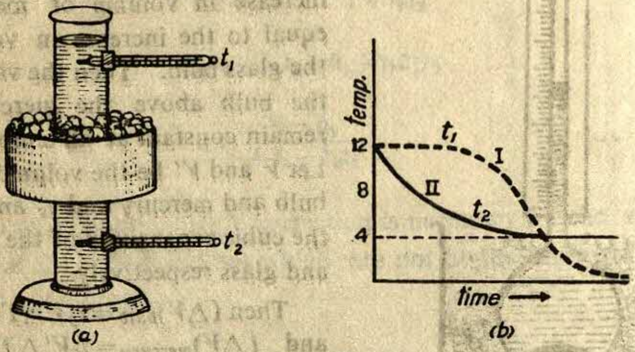


Fig. 2.11

the vessel. The vessel is filled with water pre-cooled to about 10°C, and the trough is packed with a freezing mixture of ice and salt. It is found that while the reading of the upper thermometer remains almost stationary at 10°C, the lower thermometer records a gradual fall in the temperature and becomes stationary at 4°C (curve II of Fig. 2.11 b). Then the upper thermometer also begins to record a fall in temperature and gradually comes t› 0°C, at which it remains steady (curve I). Small crystals of ice are found to float on water at this stage. In the beginning the water in the middle of the vessel cools, becomes heavier and sinks to the bottom. Water from the bottom, being lighter, rises up, gets cooled and sinks down to the bottom. In this way a convection current is first set up in the lower part of the vessel till the water at the bottom becomes **heaviest**. There is a gradual distribution of density from the bottom up to the central part of vessel. After this, a convection current is set up locally in the upper part of the vessel because water from the central part cannot rise up to the top, for it is heavier than water at the upper part, which is at about 10°C. With the progress of cooling the local convection currents gradually shift to the upper part, when the water at the top comes down to 0°C. With further cooling small crystals of ice are formed and float on the surface. In the final steady state there is no convection current anywhere in the liquid i.e. the liquid is at rest. In a liquid at rest the heaviest part of it must occupy the

lowest position and the lightest part the top most position. Hence we can conclude that water has got the highest density at 4°C.

IMPORTANCE IN NATURE. The anamalous behaviour of water has a great significance in Nature in preservation of aquatic lives in cold countries. If the density of water were not maximum at 4°C but at 0°C ponds in cold countries would freeze from top to bottom as a single continuous solid mass of ice and no aquatic animal would survive. But this does not happen due to the anamalous behaviour of water. Water on top freezes into ice, but it remains in the liquid form at the bottom, where aquatic animals can easily move about.

2.9. Thermal Expansion of Gases

Unlike solids and liquids, gases expand considerably on adding heat to them with change of **pressure** and **volume.** Hence in the thermal expansion of gases two cubic expansivity are to be defined—one at constant pressure (γ_p) and the other at constant volume (γ_v)

$$\gamma_p = \frac{1}{V}\ \frac{\Delta V}{\Delta T} \quad .. \quad (2.9)$$

$$\gamma_v = \frac{1}{P}\ \frac{\Delta P}{\Delta T} \quad .. \quad (2.9a)$$

If V_0 is the volume of a gas at 0°C and V_t is the volume at t°C, then $\Delta V = \gamma_p V_0 t$.

$$\therefore \quad V_t = V_0 + \Delta V = V_0 + \gamma_p V_0 t$$

$$V_t = V_0(1 + \gamma_p t) \quad .. \quad (2.9b)$$

Similarly if P_0 is the pressure of a gas at 0°C and P_t is the pressure at t°C, the volume remaining constant

$$P_t = P_0(1 + \gamma_v t) \quad .. \quad (2.9c)$$

In fact the relations 2.9 (b) and (c) are similar to that in the case of thermal expansion of solids and liquids The coefficients of expansion of gases at constant pressure and volume were determined by Charles and he found them to be the same for all gases. It was found to be ·00366 per °C corresponding to $\frac{1}{273{\cdot}15}$.

Hence relations 2.9 (b) or 2.9 (c) may be written as

$$V_t = V_0\left(1 + \frac{t}{273{\cdot}15}\right) \quad .. \quad (2.10)$$

$$P_t = P_0\left(1 + \frac{t}{273{\cdot}15}\right) \quad .. \quad (2.10a)$$

These relations are known as Charles' Law at constant pressure and Charles' Law at constant volume. The latter is also called the Pressure law.

Charles' law. The pressure remaining constant, the volume of a given mass of any gas increases or decreases by the constant fraction $\frac{1}{273 \cdot 15}$ of its volume at 0°C for each degree centigrade rise or fall in temperature.

Pressure law. The volume remaining constant, the pressure of a given mass of any gas increases or decreases by the constant fraction $\frac{1}{273 \cdot 15}$ of its pressure at 0°C for each degree centigrade rise or fall in temperature.

2.10. Determination of γ_p and γ_v

The two cubic expansivities of a gas are determined by the constant pressure air thermometer and the constant volume air thermometer.

(*a*) *Determination of γ_p by the Constant Pressure Air Thermometer or Regnault's Apparatus.* The constant pressure air thermometer or the Regnault's apparatus consists of a glass bulb (cylindrical) *B* graduated in c.c attached to one limb of a U-tube. The other limb of the tube is longer and it ends in a funnel. At the bend of the U-tube a third short tube is attached which is provided with a stop cock. The U-tube contains strong sulphuric acid (conc. H_2SO_4). The whole apparatus is immersed in water contained in an outer glass jacket. A copper tube is introduced through the bottom rubber stopper of the jacket. The bath is heated by passing steam through this copper tube. A thermometer introduced into the bath from top records the temperature of the bath. First of all pieces of ice are added in sufficient quantity to

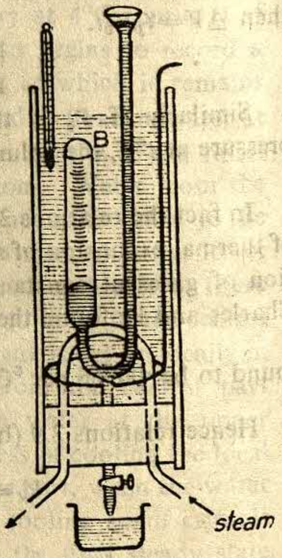


Fig. 2.12

the bath and it is stirred thoroughly well to make the temperature uniform from top to bottom. Now sulphuric acid is poured into the funnel or run out by opening the stop-cock until its levels are on the same horizontal line. The volume of the air in the bulb is then read out from the graduations on the body of the bulb. Steam is passed through the copper tube and the water is stirred constantly to maintain uniformity of temperature. The temperature is kept constant by regulating the supply of steam through the tube and the levels of the acid are brought to the same horizontal level by carefully draining out acid through the stop-cock or adding more acid into the funnel. The volume of the air is again read out as before The heating is continued and the volumes are noted at different temperatures.

Let V_1 and V_2 be the volumes at t_1°C and t_2°C respectively.

Then $V_1=V_0(1+\gamma_p t_1)$ and $V_2=V_0(1+\gamma_p t_2)$

$$\therefore \quad \frac{V_1}{V_2}=\frac{1+\gamma_p t_1}{1+\gamma_p t_2} \quad \text{or} \quad \gamma_p=\left(\frac{V_2-V_1}{V_1 t_2-V_2 t_1}\right) \qquad .. \quad (2.11)$$

(*b*) *Determination of γ_v by the Constant Volume Air Thermometer or the Joly's Apparatus.* For figure and description of this apparatus refer to article 1.4. The procedure is the same as for the constant pressure thermometer. The working formula is

$$\gamma_v=\frac{P_2-P_1}{P_1 t_2-P_2 t_1} \qquad .. \quad (2.11a)$$

2.11. Equation of State of a Perfect Gas : Universal Gas Constant : Boltzmann Constant

The state of a gas is determined by three variables, namely, the temperature (T), the pressure (p) and the volume (V). Any relation connecting p, V and T is called an *equation of state.* The thermal expansion of a gas is governed by the relation (Eq. 2.29 b)

$$V_t=V_0(1+\gamma_p t).$$

If we put $t=\left(-\frac{1}{\gamma_p}\right)$°C, then $V_t=0$. Thus according to this relation $\left(-\frac{1}{\gamma_p}\right)$°C is the lowest possible temperature that we can imagine on the property of a gas. Assigning zero numerical value to this temperature a scale is designed. This scale is called the *perfect gas scale.*

$$\therefore \quad \left(-\frac{1}{\gamma_p}\right)^\circ C = 0^\circ \text{ absolute } (^\circ K)$$

or $$0^\circ C = \left(\frac{1}{\gamma_p}\right)^\circ K = T_0 \,^\circ K. \text{ (say)}$$

or $$t^\circ C = \left(\frac{1}{\gamma_p} + t\right)^\circ K = T \,^\circ K \text{ (say)}$$

$$\therefore \quad V_t = V_0(1+\gamma_p t) = V_0 \gamma_p \left(t + \frac{1}{\gamma_p}\right)$$

$$= V_0 \frac{\left(t + \frac{1}{\gamma_p}\right)}{\frac{1}{\gamma_p}} = \frac{V_0 T}{T_0}.$$

Thus $V \propto T$ where T is the temperature on the absolute scale

or $V \propto T$ when the pressure is constant. .. (i)

At a constant temperature the expansion or contraction of a gas is governed by Boyle's law, which states that at a constant temperature the volume of a given mass of a gas is inversely proportional to the pressure of the gas.

That is $V \propto \frac{1}{p}$ when T is constant. .. (ii)

Combining (i) and (ii) we have,

$$V \propto \frac{T}{p} \text{ when both } p \text{ and } T \text{ vary}$$

or $$V = \text{a constant} \times \frac{T}{p}$$

or $$\frac{pV}{T} = \text{a constant.}$$

This constant depends on the quantity of the gas. If one 'mole of a gas' is considered then the constant will be a **universal constant** called the *Universal Gas Constant*. This is denoted by R. Thus if V is the volume of a 'mole' of any gas then

$$pV = RT. \qquad .. \quad (2.12)$$

This equation is called the equation of state of a perfect gas. A gas obeying Boyle's law and Charles' law is called a perfect gas.

Gases in real existence such as hydrogen, oxygen, carbon dioxide, nitrogen etc. do not obey Boyle's law and Charles' law completely. These are called real gases. The behaviour of perfect or ideal gases is explained thoroughly well by the kinetic theory of gases (Chapter 10). The standard value of R is 8·3 joule per mole per kelvin ($\text{J mol}^{-1}\text{K}^{-1}$).

The number of molecules in a 'mole' of a gas is called Avogadro's number denoted usually by N.

N (Avogadro's number) $= 6{\cdot}022 \times 10^{23}$ per gramme-mole (gm mol^{-1})

$= 6{\cdot}022 \times 10^{26}\ \text{kgmol}^{-1}$.

The universal gas constant per molecule is called Boltzmann constant usually denoted by k.

$$\therefore \qquad k = \frac{R}{N} \qquad \qquad (2.13)$$

$$= \frac{8{\cdot}3}{6{\cdot}022 \times 10^{23}} = 1{\cdot}38 \times 10^{-23} \text{ joule per kelvin } (\text{JK}^{-1}).$$

The number of molecules per unit volume of a gas is called Loschmidt's number usually denoted by n. It is not a universal constant. It varies from gas to gas.

Thus $\quad n = \dfrac{N}{V}$ where N = Avogadro's number. and V = molar volume.

Since $\quad pV = RT$,

$$\therefore \qquad p = \frac{R}{V}T = \frac{Nk}{V}T \quad (\because \ R = Nk)$$

$$= \left(\frac{N}{V}\right)kT$$

$$= nkT \quad \left(\because \ n = \frac{N}{V}\right)$$

or $$p = nkT. \qquad \qquad (2.14)$$

This is known as Clapeyron's equation. This equation is helpful to know the number of molecules of a gas contained is a vessel at a given temperature and pressure.

Examples :

1. *Show that the superficial expansivity of a solid is twice its linear expansivity and the cubic expansivity is three times the linear expansivity.*

Sol. Consider a square of side l. Then $S = l^2$.

Differentiating we have, $\triangle S = 2l\triangle l$.

$$\therefore \quad \beta = \frac{1}{S} \cdot \frac{\triangle S}{\triangle T} = \frac{1}{l^2} \cdot \frac{2l\triangle l}{\triangle T} = 2\frac{1}{l} \cdot \frac{\triangle l}{\triangle T} = 2\alpha. \textbf{ Proved.}$$

Consider a cube of side l on each edge. Then $V = l^3$.

Differentiating $\qquad \triangle V = 3l^2\triangle l$.

$$\therefore \quad \gamma = \frac{1}{V} \cdot \frac{\triangle V}{\triangle T} = \frac{1}{l^3} \cdot \frac{3l^2\triangle l}{\triangle T} = 3\frac{1}{l} \cdot \frac{\triangle l}{\triangle T} = 3\alpha. \textbf{ Proved.}$$

2. *There are 3 iron rods, each 1 metre long on average and 2 brass rods in a Grid-iron pendulum. What is the length of each brass rod?* (*linear of expansivity of iron* $= 12 \times 10^{-6} K^{-1}$ *and that of brass* $= 19 \times 10^{-6} K^{-1}$).

Sol. Here the downward expansion of 2 iron rods is compensated by the upward expansion of 1 rod.

The effective length of iron rods $= 2 \times 1 = 2$ m.

Let l be the effective length of each brass rod.

Then $\qquad (\triangle l)_{iron} = 2 \times 12 \times 10^{-6} \times \triangle T$

and $\qquad (\triangle l)_{brass} = l \times 19 \times 10^{-6} \times \triangle T$

$\therefore \qquad 2 \times 12 \times 10^{-6} \triangle T = l \times 19 \times 10^{-6} \times \triangle T$

or $\qquad l = \frac{24}{19} = 1{\cdot}263$ m. **Ans.**

3. *A brass scale reads correctly in mm at* 0°C. *If it is used to measure a length at* 33°C, *the reading on the scale is* 40·5 *cm. What is the correct length at* 33°C?

(*linear expansivity of brass* $= 18 \times 10^{-6} K^{-1}$).

Sol. 1 mm of scale at 0°C = 1 mm

1 mm of scale at 33°C $= (1 + 18 \times 10^{-6} \times 33)$ mm.

$\therefore$ 40·5cm of scale at 33°C $= 40{\cdot}5(1 + 18 \times 33 \times 10^{-6})$ cm

$$= (40{\cdot}5 + 40{\cdot}5 \times 18 \times 33 \times 10^{-6}) \text{ cm}$$
$$= 40{\cdot}5 + {\cdot}025$$
$$= 40{\cdot}525 \text{ cm.} \quad \textbf{Ans.}$$

4. *The loss of weight of a sinker in a liquid at* 0°C *is* W_0. *Show that the loss* W *at* t°C *is given by*

$$W = W_0[1 + (\alpha - \beta)t]$$

where α *and* β *are the cubic expansivity of the sinker and the liquid respectively.*

Sol. W_0 = loss in weight of the sinker at 0°C

= weight of liquid displaced

$= V_0\rho_0 g$ where V_0 = volume of sinker at 0°C

and $\quad \rho_0$ = density of the liquid at 0°C.

$W = V_t\rho_t g$.

$\therefore \quad \frac{W}{W_0}=\frac{V_t\rho_t}{V_0\rho_0}$. Now $V_t=V_0(1+\alpha t)$ and $\rho_0=\rho_t(1+\beta t)$

$$\therefore \quad \frac{W}{W_0}=\frac{V_0(1+\alpha t)\rho_t}{V_0\rho_t(1+\beta t)}=(1+\alpha t)(1+\beta t)^{-1}$$

Expanding and neglecting small terms

$$=(1+\alpha t)(1-\beta t)$$

$$=(1+\alpha t-\beta t)$$

$\therefore \quad W=W_0[1+(\alpha-\beta)t]$. **Proved.**

5. *A glass weight thermometer has a mass of* 6·34 *gm when empty, and* 153·81 *gm when filled with mercury at* 0°C. *If* 2·08 *gm are expelled when it is heated to* 100°C, *find the apparent expansivity of the liquid.*

Sol. Mass remaining $=(153{\cdot}81-6{\cdot}34)-2{\cdot}08$

$$=145{\cdot}38.$$

$$\gamma_a=\frac{\text{mass expelled}}{\text{mass remaining}\times\text{rise in temperature}}=\frac{2{\cdot}08}{145{\cdot}38\times 100}$$

or $\gamma_a={\cdot}000143\ \text{K}^{-1}$. **Ans.**

6. *A U-tube containing a liquid has two limbs maintained at* 20°C *and* 100°C *respectively. On reachirg a steady state the lengths of the liquid columns are* 98 *cm and* 102 *cm. What is the cubic expansivity of the liquid?*

Sol. We have by law of fluid mechanics

$$102\times\rho\times g=98\times\rho'\times g.$$

or $$102\times\frac{\rho_0}{1+\gamma.100}=98\times\frac{\rho_0}{1+\gamma.20}$$

or $$102(1+\gamma.20)=98(1+\gamma.100)$$

or $$\gamma=\frac{102-98}{98.100-102.20}=\frac{4}{9800-2040}={\cdot}00052\ \text{K}^{-1}.\ \textbf{Ans.}$$

7. *Show that for a perfect gas* $\gamma_p=\gamma_v$.

Sol. Let us consider a certain mass of a perfect gas at 0°C when its pressure and volume are respectively P_0 and V_0. Let us heat it through t°C keeping its pressure constant at P_0. Let the volume become V.

Then $$V=V_0(1+\gamma_p t). \quad \text{.. (i)}$$

Next keeping the volume fixed at V_0, let us heat the same gas from 0°C to t°C. Let its pressure become P. Then

$$P=P_0(1+\gamma_v t). \quad \text{.. (ii)}$$

Note that P_0 and V are the pressure and volume of the gas at t°C and P and V_0 are the pressure and volume of the same gas at the same temperature.

Therefore by Boyle's law, $P_0V=PV_0$

or $$P_0V_0(1+\gamma_p t)=P_0(1+\gamma_v t)V_0$$

or $$\gamma_p=\gamma_v.$$ **Proved.**

QUESTIONS

(A)

1. When a bimetallic strip of copper and iron is cooled (a) it will bend towards copper, (b) it will bend towards iron, (c) it will not bend at all, (d) it will vibrate.

2. The unit of α is (a) mK^{-1}, (b) $m^{-1}K$, (c) K^{-1}, (d) K.

3. If α is the linear expansivity and γ is the cubic expansivity of a solid then (a) $\alpha=\gamma$, (b) $\alpha=2\gamma$, (c) $\gamma=3\alpha$, (d) $\gamma=2\alpha$.

4. If β is the superficial expansivity and γ is the cubic expansivity of a solid then (a) $\beta=\gamma$, (b) $2\beta=3\gamma$, (c) $3\beta=2\gamma$, (d $\beta=2\gamma$.

5. If α is the linear expansivity and β s the superficial expansivity of a solid then (a) $\alpha=\beta$, (b) $\alpha=2\beta$, (c) $\beta=2\alpha$, (d) $\beta=3$.

6. If α, β and γ are the cubic expansivities of a substance in the three states, solid, liquid and gaseous, respectively then
(a) $\alpha=\beta=\gamma$, (b) $\alpha>\beta>\gamma$, (c) $\alpha<\beta<\gamma$, (d) $\beta>\alpha$ and $\beta>\gamma$.

7. A constant volume gas thermometer works on (a) Archimedies' principle, (b) Boyle's law, (c) Pascal's Law, (d) Charles' Law. **(I. I. T. 1980)**

(Ans. 1. a, 2. c, 3. c, 4. c, 5. c, 6. c, 7. d.)

(B)

1. Explain the working f a bimetallic thermometer. Mention some other uses of bimetallic strips.

2. Give the theory of compensated grid-iron pendulum.

3. Es ablish the relation $\gamma_{real}=\gamma_{apparent}+\gamma_{vessel}$.

4. Establish the relation $\rho_0=\rho_t(1+\gamma t)$.

5. Show that for a real gas $\gamma_p=\gamma_v$

6. Describe Hope's experiment and indicate its significance.

(C)

1. Define vagadro's number, Loschmidt's number and Boltzmann constant. Establish Clayperon's equation $p=nkT$ from Charles' law and Boyle's law.

2. Describe a weight thermometer and deduce the expression you would use to determine the coefficient of real expansion of a liquid with it.

3. How can you measure the real expansion of a liquid when the coefficient of expansion of containing vessel is not known ? **(Mag. 1973 S; Mithila '79 A)**

4. State and explain Charles' law. Describe how it is verified experimentally.

(D)

1. If the coefficient of linear expansion of glass and the coefficient of real expansion of mercury $8\times10^{-6}K^{-1}$ and $1{\cdot}8\times10^{-4}K^{-1}$ respectively, what fraction of the whole volume of a glass flask should be filled with mercury in order that the volume of the empty part of the flask may remain constant at all temperatures ?

(Ans. 2/15)

2. A flask which contains $\cdot25\times10^{-3}$ m³ of air at atmospheric pressure is heated at 100°C and then corked up. It is afterwards immersed mouth downwards in a vessel of water at 10°C and the cork removed. What volume of water will enter the flask ? (Ans. $\cdot06\times10^{-3}$ m³)

3. A weight thermometer contains ·5 kg of mercury at 0°C. If the temperature is raised to 100°C, how much mercury will be expelled ? Cubical expansivity of mercury$=18{\cdot}2\times10^{-5}$ K^{-1} and that of glass$=2{\cdot}7\times10^{-5}$ K^{-1}, (Ans. $7{\cdot}63\times10^{-3}$ kg.)

4. The density of air at STP is 1·293 kgm⁻³ Calculate its density when its temperature is 16·6°C and pressure 75 cm of mercury. (Ans. 1·203 kgm⁻³)

5. A metallic bob weighs 50 gm in air. If it is immersed in a liquid at a temperature of 25°C it weighs 45 gm. When the temperature of the liquid is raised to 100°C, it weighs 45·1 gm. Calculate the coefficient of cubical expansion of the liquid assuming the linear expansivity of the metal to be 12×10^{-6} K^{-1}. (I. I. T. 1973)

(Ans. 3×10^{-4} K^{-1})

6. The brass scale of a barometer gives correct reading at 0°C. The coefficient of linear expansivity of brass is 20×10^{-6} K^{-1}. The barometer reads 75 cm at 27°C. What is the atmospheric pressure at 27°C ? (I. I. T. 1977)

(Ans. 75·04 cm.)

(E)

1. For a perfect gas the pressure coefficient and volume coefficient are............ (equal, unequal)

2. A brass disc fits snugly in a hole in a steel plate. Should you heat or cool the system to loosen the disc from the hole ? (I. I. T. 1973)

3. In a constant volume gas thermometer............(pressures, volumes) are measured at different temperatures.

4. Does the external volume of a solid sphere when its temperature is raised depend on whether the sphere has cavities inside ?

5. Does the diameter of a hole punched in a flat plate increase when its temperature is raised ?

6. A long cylindrical vessel having a linear expansivity α is filled with a liquid up to a certain level. On heating it is found that the level of the liquid in the cylinder remains the same. What is the volume coefficient of expansion of the liquid ? (I. I. T 1976)

(Ans. 1. Equal, 2. cool the system, 3. pressures, 4. No, 5. increase because $\Delta l/l$ is the same for all lines in the solid. 6. 3α.)

CHAPTER 3

CALORIMETRY

3.1. Heat and Its Unit

When two bodies at different temperatures are placed in contact some energy flows from the hot body to the cold body. **This energy in transit due to a difference in temperature is called** *Heat.* All material bodies possess energy in the mechanical form. A solid possesses energy due to rapid vibrations of its atoms at the lattice sites, a fluid possesses energy due to the random motion of its molecules or atoms. The higher the temperature of a solid body, the more vigorous are the vibrations of its atoms or the more chaotic is the motion of the atoms or molecules of fluids. This energy of a body is called its *internal energy.* When two bodies are in contact this internal energy flows from the one at higher temperature to the one at lower temperature. This energy in flow is called heat. The moment it ceases to flow it cannot be called heat. Internal energy differs from heat in the same way as rain water differs from 'water in a lake'. Water in the form of droplets in motion is called 'rain'. After it (rain water) mixes with lake water it loses its identity as 'rain water' and becomes 'lake water'. Exactly in the same way heat differs from internal energy. · Heat is analogous to 'rain' and 'water in lake' is the analog of internal energy. Basically they are the same thing, namely, energy. When internal energy is in flow it is called 'Heat'. The internal energy loses its identity as heat when it ceases to flow. The unit of heat is **joule** (J) in SI. Other units are also used, notably the **kilo calorie** or **calorie.** Where quantities of heat are expressed in calories, it is recommended that the conversion factor to convert to joule be stated forthwith. When we take one kilograme of water at 14·5°C (a cold body) and place it over a hot plate (a hot body) energy (internal energy of the hot plate) will flow to the water due to the difference in temperature between the water and the hot plate. In this case we say that 'heat is flowing from the hot plate to the water. Due to the addition of heat, the molecules of the water will be more agitated and so their temperature will rise. Let it go up by 1°C. Then the amount of

energy that flows to the water till its temperature rises by 1°C from 14·5°C to 15·5°C is called a *kilo calorie.*

DEFINITION OF KILO CALORIE. *The amount of energy that will flow from a hot body to a kilogram of water due to the difference of its temperature with that of the hot body till its temperature rises by 1°C from 14·5°C to 15·5°C is called a kilocalorie.* One-thousandth of a kilocalorie is a calorie. A calorie is the amount of energy that will flow from a hot body to a gram of water due to the difference of its temperature with that of the hot body till its temperature rises by 1°C from 14·5°C to 15·5°C. One calorie is equal to 4·2 joules.

3.2. Specific Heat Capacity

Substances differ from one another in the quantity of heat needed to produce a given rise of temperature in a given mass. If $\triangle Q$ is the heat supplied to a body when its temperature rises by $\triangle T$, then the 'heat capacity' or 'thermal capacity' of the body is defined as

$$C \text{ (heat capacity)} = \frac{\triangle Q}{\triangle T} \quad \ldots \quad 3.1$$

DEFINITION. When $\triangle T = 1°\text{K}$, $C = \triangle Q$. Thus the *heat capacity of a body is the heat required to raise its temperature by* 1°K (*or* 1°C). Units, JK^{-1} or cal K^{-1}.

The heat capacity per unit mass of a body is called *specific heat capacity*. It is characteristic of the material of which the body is composed. If $\triangle Q$ is the heat supplied to a body of mass m when its temperature rises by $\triangle T$, then the specific heat capacity c or s is given by

$$\text{or} \quad s \text{ or } c \text{ (specific heat capacity)} = \frac{\triangle Q}{m\triangle T} \quad \ldots \quad 3.2$$

$$\text{or} \quad \triangle Q = mc\triangle T \quad \ldots \quad 3.2\text{a}$$

DEFINITION. When $\triangle T = 1°\text{K}$ and $m = 1$ kg. $c = \triangle Q$. Thus *specific heat capacity is the heat required to raise the temperature of one kilogram of the substance through* 1°K (or 1°C).

Units, kcal $\text{kg}^{-1}\ \text{K}^{-1}$ or J $\text{kg}^{-1}\ \text{K}^{-1}$.

Molar Heat Capacity. This is the amount of heat required to raise the temperature of 1 mole of a substance through 1°K. Unit,

J $\text{mol}^{-1}\ \text{K}^{-1}$ or cal $\text{mol}^{-1}\ \text{K}^{-1}$.

3.3. Water equivalent

Water is the most common liquid used in calorimetry. This is why the kilocalorie is defined in terms of heat required to raise the temperature of 1 kg of water through 1°K. It is always useful in calorimetry to know how much water will have the same thermal capacity as a given body. This is called its water equivalent. Thus the *water equivalent of a body is the mass of the water having the same thermal capacity as the given body.* Unit, kilogram (kg).

Water equivalent $= C$ kg where C is the thermal capacity of the body.

3.4. Principle of Calorimetry

The principle of calorimetry is the **'principle of conservation of energy'**. When two bodies at different temperatures are put in contact, heat flows from one body to the other. It follows at once from the conservation principle that one will gain as much energy as the other will lose. If the two bodies are considered as a closed system, i.e., a system to which heat from the environment can neither flow in nor out, the net change of heat must be zero. Hence the principle of calorimetry is

either Heat lost = Heat gained .. 3.3

or Net change of heat = 0 .. 3.3a

3.5. Determination of Specific Heat Capacities of Solids

(a) *Method of mixture.* In this method a solid body of known mass is heated to a steady temperature in a steam-heater and then it is quickly mixed with a known mass of water contained in a copper vessel of known water equivalent. From the knowledge of the temperature of the mixture the specific heat capacity of the solid is calculated by applying the principle of calorimetry, that is, Heat lost = Heat gained.

For quick mixing of the hot solid with cold water an apparatus is used which was designed by Regnault and is known as Regnault's apparatus The complete outfit of the apparatus is shown in the Fig. 3.1. It consists of a steam heater H seated on a raised platform exactly above a hole cut in the platform and a calorimeter box containing a copper calorimeter C screened from the heater by a wooden

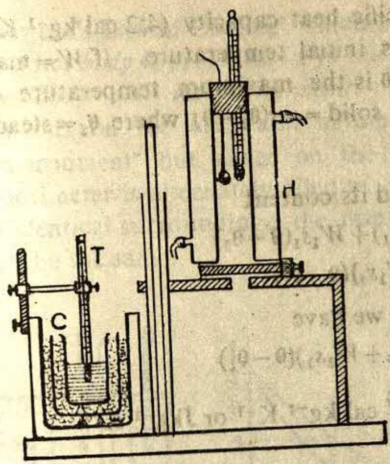


Fig. 3.1

plank. The calorimeter (C) is simply a copper cylindrical vessel placed inside another copper vessel. The inner surface of the latter is made shining to reflect heat lost by radiation back to the calorimeter The space between the two vessels is loosely packed with felt or cotton-wool. This reduces heat loss due to convection. A thermometer clamped to the stand provided with the calorimeter box, records the temperature of the calorimeter and its contents.

The steam heater H is a double-walled metallic vessel. The inner chamber is closed by a cork at the upper end and by a sliding shutter at the lower end. It is heated by circulating steam from a boiler through the annular space between the outer and inner walls of the vessel. The solid to be heated is suspended inside the inner chamber by means of a cotton thread and the temperature of the solid is recorded by a thermometer placed close to the solid.

In the actual procedure, the solid is weighed and suspended inside the steam-heater as explained. The calorimeter is cleaned, dried and weighed. Then some water, sufficient to immerse the solid, is taken and weighed again. The difference gives the mass of water taken. The temperature of the calorimeter is recorded by a sensitive thermometer. By this time the temperature of the solid becomes steady as is shown by the constancy of the reading of the thermometer, placed close to the solid. The partition is lifted and the calorimeter box is pushed below the hole of the platform. After removing the sliding shutrer the suspending thread is cut when the solid quickly drops into the liquid in the calorimeter. The calorimeter box is then quickly moved back to its original position and the partition is lowered. The recording thermometer which was removed to receive the solid is now re-introduced and the maximum temperature of the mixture is noted after stirring the liquid well.

CALCULATION. Let m and s be the mass and specific heat capacity of the solid respectively. Suppose that the water in the

calorimeter is W_1 and s_1, its specific heat capacity (4·2 cal kg^{-1} K^{-1} or 4200 joule kg^{-1} K^{-1}) and θ_1, its initial temperature. If W = mass of calorimeter and its stirrer and θ is the maximum temperature of the mixture then heat lost by solid $= ms(\theta_2 - \theta)$ where θ_2 = steady temperature of the hot solid.

Heat gained by calorimeter and its content

$$= Ws_{cu}(\theta - \theta_1) + W_1 s_1(\theta - \theta_1)$$
$$= (Ws_{cu} + W_1 s_1)(\theta - \theta_1)$$

By the principle of calorimetry we have

$$ms(\theta_2 - \theta) = (Ws_{cu} + W_1 s_1)(\theta - \theta_1)$$

or $$s = \frac{(Ws_{cu} + W_1 s_1)(\theta - \theta_1)}{m(\theta_2 - \theta)} \text{ cal kg}^{-1}\text{ K}^{-1} \text{ or Jkg}^{-1}\text{ K}^{-1}$$

according as the values of s_{cu} or s_1 are taken in heat unit or 'work unit'.

PRECAUTIONS. (i) Some heat is lost by radiation. This may be corrected to some extent by taking cold water as much below the room temperature as the final temperature would be above the temperature of the room. So, the loss of heat by the calorimeter during the second half of the experiment is compensated for, by an equal gain in the first half. This method is known as Rumford's compensation methods.

(ii) The final temperature of the mixture must be recorded by a sensitive thermometer.

(iii) The thermometer used in the steam heater should be corrected for the boiling point.

(b) By Bunsen's ice-calorimeter : See Art. 3.8 (b)

(c) By Joly's steam calorimeter : See Art. 3.9 (b)

3.6. Determination of Specific Heat Capacities of Liquids

(a) *By the Method of Mixtures.* The method is the same as the one described above for solids. Here a solid which is not chemically acted on by the liquid and whose specific heat capacity is known is to be taken. The liquid is taken in the calorimeter instead of water. Using the same notations we have,

$$s_1 = \left\{ \frac{ms(\theta_2 - \theta)}{W_1(\theta - \theta_1)} - \frac{Ws_{cu}}{W_1} \right\} J \text{ kg}^{-1}\text{ K}^{-1}.$$

(b) *By the Method of Cooling.* This method is based on

'Newton's Law of Cooling' which states that *the rate of loss of heat by a hot body is proportional to the difference of temperature between the body and its environment.* The constant of proportionality depends on the area and nature of the surface exposed to the 'environment' but never on the 'contents of the body'. So if a calorimeter is alternately filled with two liquids and allowed to cool in identical surroundings, the rate of loss of heat by the calorimeter will be the same.

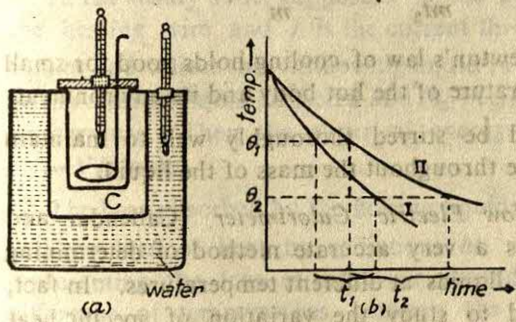


Fig. 3.2

EXPERIMENT. A small calorimeter is weighed with its stirrer and then it is half filled with the experimental liquid prewarmed to a temperature, say, 70°C, in a beaker. Quickly transfer the calorimeter to a water jacket. Stir the liquid well and record temperature at the interval of ½ minute from 60° up to, say, 35°C. Weigh the calorimeter and its contents. The difference of the two weights gives the mass of the liquid taken. Repeat the process by taking an equal volume of water. Draw cooling curves for water and liquid and from graph estimate the times taken by water and liquid respectively to cool through the same range of, say, 10°C.

CALCULATION. Suppose, mass of liquid taken $= m$

mass of water taken $= m'$

mass of calorimeter and stirrer $= W$

Time taken by the liquid to cool from θ_1 to $\theta_2 = t_1$ sec.

Time taken by the water to cool from θ_1 to $\theta_2 = t_2$ sec.

Rate of loss of heat by calorimeter containing liquid

$$= \frac{(Ws_{cu} + ms_l)(\theta_1 - \theta_2)}{t_1}$$

where s_{cu} and s_l are the specific heat capacities of copper and liquid respectively.

Rate of loss of heat by the same calorimeter containing water

$$= \frac{(Ws_{cu} + m's_w)(\theta_1 - \theta_2)}{t_2}$$

By Newton's law of cooling, the two rates are equal;

$$\therefore \quad \frac{(Ws_{cu}+ms_l)(\theta_1-\theta_2)}{t_1} = \frac{(Ws_{cu}+m's_w)(\theta_1-\theta_2)}{t_2}$$

or
$$s_l = \frac{(Ws_{cu}+m's_w)t_1}{mt_2} - \frac{Ws_{cu}}{m}.$$

PRECAUTIONS. (i) Newton's law of cooling holds good for small difference between temperature of the hot body and its environment.

(ii) The liquid should be stirred thoroughly well to maintain uniformity of temperature throughout the mass of the liquid.

(c) *By Continuous Flow Electric Calorimeter* (Callendar and Barnes' method). This is a very accurate method of determining specific heat capacities of liquids at different temperatures. In fact, this is the only method to study the variation of specific heat capacities of liquids with temperatures.

A steady current of the experimental liquid flowing through a narrow glass-tube is heated by an electric current flowing through the central conductor of platinum. The steady difference of temperature between the in-flowing and out-flowing water is measured by a pair of platinum resistance thermometers. The thermometer coils are surrounded intimately by thick copper tubes attached to the platinum wire. Copper, by virtue of its good thermal conductivity, keeps the thermometer coils exactly at the temperature of the adjacent water. The leads L, L and L', L' are provided with the copper tubes, the former for introducing the heating current and the

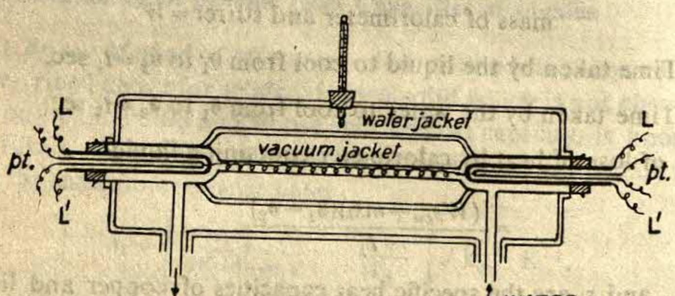


Fig. 3.3

latter for measuring the potential difference across the central conductor by means of an accurately calibrated potentiometer. The current through the conductor is also measured by the same potentiometer by measuring the potentia' difference across a standard resistance included in the circuit. In order to diminish the external loss of heat the flow tube is surrounded by a vacuum jacket, which, in its turn is surrounded by a constant temperature water-jacket.

In the steady state, suppose, V is the potential difference across the heating wire and I is the current through it. Take a weighed beaker and collect the outflowing liquid in it for t seconds. Let m be the mass of liquid collected and $(\theta_2-\theta_1)=\triangle\theta$ be the difference in temperature. Then the electrical energy supplied to the heating coil $=VIt$ joules.

The heat absorbed by the flowing liquid $=ms_l\triangle\theta$,

and the loss of heat by radiation $=ht$ where h is a constant representing the loss of heat per second by radiation for a given difference of temperatures between the flowing liquid and its environment.

$$\therefore \quad VIt=ms_l\triangle\theta+ht.$$

To eliminate h, two sets of observations are taken by suitably adjusting the electric current and the rate of flow of liquid so as to secure the same difference in temperature. Thus for the two rates of flow we have

$$V_1I_1t=m_1s_l\triangle\theta+ht$$

$$V_2I_2t=m_2s_l\triangle\theta+ht$$

$$\therefore \quad (V_1I_1-V_2I_2)t=(m_1-m_2)s_l\triangle\theta.$$

or $$s_l=\frac{(V_1I_1-V_2I_2)t}{(m_1-m_2)\triangle\theta}\ \mathrm{J\ kg^{-1}\ K^{-1}}.$$

This is the specific heat capacity at the mean temperature $\theta=\frac{\theta_1+\theta_2}{2}$. By changing the mean temperatures, the specific heat capacities at different temperatures can be determined. The great advantage of this method is that no correction is necessary for the thermal capacity of the calorimeter. Since all conditions are steady the observations can be taken with the highest degree of accuracy. As there is ample time there is no question of thermometeric lag. This is the only method available for studying the variation of

specific heat capacity with temperature. Callendar and Barnes used this method to study the variation of specific heat capacity of water with temperature. They found water to have a minimum specific heat capacity near about 34°C.

3.7. Latent Heat and Specific Latent Heat

During a change of state a substance absorbs or releases heat **without change of temperature.** The heat so absorbed or given out by a substance during change of state without change of temperature is called the *Latent heat of the substance.* If $\triangle m$ is the mass of the substance and $\triangle Q$ is the amount of heat absorbed or released by it, then the latent heat of the substance $= \triangle Q$ joule or calorie according as the energy is taken in 'work unit' or 'heat unit.'

The latent heat per unit mass of the substance is called the *specific latent heat.*

$$\therefore \quad \text{Specific latent heat } (L) = \frac{\triangle Q}{\triangle m} \text{ J kg}^{-1} \text{ or cal kg}^{-1} \quad .. \quad 3.4$$

DEFINITION. When $\triangle m = 1$, $L = \triangle Q$. Thus the *specific latent heat is the amount of heat absorbed or released by one kilogram of a substance during change of state without change of temperature.*

There are three types of specific latent heats corresponding to three types of changes of state from solid to liquid and vice versa, liquid to vapour and vice versa and solid to vapour and vice versa called specific *latent heat of fusion*, specific *latent heat of vaporisation* and specific *atent heat of sublimation* respectively. The specific latent heat of fusion of ice is **336 × 10³ J kg⁻¹** and the specific latent heat of vaporisation of water at 100°C is **2250 × 10³ J kg⁻¹. The heat is latent in the sense that a thermometer fails to indicate its addition or subtraction. In contrast to this we have 'sensible heat' which is indicated by a thermometer.**

3.8. Determination of the Specific Latent Heat of Fusion of Ice

(a) *Method of mixtures.* A copper calorimeter with a netted stirrer is weighed. It is then half-filled with water and again weighed Put the calorimeter in the calorimeter box and note its temperature by a sensitive thermometer. Take a piece of ice and dry it by means of blotting paper and drop it into the calorimeter after lifting the netted stirrer. Press the ice well below the surface of

water by means of the stirrer and gently stir the water. Note the lowest temperature attained by the mixture. Weigh the calorimeter and its contents again.

Let m = mass of ice taken, W = weight of calorimeter and stirrer and W' = mass of water taken. Further let θ_1 be the initial temperature and θ_2 final lowest temperature of the mixture.

Then the heat lost by the calorimeter and its contents

$$=(Ws_{cu}+W's_w)(\theta_1-\theta_2)$$

where s_{cu} and s_w are specific heat capacities of copper and water respectively.

The heat gained by the ice $= mL + ms_w\theta_2$

By the principle of calorimetry

$$(Ws_{cu}+W's_w)(\theta_1-\theta_2)=mL+ms_w\theta_2$$

$$\text{or} \quad L=\left[\frac{(Ws_{cu}+W's_w)(\theta_1-\theta_2)}{m}-s_w\theta_2\right] \text{J kg}^{-1} \text{ or cal kg}^{-1}$$

according as s_{cu} and s_w are taken in 'work unit' or heat unit.

PRECAUTIONS. (i) Before dropping keep the ice piece on the blotting paper and never touch it with the fingers.

(ii) Apply Rumford's Compensation Method to minimise the loss of heat by radiation.

(iii) Do not add too much ice to bring down the final temperature below the dew-point corresponding to the present condition of the atmosphere. Do not add too little ice as well. Add suitable mass of ice to bring the temperature down by about 4°C to 5°C.

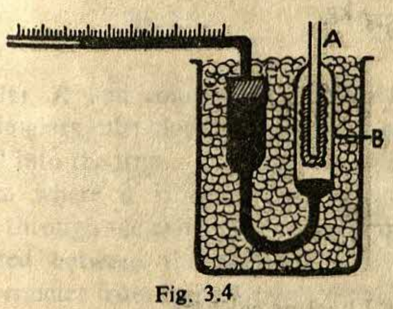


Fig. 3.4

(b) *Bunsen's ice calorimeter*. This is a very sensitive instrument which gives very accurate values of the specific latent heat of ice and the specific heat capacity of solids available in a very small quantity. The basis of working of this apparatus is the fact that when one kilogram of ice melts there is a contraction in volume by 9×10^{-5} cubic metre.

It consists of a thin walled test tube A sealed into a wider tube which is provided with the bent glass stem and is nearly fined with boiled air free water and the remaining space and the stem are filled with mercury. The stem ends in an iron-collar containing mercury. A horizontal graduated capillary tube is pushed in the collar so that mercury stands at a certain graduation in the horizontal portion. The apparatus is kept in a box packed with pieces of ice. In conducting an experiment, a stream of alcohol precooled by a freezing mixture is passed through A until a cap of ice is formed round it. The whole instrument is then left to itself for a long time to allow it to attain **a steady state**. It is then ready for use. Read the position of the 'head' of the mercury thread in the capillary tube. A known mass of the experimental solid is then heated to a steady temperature and dropped into A. This melts some of the ice in the cap of ice surrounding A and causes a contraction in volume. Due to the hydrostatic pressure difference, the mercury contracts in the capillary tube. When the position of the 'head' of the mercury thread becomes 'steady' again its position is read on the scale.

CALCULATION. Let m be the mass of the solid heated to a steady temperature θ and l is the contraction of mercury thread in the capillary tube.

Then heat lost by solid $= ms\theta$

Let α be the area of cross-section of the capillary tube. Then contraction of mercury = contraction of ice melted

$$= l\alpha$$

$\therefore$ mass of the ice melted by the heat of the solid

$$= \frac{l\alpha}{9\times 10^{-5}} \text{ kg.}$$

$\therefore$ Heat absorbed by the ice $= \dfrac{l\alpha}{9\times 10^{-5}} L.$

By the principle of calorimetry

$$ms\theta = \frac{l\alpha L}{9\times 10^{-5}}$$

or
$$L = \frac{9ms\theta}{l\alpha} \times 10^{-5} \text{ J kg}^{-1} \text{ or cal kg}^{-1}$$

according as s is taken in 'work unit' or 'heat unit'.

or $$s = \frac{Lal}{9m\theta} \times 10^5 \text{ J kg}^{-1}\text{ K}^{-1} \text{ or cal kg}^{-1}\text{ K}^{-1}$$

according as L is taken in 'work unit' or 'heat unit'.

Thus by this apparatus L can be determined by taking a solid of known specific heat capacity or specific heat capacity if L is given.

ADVANTAGES. (i) The arrangement is very sensitive, (ii) there is no loss of heat by radiation, (iii) no thermal capacity of the calorimeter is to be considered in the calculation, (iv) since all conditions are steady, observations can be taken with the highest degree of accuracy, (v) the specific heat capacity of precious metals available in small quantity can be determined by this apparatus.

The only disadvantage of this apparatus is that it is difficult to set it.

3.9. Determination of the Specific Latent Heat of Vaporisation of Water

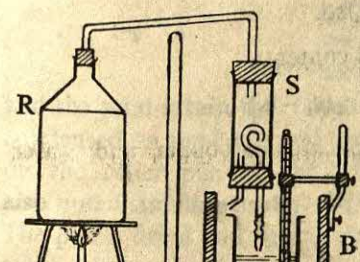


Fig. 3.5

(a) *Method of mixtures.* A clean dry copper calorimeter is taken and weighed. It is then filled with water up to about two-thirds and again weighed. The difference of the two weights gives the mass of water taken. The calorimeter is put in the calorimeter box B and the temperature of water is noted by a sensitive thermometer. Now generate steam in a boiler R and connect its delivery tube to a 'steam-trap' S. It is a wide glass tube closed by steam-tight corks. The delivery tube fits well into the trap. The wet steam settles at the bottom of the trap from where it is drained off by the side tube and dry steam passes out through the exit tube ending in a nozzle. A wooden plank is placed between the calorimeter box and the boiler to protect the calorimeter from being heated directly by the burner. The delivery tube and the trap are well lagged with cotton-wool or asbestos to keep the steam as dry as possible in them.

Bring the calorimeter under the nozzle and pass steam for some

time. Steam condenses and heat released by it, is absorbed by the calorimeter and its contents. So the temperature of the calorimeter and its contents increases progressively. When it goes up by about 10°C, remove the nozzle and stir the mixture well. Note down the **highest** temperature attained by the calorimeter and its contents. After cooling the calorimeter to room temperature weigh it again to know the mass of steam condensed. Lastly, read the Fortin's barometer to know the atmospheric pressure at the time of performing the experiment.

CALCULATION. Let W be the weight of the calorimeter and its stirrer and W' is the mass of water taken. Further let θ_1 be the initial temperature of water and θ_2 the highest temperature of the mixture; m is the mass of steam condensed.

If H cm be the barometer reading at the time of performing the experiment, the temperature of the steam is

$$\theta_3=\{100-{\cdot}37(76-H)\}°\text{C}.$$

Heat lost by steam $=mL+m(\theta_3-\theta_2)s_w$.

Heat gained by calorimeter and its contents

$$=Ws_c(\theta_2-\theta_1)+W's_w(\theta_2-\theta_1)$$

where s_c and s_w are the specific heat capacities of copper and water respectively.

By the principle of calorimetry

$$mL+m(\theta_3-\theta_2)\,s_w=Ws_c(\theta_2-\theta_1)+W's_w(\theta_2-\theta_1)$$

$$\text{or}\quad L=\left[\frac{(Ws_c+W's_w)(\theta_2-\theta_1)}{m}-(\theta_3-\theta_2)\,s_w\right]\text{J kg}^{-1}\text{ or cal kg}^{-1}$$

according as s_c and s_w are in 'work units' or 'heat units'.

PRECAUTIONS. (i) The rate of issue of steam from the exit tube must be controlled, otherwise if it is too rapid, some water may be lost by splashing.

(ii) Apply Rumfords's compensation method to reduce the loss of heat by radiation.

(iii) The final temperature of the mixture should not be allowed to increase by more than 15°C, otherwise water and heat will be lost by vaporisation.

(b) *Joly's Steam Calorimeter.* In 1886 Prof. Joly devised a very simple and accurate method of determining the specific heat capacity of a **substance or specific latent heat of vaporisation of water.**

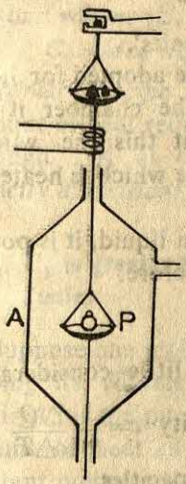


Fig. 3.6

The apparatus consists of a metal enclosure A placed beneath a sensitive balance. From one end of the beam of the balance hangs freely a wire supporting a platinum pan P inside the enclosure. The substance whose specific heat capacity is required is placed on this pan and weights are added on the other pan, till balance is attained. If the plan is to determine the specific latent heat of water a solid of known specific heat capacity is placed on the pan. The temperature of the enclosure A is noted after the solid is placed inside the chamber. Then steam from a boiler is admitted into the chamber. Steam condenses on the body and the pan till a steady state is attained, i.e., both the pan and the solid attain the temperature of steam. The mass of water condensed on the body and the pan is determined by placing weights on the other pan of the balance. The final steady temperature is also noted simultaneously. The body and the pan are then taken out. The pan is dried and again introduced into the chamber but not the solid. Steam is admitted once again into the chamber. This time steam condenses on the pan only. The mass of the steam condensed is determined as before. If m_1 is the mass of the steam condensed on the body and the pan and m_2 is the mass of the steam condensed on the pan only, then the mass of the steam condensed on the body is (m_1-m_2).

Let θ_1 be the initial temperature of the chamber and θ_2 be the final steady temperature.

Heat lost by steam $=(m_1-m_2)L$ where L is the specific latent heat of water.

Heat gained by the solid $=ws(\theta_2-\theta_1)$ where w is the mass of the solid and s is its specific heat capacity.

By the principle of calorimetry

$$(m_1-m_2)L=ws(\theta_2-\theta_1)$$

or $$L=\frac{ws(\theta_2-\theta_1)}{m_1-m_2}\ \text{J kg}^{-1}\text{ or cal kg}^{-1}$$

according as s is taken in 'work units' or 'heat units'.

or $$s=\frac{(m_1-m_2)L}{w(\theta_2-\theta_1)}\ \text{J kg}^{-1}\text{ K}^{-1}\text{ or cal kg}^{-1}\text{ K}^{-1}$$

according as L is taken in 'work units' or 'heat units'.

With slight changes, the method can also be adopted for liquid and gases. When steam is passed through the chamber it may condense on the suspension wire. To prevent this the wire is surrounded by a small spiral of platinum wire which is heated by passing electric current through it.

For determining the specific heat capacity of a liquid, it is poured on the pan and the experiment is carried out as before.

3.10. Specific Heat Capacities of Gases

Gases have two specific heat capacities. A little consideration of the functional formula for specific heat capacity $c=\frac{1}{m}\frac{\triangle Q}{\triangle T}$ will make it clear why gases have two specific heat capacities.

Imagine a quantity of gas to be suddenly compressed. The temperature of the gas will be found to rise though no heat has been added. According to the above formula the specific heat capacity of the gas is zero. Again, let the same gas be compressed slowly. Heat will flow out to the environment but there will be no change in temperature. In this case the specific heat capacity becomes infinite. Thus we see the original definition gives an infinite range of values for the specific heat capacities of gases. Hence external conditions are very important in considering the specific heat capacities of gases. The two external conditions that are usually considered are 'constant volume' and constant pressure'. This is why gases have two specific heat capacities : **the specific heat capacity at constant volume denoted by c_v and the specific heat capacity at constant pressure denoted by c_p.**

DEFINITIONS. *Specific heat capacity at contsant volume (c_v)* The amount of heat required to raise the temperature of one kilogram of a gas through 1°K at constant volume is called its *specific heat capacity at constant volume.*

Specific heat capacity at constant pressure (c_p). The amount of heat required to raise the temperature of one kilogram of the gas through 1°K at constant pressure is called *specific heat capacity of*

the gas a constant pressure.

Molar specific heat capacity at constant volume (C_v). The amount of heat required to raise the temperature of 1 mole of a gas through 1°K at constant volume is called *Molar specific heat capacity at constant volume* (C_v).

$C_v = Mc_v$ where M is the molecular weight of the gas.

Molar specific heat capacity at constant pressure (C_p). The amount of heat required to raise the temperature of 1 mole of a gas through 1°K at constant pressure is called *Molar specific heat capacity at constant pressure* (C_p).

$$C_p = Mc_p.$$

3.11. C_p is greater than C_v and $C_p - C_v = R$ when C_p and C_v are in Joules

Suppose one mole of a gas is taken and is heated through 1°K keeping its volume constant. A definite quantity of heat will be required for the purpose. This quantity of heat is C_v. If the same amount of heat is supplied to the same quantity of the gas at constant pressure, i.e., the gas is allowed to expand, its temperature will not rise by 1°K, because a part of the energy supplied is used in doing work against the external pressure. Hence some additional amount of heat required to do the external work must be supplied to the gas. This is why the specific heat capacity at constant pressure is greater than the specific heat capacity at constant volume and $C_p - C_v =$ the external work done by the gas.

Let us consider a mole of a gas in a cylinder fitted with a frictionless piston. Let p, V and T be the pressure, volume and temperature of the gas respectively.

Assuming that the gas is perfect we have

$$pV = RT. \quad \ldots \quad \text{(i)}$$

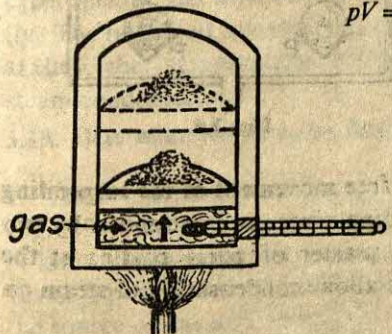


Fig. 3.7

The weight of the piston and sand is balanced by the upward thrust of the gas. Let us now heat the gas till its temperature rises by 1°K. Let the piston be pushed up by x.

Then the work done by the gas = force × distance

$$= pA \times x = pAx$$

where A is the area of the piston.

$$\therefore \quad C_p - C_v = pAx \quad \ldots \quad \text{(ii)}$$

By the equation of state of a perfect gas $pV = RT$,

we have $$p(V+Ax) = R(T+1) \quad \ldots \quad \text{(iii)}$$

or $$pV + pAx = RT + R. \quad \ldots \quad \text{(iiia)}$$

Substracting (i) from (iii a) we have

$$pAx = R;$$

$$\therefore \quad C_p - C_v = R. \quad \ldots \quad (3{\cdot}5)^{**}$$

If C_p and C_v are in calories, then $C_p - C_v = \dfrac{R}{J}$ where J is the mechanical equivalent of heat.

If the mass of the gas taken be unit mass then

$$c_p - c_v = r \quad \ldots \quad (3.5a)$$

where r is the gas constant per unit mass. This is not a universal constant like R.

3.12. Determination of c_v by Joly's Differential Steam Calorimeter

This apparatus is a slight modification of Prof. Joly's Steam Calorimeter made suitable for measuring the specific heat capacities of gases at constant volume. Here two equal hollow spheres of copper provided with 'catchwaters' P, P' are suspended inside a double-walled steam chamber from the pans of a sensitive balance placed over the steam chamber. To prevent the collection of steam condensed on the ceiling of the steam chamber on the 'catchwaters' there are two umbrella-like shields S and S'. Further to ensure free movement of the suspending wires through the holes, the suspending wires are surrounded by two heating coils C, C and there is the plaster of paris coating at the holes. These arrangements do not allow condensation of steam on

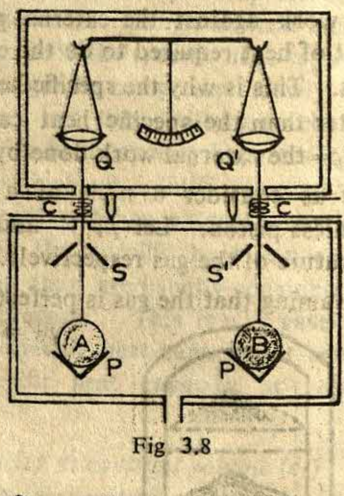


Fig 3.8

**See art. 8.6 for the same deduction thermodynamically.

the suspension wires. First of all the two spheres are counterpoised. Now one of the spheres is completely evacuated and the other is filled with the experimental gas under high pressure. Again, the balance is counterpoised. The additional weights required to restore balance gives the mass of the gas enclosed in the sphere. Now steam is introduced into the chamber and the supply of steam is maintained, till all the parts attain a steady state. Steam condenses on the pans. A greater quantity of steam condenses on the sphere containing the gas. When the steady state is attained, the balance is again counterpoised. The additional weight required to restore the balance gives the mass of steam condensed due to the exchange of heat with the gas only. Let θ_1 and θ_2 be the temperatures of the steam chamber before the introduction of the steam and after the steady state is attained. If m is the mass of the enclosed gas, then the heat gained by it $= m\, c_v(\theta_2 - \theta_1)$ and if w is the mass of the additional steam condensed then the heat released by the steam $= wL =$ heat gained by the gas.

$$\therefore \qquad mc_v(\theta_2 - \theta_1) = wL$$

or $c_v = \dfrac{wL}{m(\theta_2 - \theta_1)}$ J kg^{-1}K^{-1} or cal kg^{-1}K^{-1} according as L is in 'work units' or 'heat units'.

The following corrections are needed for accurate determination of specific heat capacities—(i) The expansion of the sphere due to increased temperature and the consequent work done by the gas in expanding to this volume. (ii) The expansion of the sphere due to the increased pressure of the gas at the higher temperature. (iii) The two spheres may not be exactly identical. A correction for this fact can easily be applied by alternately filling the two spheres with the experimental gas and taking the mean of the two determinations. (iv) The increased buoyancy of the sphere due to its increased volume at the higher temperature. (v) Buoyancy correction for the weight of steam condensed.

3.13. Determination of c_p by Regnault's Method

The complete experimental set-up is shown in the figure given in Fig. 3.9. The experimental gas is stored under pressure in a vessel A of known volume immersed in a bath of constant temperature. The vessel is provided with a stop-cock S_1 in the pipe connecting it with the source of gas and another stop-cock S_2 in the delivery pipe. It is

also provided with a manometer to record pressure of the gas. The flow of gas through the whole set-up is controlled by the controlling valve' V and the constancy of pressure at which the gas is allowed to pass is ensured by observing the difference of liquid levels in the manometer M placed immediately after the controlling valve. The gas then flows through a spiral of copper tube immersed in hot-oil bath and then into a similar spiralling copper tube in the calorimeter C finally escaping into the air. The calorimeter is protected from direct heating by means of a screen. In conducting an experiment the initial temperature and pressure of the resorvoir A are noted and the bath is strongly heated by burner. In the mean time, a copper

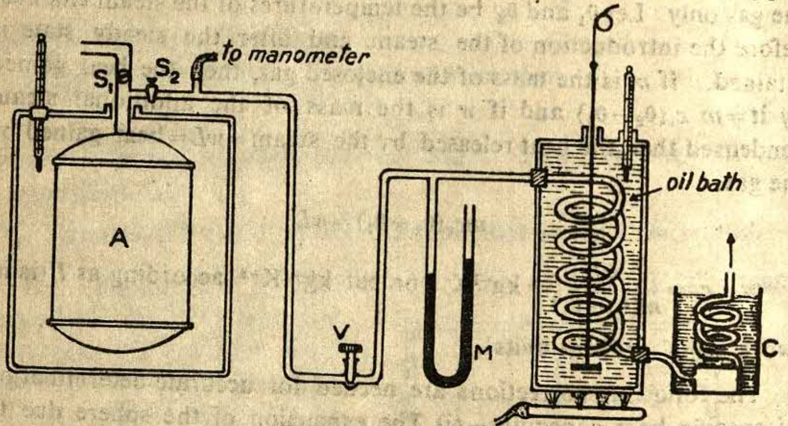


Fig. 3.9

calorimeter provided with a spiralling copper tube is weighed empty and then with water almost filling the entire calorimeter. The difference gives the mass of water taken. The calorimeter is then connected to the heating tube of the oil bath. Stop-cock S_2 is opened and the gas is allowed to pass slowly through the whole set-up with the help of the controlling valve. While passing slowly through the spiralling heating tube, the gas is heated to the temperature of the bath recorded by a thermometer. The hot gas then passes through the spiralling tube of the calorimeter and gives away its heat to the calorimeter and its contents. When the temperature of the water in the calorimeter rises by about 10°C, the supply of gas is cut off. The final pressure of the gas left in the reservoir is again noted.

Let m be the mass of gas that has escaped into the air and θ_1 and θ_2 are the initial and final temperatures of the bath. Further suppose that W is the mass of the calorimeter and W' is the mass of water

taken. Heat lost by gas $= mc_p\left(\theta_3 - \dfrac{\theta_1+\theta_2}{2}\right)$ where θ_3 is the temperature of the oil bath.

Heat gained by calorimeter and its contents

$$= (Ws_c + W's_w)(\theta_2 - \theta_1)$$

where s_c and s_w are the specific heat capacities of copper and water.

$$\therefore \quad mc_p\left(\theta_3 - \frac{\theta_1+\theta_2}{2}\right) = (Ws_c + W's_w)(\theta_2 - \theta_1).$$

Hence c_p can be calculated if m is known. The mass (m) of the gas that has escaped is calculated from the knowledge of the temperature, pressure and volume of the gas in the reservoir before and after the experiment. The calculation is very simple, if the gas be assumed to be 'perfect'. But real gases are hardly 'perfect'. The perfect gas equation for any mass of gas is

$pV = mrT$ where m is the mass of the gas, r is the 'gas constant' per unit mass.

or $\qquad p = \rho\, rT$ where $\rho = \dfrac{m}{V}$ = density of the gas.

If ρ_0, ρ_1 and ρ_2 be the densities of the gas at N.T.P., at temperature T and pressure P_1 and at temperature T and pressure P_2 respectively then

$$76 = \rho_0\, r\, 273$$
$$P_1 = \rho_1\, r\, T$$
$$P_2 = \rho_2\, r\, T.$$

Let V be the volume of the reservoir

then $m = V\rho_1 - V\rho_2 = V(\rho_1 - \rho_2) = \dfrac{V(P_1 - P_2)}{rT}$

$$= \frac{V(P_1 - P_2)\rho_0}{76} \cdot \frac{273}{T}.$$

If the gas is not perfect, we cannot use this simple formula to find m.

Examples

1. *A 100 gm block of copper, taken from a furnace, is dropped into a 300 gm copper vessel containing 200 gm of water. The temperature of the water rises from 20°C to 50°C. What was the temperature of the furnace ?* ($s_c = 120$ cal kg^{-1}K^{-1} and $s_w = 1000$ cal kg^{-1}K^{-1})

Sol. Heat lost $= \cdot 1 \times 120(\theta - 50)$ where θ = temperature of the furnace

$$= 12(\theta - 50) \text{ cals.}$$

Heat gained $= \cdot 3 \times 120(50-20) + \cdot 2 \times 1000 \times (50-20) = 7080$ cals.

$\therefore \quad 12(\theta - 50) = 7080$ or $\theta = 640°C$. **Ans.**

2. *Find the result of mixing 1 kg of ice at 0°C with 1·5 kg of water at 45°C. (Specific heat capacity of water = 1000 cal kg^{-1}K^{-1} and specific latent heat of ice = 80,000 cal kg^{-1}).*

Sol. Let t be the final temperature.

$$1 \times 80{,}000 + 1 \times 1000t = 1 \cdot 5 \times 1000(45 - t).$$

Hence t = something negative which is absured. Hence we conclude that the final temperature will be 0°C with x kg of ice and $(1 \cdot 5 + 1 - x)$ kg of water.

$$\therefore \quad (1-x)80{,}000 = 1 \cdot 5 \times 1000(45-0)$$

or
$$(1-x) = \frac{1500 \times 45}{80{,}000} = \cdot 8475$$

or
$$x = \cdot 1525 \text{ kg.}$$

Ans. The result is ·1525 kg of ice and 2·3457 kg of water at 0°C.

3. *A calorimeter of water equivalent* 20 *gm takes* 7·5 *minutes to cool from* 55°C *to* 40°C *with* 40 *gm of a liquid and* 10 *minutes to cool through the same range of temperature with* 50 *gm of water in the same environment. Calculate the specific heat capacity of the liquid. Given that specific heat capacity of water is* $4 \cdot 2 \times 10^3$ *J kg^{-1} K^{-1}.*

Sol. Rate of loss of heat by the calorimeter with liquid as its content $= \dfrac{(20 \times 10^{-3} \times 4 \cdot 2 \times 10^3 + 40 \times 10^{-3} s)(55-40)}{7 \cdot 5 \times 60}$.

Rate of loss of heat by the calorimeter with water as its content

$$= \frac{(20 \times 10^{-3} \times 4 \cdot 2 \times 10^3 + 50 \times 10^{-3} \times 4 \cdot 2 \times 10^3)(55-40)}{10 \times 60}$$

By Newton's law of cooling the two rates are equal.

$$\therefore \quad \frac{(20 \times 4 \cdot 2 + 40 \times 10^{-3} s)}{7 \cdot 5} = \frac{70 \times 4 \cdot 2}{10}$$

$$84 + 40 \times 10^{-3} s = 220 \cdot 5, \text{ or } s = 3 \cdot 4 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1}. \textbf{ Ans.}$$

4. *42 gm of copper heated to* 100°C *is placed on a large block of ice and it is found that* 5 *c.c. of ice has melted. Calculate the specific heat capacity of copper. Specific latent heat of ice* = 80000 *cal kg^{-1}.*

(*This illustrates the principle of working of a calorimeter known as Black's calorimeter.*)

Sol. Since the block is large, the final temperature would be 0°C.

Heat lost $=42\times10^{-3}\times s\times100=4{\cdot}2s$.

Heat gained by ice $=(5\times10^{-6}\times1000)\times80000$ cal.

$$\therefore \quad s=\frac{5\times10^{-6}\times8\times10^{7}}{4{\cdot}2}=95{\cdot}2 \text{ cal kg}^{-1}\text{K}^{-1}. \quad \textbf{Ans.}$$

QUESTIONS

(A)

1. For a gas (a) $C_p>C_v$, (b) $C_p<C_v$, (c) $C_p=C_v$, (d) $C_p=\sqrt{C_v}$.

2. For the determination of specific heat capacity of a precious metal available in small quantity, the method recommended is (a) Bunsen's ice calorimeter, (b) Joly's steam calorimeter, (c) Regnault's method of mixture, (d) Black's ice calorimeter.

3. (a) Ice contracts on melting by 9×10^{-5} cubic metre when 1 kg of it melts, (b) ice expands on melting by 9×10^{-5} cubic metre per kg, (c) water contracts by $9\times10^{-5}\text{m}^3$ per kg, (d) water expands by $9\times10^{-5}\text{m}^3$ per kg.

4. The unit of the specific latent heat is (a) J kg^{-1} K^{-1}, (b) J kg K^{-1}, (c) J kg^{-1}, (d) J K^{-1}.

5. The unit of the latent heat capacity is (a) J kg^{-1} K^{-1}, (b) J kg K^{-1}, (c) J kg^{-1}, (d) J K^{-1}, (e) J.

6. A quantity of hot water is mixed with a large quantity of ice. The final temperature (a) may be above 0°C, (b) may be below 0°C, (c) equal to 0°C, (d) none of these.

7. It is most efficient to heat an ideal gas by keeping its (a) pressure constant, (b) density constant, (c) volume constant, (d) mass constant.

8. Heat flows from a body A to another body B in contact. This indicates that the body A (a) contains a large amount of heat, (b) is at a higher temperature, (c) has a larger size, (d) has a larger specific heat capacity.

9. The unit of the specific heat capacity is (a) J kg^{-1}, (b) J kg^{-1} K^{-1}, (c) J kg K^{-1}, (d) J.

10. The unit of the heat capacity of a body is (a) J kg^{-1}, (b) J kg^{-1} K^{-1}, (c) J kg K^{-1}, (d) J K^{-1}.

11. The unit of water equivalent of a body is (a) J kg^{-1}, (b) J, (c) kg, (d) J K^{-1}.

[**Ans.** **1.** (a), **2.** (a), **3.** (a), **4.** (c), **5.** (e), **6.** (c), **7.** (a), **8.** (b), **9.** (b), **10.** (d), **11.** (c).]

(B)

1. Why do gases have two specific heat capacities ?

2. Show that $C_p > C_v$ and $C_p - C_v = R$ where all the symbols have their usual significance.

3. Write notes on Rumford's compensation method.

4. Explain why burns by steam are more painful than those from boiling water.

(**Hint :** High specific latent heat of steam,)

(C)

1. Define the specific heat capacity of a substance and state its unit in SI. State the principle of calorimetry. Describe Joly's steam calorimeter and show how it can be used to determine the specific heat capacity of a solid.

2. Describe the continuous flow method for determining the specific heat capacity of a liquid. What are the advantages of this method ?

3. Explain Newton's Law of cooling. Describe with theory a method of determining the specific heat capacity of a liquid based on this law.

4. Describe how the specific heat capacity of a gas at constant pressure is determined by Regnault's method.

5. Describe Joly's differential steam calorimeter method to determine the specific heat capacity of a gas at constant volume.

6. Explain the terms latent heat capacity and the specific latent heat capacity. Describe a method for determining the specific latent heat of steam and point out the precautions which are necessary in order to obtain an accurate result.

7. Describe Bunsen's ice calorimeter. Explain with theory how it is used for the determination of the specific heat capacity of a metal. Discuss the merits and demerits of this method.

(D)

1. A calorimeter whose water equivalent is ·01 kg contains ·05 kg of water at 30°C. A piece of copper whose temperature is 40°C and whose weight is ·02 kg is dropped into the calorimeter along with a dry piece of ice weighing ·01 kg. If there is no loss or gain of heat from outside, calculate what will be the minimum temperature of the mixture. (specific heat capacity of copper $= 420$ J kg^{-1} K^{-1} and specific latent heat capacity of ice $= 336 \times 10^3$ J kg^{-1}).

[Ans. 15°C]

2. What would be the result of mixing 5 gm of ice at 0°C with 10 gm of water at 30°C. (Latent heat of ice $= 336 \times 10^3$ J kg^{-1}).

[Ans. 13·75 gm of water + 1·25 gm of ice at 0°C]

3. How would you divide 1·2 kg of water at 50°C into two parts so that one part of it when turned into ice at 0°C should by this change of state give out a quantity of heat sufficient to vaporise the other part to steam at 100°C ?

(The specific latent heat capacity of ice $=336\times10^3$ J kg^{-1} and that of steam $=2{\cdot}255\times10^6$ J kg^{-1})

[Ans. ·9824 kg and ·2176 kg]

4. Find the amount of heat required to convert 15 gm of ice at $-15°$C into steam of 100°C. (The specific heat capacity of ice $=2100$ J kg^{-1} K^{-1} and the specific latent heat of steam $=2{\cdot}25\times10^6$ J kg^{-1}.

[Ans. $4{\cdot}56\times10^4$ joule]

5. The diameter of the capillary tube of a Bunsen's ice calorimeter is $1{\cdot}4\times10^{-3}$ m. On dropping into the calorimeter a piece of metal whose temperature is 100°C and mass $1{\cdot}12\times10^{-3}$ kg, the mercury thread is observed to move through 10×10^{-2} m. Calculate the specific heat capacity of the metal. (The specific latent heat of fusion of the ice $=336\times10^3$ J kg^{-1} and density of ice $=900$ kgm^{-3}.
[Ans : 420 J kg^{-1} K^{-1}]

6. A liquid cools from 70°C to 60°C in 5 minutes. What is the time taken by it in cooling from 60°C to 50°C, if the temperature of the surrounding air is assumed to be constant at 30°C ? [Ans. 7·04 min.]

7. A body cools from 50°C to 40°C in 5 minutes, when the temperature of the surroundings is 20°C. What will be its temperature after further 5 minutes ?

[Ans : 33·33°C]

8. A copper globe of ·4 kg contains 6×10^{-3} m^3 of air at 0°C and ·76 m of mercury pressure. The globe is kept immersed for long in dry steam at 100°C and 7·71 gm of water condensed on the globe. Calculate the specific heat capacity of air at constant volume. (Density of air at NTP $=1{\cdot}293$ kgm^{-3}; the specific heat capacity of copper $=420$ J kg^{-1} K^{-1} and the specific latent heat of steam $=2{\cdot}251\times10^6$ J kg^{-1}.
[Ans : 718 J kg^{-1} K^{-1}]

9. In an experiment of Callendar and Barnes when the potential difference across the wire was 3 volts, the current 2 amperes and the rise of temperature 2·7°C, the rate of flow of water was found to be 30 gm per minute. When the rate of flow was increased to 48 gm per minute, the potential difference to 3·75 volts and the current to 2·5 ampere, the rise of temperature was found to be the same. Calculate the value of the mechanical equivalent of heat. Specific heat capacity of water $=1000$ cal kg^{-1} K^{-1}.

[Ans : 4·167 joule per calorie]

10. In an industrial process 10 kg of water per hour is to be heated from 20°C to 80°C. To do this, steam at 150°C is passed from a boiler into a copper coil immersed in water. The steam condenses in the coil and is returned to the boiler as water at 90°C. How many kg of steam are required per hour ? (Specific heat of steam $=4200$ Jkg^{-1} K^{-1} and the specific latent heat of steam $=2{\cdot}268\times10^6$ Jkg^{-1}.
[Ans. 1 kg]

11. An aluminium container of mass 100 gm contains 200 gm of ice at $-20°$C. Heat is added at the rate of 100 calories per second. What is the temperature of the system after 4 minutes ? Draw a rough sketch showing the variation of the temperature of the system as a function of time. The specific heat capacity of aluminium $=840$ J kg^{-1} K^{-1} and $J=4{\cdot}2$ joule per calorie.

(I. I. T. 1973) [Ans. 25·5°C]

12. The temperature of equal mass of three differnt liquids A, B and C are 12°, 19° and 28° respectively. The temperature when A and B are mixed is 16°C; and when B and C are mixed, it is 23°C. What would be the temperature when A and C are mixed ? **(I. I. T. 1976)** **[Ans. 20·25°C]**

(E)

1. The unit of the specific heat capacity is...............
2. The unit of the heat capacity of a body is............
3. The unit of the latent heat capacity is...................
4. The unit of the specific latent heat capacity of a substance is...................
5. A solid is available in small quantity. Which method would you use to determine its specific heat capacity ?
6. Can you give heat to a substance without change of temperature ? Does it violate the concept of heat ?
7. There is some rise in temperature. Can you distinguish whether it is due to 'work flow' or 'heat flow' to the system ?
8. Is 'r' the gas constant for unit mass a universal constant ? Is R ?
9. When ice melts at 0°C it absorbs heat without rise in temperature. From this is it justified to say that specific heat capacity of ice at 0°C is infinite ?

[Ans : 1. J kg^{-1} K^{-1}, 2. J K^{-1}, 3. J, 4. J kg^{-1}, 5. Bunsen's ice calorimeter, 6. Yes, No, 7. No, 8. No. Yes.
9. No, because the heat absorbed is its latent heat and not the sensible heat in terms of which specific heat capacity is defined.]

CHAPTER 4

CHANGE OF STATE

4.1. Fusion : Vaporisation and Sublimation

It is a matter of common experience that on the application of heat substances change their state of aggregation, passing from the solid to the liquid state (fusion), from the liquid to the vapour state (vaporisation) and the solid to the vapour state (sublimation) with the absorption of a large amount of heat, but without any change in temperature.

4.2. (a) Fusion (Solidification)

When a substance changes from the solid state to the liquid state with absorption of heat, the process is called *fusion*, and when it changes from the liquid to the solid state after rejecting heat the process is called *freezing* or *solidification*. The amount of heat absorbed or rejected by the substance during its change of state from solid to liquid without change of temperature or vice versa is called its latent heat of fusion. The particular temperature at which it changes from the solid to the liquid state is called its *melting point* and the temperature at which it changes from the liquid to the solid state is called the *freezing point* or the *solidification temperature.* Generally, the temperature at which a solid melts is the same as the one at which it freezes. But for certain *fats* like butter, this is not true. Butter melts at about 33°C, but it is solidified at about 20°C.

(b) Laws of fusion

(i) A solid under a constant pressure melts at a definite temperature called the melting point of the solid.

(ii) The rate at which fusion takes place is proportional to the supply of heat, but the temperature remains constant until the whole of the solid melts.

(iii) Substance like ice which contracts on melting, have their melting points lowered by an increase of pressure; while substances like paraffin, wax which expand on melting, have their melting points raised by increase of pressure.

(iv) During fusion every substance absorbs a definite amount of heat called the latent heat of fusion.

(c) Determination of the melting point of a substance

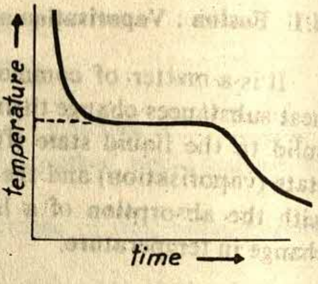


Fig. 4.1

The melting point of a substance under normal condition can be determined in a very simple way The substance is taken in a crucible of a material strong enough to withstand the high temperature to which it will be heated to melt down the substance. For ordinary wax, naphthalene etc. a glass tube will do but for iron, copper, etc. which melt at very high temperature a porcelain crucible is essential. Secondary thermometers such as mercury-in-glass thermometer for ordinary temperatures and thermocouple thermometers for very high temperature are used. After melting the entire substance electrically or otherwise, it is allowed to cool and its temperature or thermo-emf is noted as the cooling proceeds at regular intervals. The reading will remain stationary during the process of solidification after which it will fall. A curve is plotted with temperature as ordinate and time as abscissa. The horizontal part represents the melting point of the substance.

(d) Melting point of alloys

Metals have generally high melting points but their alloys have melting points lower than those of the constituents. It is for this reason that 'flux' is added in soldering of radio parts to melt down lead at a lower temperature. An alloy of tin (232°C), lead (327°C), cadmium (232°C) and bismuth (271°C) has a melting point of 60°C only. This is called *Wood's metal*. Another alloy called *Rose's Metal* made of tin (232°C), lead (327°C) and bismuth (271°C) has a melting point of 95°C. These are readily fusible and so they find many applications in safety devices against fire in our daily life. They are used in automatic sprinklers for buildings, in electrical circuit as fuse etc. For automatic sprinkling of water in the event of fire a plug made of Rose's or Wood's metal is inserted in the water pipe of the building. When fire breaks out, the fusible plug melts releasing a gush of water from the main water pipe. In electrical circuits, 'fuse

wire' of suitable amperage is put in series with the device to be protected. The moment the current exceeds the stipulated values, the fuse wire melts.

(e) Regelation

The melting point of a substance depends on pressure. The melting point of ice is lowered on increasing the pressure at the rate of ·00769 per atmosphere. When two pieces of ice are pressed together and then released, the two are frozen into one. Such phenomenon of melting by pressure and solidification on release of pressure is called *Regelation*. The process of *welding* is, in fact, a case of regelation. In welding two pieces are joined together rigidly by strong heating. The two pieces are first heated to a temperature a little below the melting point and then a pressure is applied. Due to pressure, the melting point is lowered and the two pieces melt at the junction. On removal of the pressure the melting point (and hence solidification point as well) is restored to the previous value and so the two pieces are frozen into one.

DEMONSTRATION OF REGELATION OF ICE

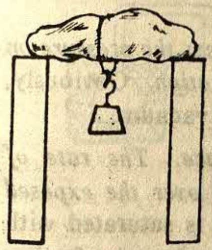


Fig. 4.2

Bottomley's Experiment. A large block of ice is placed over two supports and a turn of copper wire is wound round the block. A heavy weight is attached to the wire. The wire cuts its way through the block but the block remains intact. This is due to regelation of ice. Due to heavy pressure the melting point of ice beneath the wire is lowered and so the ice there melts and the wire passes through water which on being relieved of pressure freezes into ice again. It is to be noted that the ice below the wire needs latent heat for fusion and the water above gives out latent heat on freezing. Hence the wire must be a good conductor to conduct the heat released by the water to the ice below. Another important point to be noted is the condition of the 'environment'. The temperature of the environment must be above 0°C. If the temperature of the environment be below 0°C, say, – 1°C, then 130 atmospheric pressure has to be applied first to lower the melting point to – 1°C and then only regelation may take place.

4.3. Vaporisation : Evaporation and Boiling

The change of a substance from the liquid to the vapour state is *called vaporisation.* Vaporisation may take place at any temperature. When the change takes place from the liquid to the vapour state *slowly at any temperature* and that too *only from the exposed surfaces,* the process of vaporisation is specially called *evaporation.* But when the change takes place rapidly at a fixed temperature called the boiling point from throughout the mass of the liquid, the process of vaporisation is specially called *ebullition* or *boiling.* In evaporation the latent heat of vaporisation of liquid is derived from the liquid itself and hence the liquid loses heat progressively as evaporation proceeds. This is why evaporation causes cooling. But in boiling the latent heat of vaporisation is supplied by the source of heat over which the liquid is boiled.

4.4. Factors Governing Evaporation

(i) *The temperature of the liquid. The higher the temperature the greater is the rate of evaporation.*

(ii) *The boiling point of the liquid. The lower the boiling point of a liquid, the greater is its rate of evaporation.* A quantity of ether disappears faster than the same quantity of water under the same conditions.

(iii) *The pressure on the exposed surface. The less the pressure on the exposed surface, the greater is the rate of evaporation.* Obviously, therefore, the rate of evaporation is maximum in a vacuum.

(iv) *The renewal of air over the exposed surface. The rate of evaporation is enhanced by the arrival of fresh air over the exposed surface.* The stagnant air over the liquid surface is saturated with the vapour of the liquid when saturated air refuses to take further vapour from the liquid. So evaporation becomes slow. When air is blown, saturated air is constantly replaced by fresh air and so evaporation proceeds vigorously uninterrupted.

(v) *The area of the exposed surface. The greater the area of the surface of a liquid exposed to the air, the greater is the evaporation.* This is why hot tea or milk is poured in a saucer to get it cooled quickly.

(vi) *The presence of vapour in contact with the liquid. The rate of evaporation is slowed down by the presence of vapour of the liquid.*

This is why wet clothes dry up more quickly in the winter than in the rainy season.

4.5. Evaporation Causes Cooling

Evaporation produces cooling. When evaporation takes place, the latent heat neceesary for vaporisation is supplied by the liquid itself and its temperature goes down. This is why a porous pot keeps water cooler than a non-porous pot, a drop of ether produces cold on the skin, a fan cools down human body wet with sweat etc.

EXPERIMENT TO SHOW THAT EVAPORATION CAUSES COOLING

Expt. I. Pour a few drops of water on a sun-mica sheet and then place a thin walled copper calorimeter containing some ether over that water. Now, briskly blow air from a foot bellows through the ether. It is found that after sometime the beaker is tagged to the sheet by a layer of ice. Due to the brisk blowing of air, rapid evaporation of ether takes place, which takes heat from the water under the beaker and so it ultimately freezes into ice.

Expt. II. By Wallaston's Cryophorous. This apparatus demonstrates two basic facts—evaporation causes cooling and the less the pressure of vapour, the greater is the rate of evaporation. It consists of a bent glass tube having two bulbs *A* and *B* at the two ends. The bulbs contain water and water vapour only. First all the water is transferred to the bulb *A* and the other bulb *B* is immersed in a bath containing a freezing mixture. The moment *B* is immersed in the mixture, the vapour in it condenses; the vapour pressure inside falls, so evaporation of water sets in. This goes on uninterrupted as the freezing mixture continuously condenses vapour entering into *B*. The water in *A* is gradually cooled and so ultimately it may be frozen into ice.

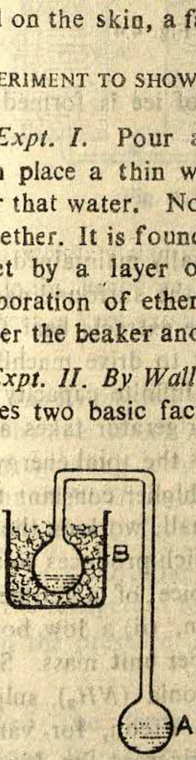


Fig. 4.3

Expt. III. Leslie Expt. A shallow dish containing a little water and another containing strong H_2SO_4 is placed on the receiver of a suction pump On working the pump a thin layer of ice is formed on the surface of water. This is due to evaporation causing cooling.

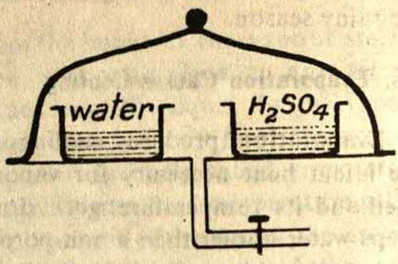


Fig. 4.4

As air is withdrawn, pressure inside falls and so rapid evaporation of water sets in. The vapour formed is immediately absorbed by conc. H_2SO_4. Thus the vapour pressure and the pressure of air are always kept low and so the water continues to evaporate rapidly. As evaporation causes cooling, the temperature of water is ultimately lowered so much so that a thin layer of ice is formed on the surface of it.

4.6. Refrigerator

It is a device by which an enclosure is artificially maintained at a constant temperature much lower than that of the surroundings. It is the converse of heat of engine. A heat engine extracts heat from a source of heat of infinite capacity, gives work to drive machinery parts and rejects the rest to another source of infinite capacity but at a lower temperature than the source. A refrigerator takes away heat from a body, work is done on it and rejects the total energy received (heat + work done) to a source of heat at higher constant temperature. All practical refrigerators, big or small, work on the fact that *evaporation causes cooling*. The liquid which produces cold by evaporation is called the *refrigerant*. The choice of a refrigerant depends on (i) large latent heat of vaporisation, (ii) a low boiling point, (iii) a small specific volume, i e., volume per unit mass. Some common refrigerants are freon (CCl_2F_2), ammonia (NH_3), sulphur dioxide (SO_2), ethyl chloride (C_2H_5Cl) etc. Freon, for various considerations, is supposed to be the best refrigerant for domestic refrigerators but in big ones such as ice machines NH_3, SO_2 etc. are found suitable as refrigerant. There are two types of refrigerators—(i) *The absorption type* (Electrolux). In it the working energy is supplied in the form of heat energy by burning a fuel such as coal gas, kerosene, etc. (ii) *The Compression type* (Frigidaire type). In it the working energy is supplied in the form of mechanical energy by an electric motor. This type of refrigerators is now of common use. So we describe here the working of this type only. The principle of working of a compression type refrigerator is as follows—

The essential parts of a refrigerator are—

(*a*) *The Pump* (*P*). It is a mechanical pump worked by an electric motor. It has two valves V_1 and V_2 at its bottom. The valves are operated by the machine itself and they are adjusted to open and remain closed at the right moments as required for the operation of the device.

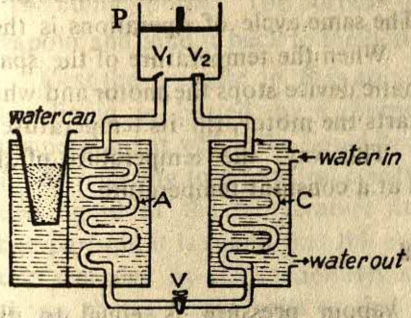


Fig. 4.5

(*b*) *The Condenser* (*C*). It is simply a coil of copper tube placed in a tank in which water is continuously circulated. This arrangement is made in heavy machines. But in domestic refrigerators the condenser is kept exposed to the atmosphere which takes away heat from it.

(*c*) *The Evaporator* (A). This is also a coil of copper tube placed in the chamber which is to be maintained at lower temperature called the refregerating chamber. This chamber contains brine water (solution of *NaCl*) in ice machines. Cans containing water are placed in the chamber to form ice.

(*d*) *The Controlling Valve* (*V*). This is a one-way valve which allows the refrigerant to pass from the condenser *C* to the evaporator *A* when the pressure difference between the two reaches a certain value.

Action. The device repeatedly goes through the same cycle of operations. In each cycle there are two strokes of the piston.

In the first stroke the piston moves outward when valve V_1 opens and vapour of the refrigerent is sucked in by the pump. Following the reduction of vapour pressure from above the liquid quick evaporation of the liquid sets in. The liquid extracts its latent heat for vaporisation from the cold storage space or the brine surroundings, whatever it is.

In the next inward stroke both the valves V_1 and V_2 remain closed and the vapour, already sucked in by the pump, is compressed till its temperature rises to the temperature of the condenser chamber when

V_2 suddenly opens and the vapour is instantaneously converted into liquid in the condenser, rejecting its latent heat to the circulating water. Immediately after condensation, the controlling valve V opens and the pump pushes the liquid to the evaporator through it, completing its inward stroke. The same cycle of operations is then repeated over and over again. When the temperature of the space reaches the desired value, automatic device stops the motor and when the temperature rises, it again starts the motor, till its temperature is lowered to the desired value. This way, the temperature of the refrigerating space is maintained at a constant temperature.

4.7. Laws of Ebullition or Boiling

(i) A liquid boils when its vapour pressure is equal to the superincumbent pressure.

(ii) At a particular superincumbent pressure a liquid boils at *that temperature* where its vapour is equal to that superincumbent pressure.

(iii) The temperature at which a liquid boils called the boiling point remains stationary, until the whole of the liquid is vaporised.

(iv) A definite amount of every liquid absorbs a definite amount of heat called its latent heat of vaporisation at its boiling point from the source over which it is boiled.

4.8. To show that a Liquid Boils at that Temperature where its Vapour Pressure is Equal to the Superincumbent Pressure

The experimental liquid is placed in a boiler A which is connected to a large flask B through a Liebig's condenser L. The flask B is immersed in a bath of constant temperature and is connected with a mercury manometer and a pump (compression pump to increase the pressure or a suction pump to reduce the pressure). A thermometer is put in the boiler A to record the

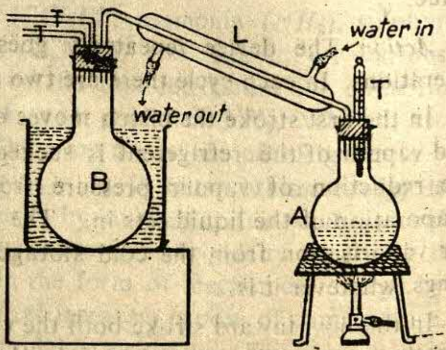


Fig. 4.6

temperature at which the liquid boils. The bulb of the thermon eter is purposely kept in the vapour, and not in the liquid, because sometimes liquid ma: boil at a temperature several degrees above the true boiling point. The purpose of the condenser is to condense the vapour and send it back to the boiler.

First the pressure in B is adjusted to a definite value and then the liquid is heated by a burner until it starts boiling. The temperature at which the liquid boils is noted from the thermometer T. By referring to a physical constant book, the vapour pressure of the liquid at the temperature at which it boiled is found. By comparison, it is found that this vapour pressure is the same as the one recorded by the manometer attached to the flask. Thus it is shown that a liquid boils at that temperature where its saturated vapour pressure is equal to the superincumbent pressure. This experiment also shows that under reduced pressure a liquid boils at lower temperature and at higher pressure it boils at higher temperature. The saturated vapour pressure of water at 100°C is 76 cm of mercury. The normal atmospheric pressure is also 76 cm of mercury. This is why the superincumbent pressure normally being 76 cm of mercury water boils at 100°C. But at Darjeeling which is 2195 m above the sea-level, water boils at only 93·6°C, and at Quito (Latin America), the highest city in the world (2900 m above the sea level), water boils at only 90°C. It is found that on an average for every 292·6 m increase in height above the sea-level there is drop of boiling point of water by 1°C and for every 2·68 cm change of pressure, the normal boiling point changes by 1°C.

4.9. Pressure Cooker and Papin's Digestor

The cooking power of boiling water depends on the temperature at which it boils. So at Darjeeling the cooking power of water is less than the water in the planes. This is why it takes longer time to cook food on the top of high mountains. If the pressure is increased water can be made to boil at higher temperature and thereby the cooking power of water can be increased. This is exactly what is done in domestic pressure cookers. Papin's digestor is also a pressure cooker of large size suitable for silk and paper industries. To make paper, pulp of bamboo, wood etc. is prepared in Papin's digestor with caustic soda under increased pressure.

QUESTIONS

(A)

1. Freon is suitable as refrigerant in domestic refrigerators because of its (a) low specific heat capacity, (b) low specific volume, (c) low specific gravity, (d) low specific latent heat capacity.

2. Water (a) always boils at 100°C, (b) boils when its saturated vapour pressure is equal to the superincumbent pressure, (c) boils at any temperature at any pressure, (d) boils at any temperature at a particular pressure.

3. Rose's metal is made of (a) tin, lead, iron, (b) tin, iron, bismuth, (c) tin, lead, bismuth, (d) lead, bismuth, cobalt.

4 An alloy of metals has melting point (a) higher than that those of the constituents, (b) equal to that of one of the constituents, (c) lower than those of the constituents, (d) any temperature.

5 The boiling point of water at a place is 97°C. The height of the place above the sea-level is approximately (a) 200 m, (b) 900 m, (c) 1800 m, (d) 2000 m.

[Ans . 1. (b), 2. (b), 3. (c), 4. (c), 5. (b).]

(B)

1. (i) Why is a man liable to catch cold by wearing wet clothes ?

(Hint Evaporation causes cooling.)

(ii) Why does a porous pot keep water cooler than a non-porous pot ?

(Hint. evaporation causes cooling.)

(iii) Why does it take longer time to cook food on the top of high mountains ?

(Hint. Lowering of boiling point of water.)

2. Explain 'regelation'.

3. What is Papin's digestor ? Explain the principle on which it works.

(C)

1. Explain the phenomenon of regelation and describe an experiment to illustrate this phenomenon.

2. Distinguish between evaporation and boiling (ebullition). State and explain the various factors that influence the rate of evaporation.

3. State the laws of ebullition (boiling). Describe an experiment to show that a liquid boils when its saturated vapour pressure is equal to the superincumbent pressure.

(D)

1. Ice has its melting point......(raised, lowered) by the increase of pressure.

2. Wax has its melting point......(raised, lowered) by the increase of pressure.

3. Fuse wire is an alloy of......(low, high) melting point.

4. Evaporation causes......(heating, cooling).

5. Rose's metal is a......(metal, alloy) of......(high, low) melting point.

6. The boiling point of water is......(lowered or raised) by 1°C for every......m increase in height above the sea-level.

7. What happens when the door of a domestic refrigerator is kept open in a closed room ?

[Ans. 1. lowered, 2. raised, 3. low, 4. cooling, 5. alloy, low, 6. lowered, 292·6, 7. The room gets *heated.* The function of the refrigerator is to take away heat (Q) from the bodies kept inside of it, receive energy (W) from the motor and reject the total ($Q+W$) to the room in every second. In the *equilibrium state* the same amount of energy is conducted to the bodies from the room through the *conducting* walls of the refrigerator. When the door is kept open, Q is taken from the room (because the door is open) and ($Q+W$) is given to it in every second. So the room gets heated.]

CHAPTER 5

SATURATED AND UNSATURATED VAPOUR : DALTON'S LAW OF PARTIAL PRESSURE

5.1. Gas and Vapour : Critical Temperature, Pressure and Volume

It was established experimentally by Dr. Andrews in 1863 that for every substance there exists a definite temperature below which it can be liquefied by applying suitable pressure and above that temperature it can never be liquefied, however great the pressure applied may be. This temperature is called *critical temperature.* The pressure needed to liquefy the substance at its critical temperature is called *critical pressure* and the volume of the substance at the critical temperature and pressure is called the *critical volume.* For CO_2 the critical temperature is 31·1°C and the critical pressure 73 atmospheres. Before Dr. Andrews people termed the gases which they failed to liquefy as permanent gases. Helium, hydrogen, nitrogen, oxygen etc. were such gases. After Dr. Andrews the whole picture about the behaviour of substances in the gaseous state became clear and it was understood that there existed no permanent gases. All gases are liquefiable. The so called permanent gases have a very low critical temperature. This is why they could not be liquefied in the beginning The terms '*gas*' and '*vapour*' are used to denote the gaseous states of a substance above and below its critical temperature. The term 'gas' means the gaseous state of a substance above its critical temperature and the term 'vapour' means the gaseous state of the same substance below its critical temperature. Thus 'vapour' means the *liquefiable gaseous state of a substance.* The gaseous state of ordinary liquids such as water, ether, mercury etc. will obviously be termed as 'vapour' of the corresponding substance

5.2. Unsaturated and Saturated Vapour

A closed space (vacuous or not) of a given volume has a **limited** capacity to retain the vapour of a liquid at a given temperature. If the space contains that limited amount of vapour, it is said to be a saturated one. If the space contains vapour less than that limited amount of vapour, it is said to be unsaturated with the vapour and

the vapour itself is said to be unsaturated one. If unsaturated vapour is exposed to its liquid, evaporation will take place until it becomes saturated with the vapour. On the other hand, if a saturated vapour is exposed to its liquid, there will be no evaporation of the liquid. Thus an unsaturated vapour possesses the capacity to suck vapour from its liquid, whereas saturated vapour has no capacity to suck vapour from its liquid. Hence this sucking power may be taken as a test of the state of 'saturation' and 'unsaturation' of a vapour.

Vapour exerts pressure as a gas does. Since at constant volume and temperature the pressure is proportional to the mass, the unsaturated vapour pressure (U. V. P.) is always less than the saturated vapour pressure (S. V P.) at the same temperature and volume. Thus the saturated vapour may be defined as *the vapour which exerts maximum pressure at a given temperature.*

5.3. Effect of Change of Volume on Saturated and Unsaturated Vapours at a Constant Temperature.

If the volume of an unsaturated vapour is increased keeping its temperature constant, its pressure decreases almost in accordance with Boyle's law. The volume may be increased to any extent, the law will never fail anywhere. But if the volume of unsaturated vapour is decreased, its pressure will increase in accordance with Boyle's law only up to a certain limit and that limit is the maximum vapour pressure corresponding to that temperature. At this stage the vapour becomes saturated, exerting a maximum pressure characteristic of that temperature. If the volume is further decreased the pressure does not increase, but remains fixed at that value. Some vapour condenses and that much vapour remains in the space which is required to saturate it at that temperature. Thus on reducing the volume of unsaturated vapour, it becomes saturated at some stage where it no more obeys Boyle's law.

The effect of the change of volume of saturated vapour is examined in two ways—in the preserce of the liquid and in the absence of the liquid If the volume of saturated vapour is increased in the presence of its liquid, more liquid evaporates and makes the enlarged space saturated with the vapour. The vapour being saturated exerts the same pressure. If the volume is decreased, some vapour condenses and the vapour continues to remain saturated exerting the same maximum pressure. Thus *saturated vapour does not obey Boyle's law.*

If the volume of saturated vapour is **increased** in the **absence** of its liquid, it becomes unsaturated and its pressure decreases in accordance with Boyle's law. If the volume is decreased, some vapour condenses and the vapour continues to be saturated exerting the same maximum pressure.

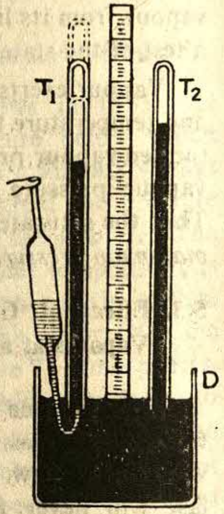


Fig. 5.1

Experimental verification : Two barometer tubes (about 1 m long) T_1 and T_2 are completely filled with mercury and inverted over a mercury vessel D. Mercury comes down in both the tubes by some distance creating a vacuous space above the free surface of the mercury. This is called Torricellian vacuum. The columns of mercury in the tubes balance the atmospheric pressure. Place a metre scale in between the tubes Now introduce a small quantity of the liquid into the vacuous space of the first tube (T_1) with the help of a bent pipette. The vapour formed (unsaturated) depresses the level of mercury in the tube. Note the length of the tube (T_1) above the mercury level in T_1 and also the difference of mercury levels in the tubes. The length (l) of the tube above mercury level in T_1 is proportional to the volume of unsaturated vapour and the difference (h) of mercury levels in the tubes is proportional to its pressure. Now raise the tube T_1 (taking care that lower end of the tube remains under mercury in D) a little and note down h and l. Repeat this process several times. It is found that

$$h_1l_1 = h_2l_2 = h_3l_3 = \ldots\ldots\ldots\ldots\ldots$$

or $$P_1V_1 = P_2V_2 = P_3V_3 = \ldots\ldots\ldots\ldots\ldots$$

or $$PV = \text{a constant.}$$

Now push the tube T_1 gradually into the vessel D and note down h and l. The product $(h \times l)$ will be found to be constant. However, this will not continue indefinitely. A stage is reached when a thin layer of liquid deposits on the surface of mercury in T_1 which indicates that the space is no longer unsaturated, but is saturated with the vapour of the liquid at that temperature. Further depression of the tube does not depress the mercury level any more. Thus for saturated vapour the volume decreases, but the pressure remains

constant. This is against Boyle's law. Thus *saturated vapour does not obey Boyle's law.*

To see the effect of the change of volume of saturated vapour, introduce sufficient liquid into the experimental tube (T_1) till a thin layer of the liquid is deposited on the surface of mercury. The presence of liquid indicates that the vapour is fully saturated. Depress the tube gradually into the vessel *D*. The level of mercury remains nearly stationary, some vapour condenses and the layer of liquid becomes thicker. Now raise the tube gradually. The volume of the space increases, but the level of mercury stands at the same level provided a thin layer of liquid always lies there on the surface of mercury. Thus there is no effect of change of volume of saturated vapour in the presence of its liquid. If the vapour is just saturated and its volume is increased by raising the tube, it becomes unsaturated and its pressure changes in compliance with Boyle's law. On depressing the tube its volume decreases, but its pressure remains the same.

5.4. Effect of Change of Temperature on Saturated and Unsaturated Vapour

If the temperature of a saturated vapour keeping its volume constant is increased in the presence of its liquid, its pressure increases, but not in compliance with Charles' law. According to Charles' law the plot of pressure (P) against temperature (t°C) is straight line. The plot of pressure against temperature for saturated vapour is a curve (Fig. 5.2). If the temperature of saturated vapour is increased out of contact of its liquid it becomes unsaturated and its pressure changes approximately in compliance with Charles' law (Fig. 5.3). The pressure of unsaturated vapour increases with

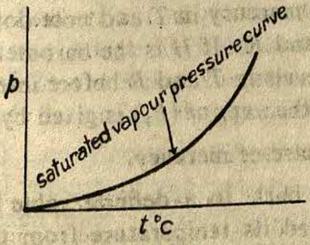


Fig. 5.2

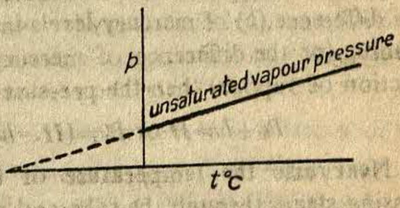


Fig. 5.3

increase of temperature in compatibility with Charles' law and decreases also with temperature in compliance with the law till the

space becomes saturated with the vapour present. After this the vapour behaves like a saturated vapour.

EXPERIMENT

Take a modified Boyle's law apparatus as shown in the Fig. 5.4. The apparatus consists of a tube T provided with two stop-cocks S_1 and S_2 and a funnel F on the top. It is immersed in a water bath whose temperature can be raised by passing steam into it through a copper tube or reduced by adding ice to the bath. The tube T is connected by a long thick-walled rubber tube to another tube R. The whole of the rubber tube and parts of the experimental tube T and the reservoir tube (R) are filled with mercury.

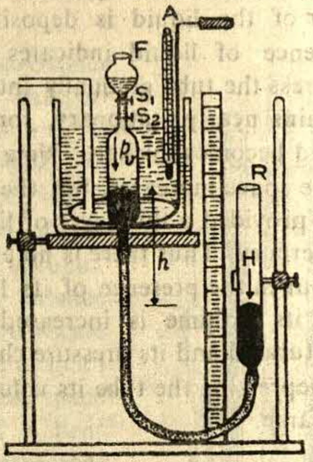


Fig. 5.4

To start with, open S_1 and S_2 and gradually raise R till the mercury level rises up to S_1. Now close S_1 and lower R till the level of mercury in the tube T comes down well below S_2 creating a suitable vacuous space. Close S_2 and take the experimental liquid in the funnel after opening S_1. Then close S_1 and open S_2 carefully to drop the liquid into the vacuous space. The liquid evaporates immediately forming an unsaturated vapour. If there is no collection of the liquid on the surface of mercury in T, that is the test of unsaturation of the vapour. Mark the level of mercury in T and note down the difference (h) of mercury levels in T and R. If H is the barometer reading (or the difference of mercury levels in T and R before introduction of vapour) then the pressure of the vapour (P_v) is given by

$$P_v + h = H \text{ or } P_v = (H - h) \text{ metre of mercury.}$$

Next raise the temperature of the bath to a definite value by passing steam through the tube and record its temperature from the thermometer A. After thoroughly stirring water of the bath to ensure uniformity of temperature, raise the tube R to bring mercury in T to the same level. This is done to keep the volume of the vapour constant. Note down h again. Repeat the above operations several

times. The experiment may be repeated for a few temperatures below the initial temperature till the vapour becomes saturated i.e., collection of liquid just sets in. Plot a graph of pressure against temperature. The plot will be a straight line as shown in Fig. 5.3. This shows that unsaturated vapour obeys **Charles' law for pressure.**

For saturated vapour, drop sufficient liquid into the vacuous space till a little of it collects on the surface of the mercury which will ensure that the vapour is fully saturated. Mark the level of mercury in T and note down the difference (h) of the mercury levels.

Then $P_{saturated\ vapour} = (H-h)$ metre of mercury.

Next raise the temperature of the bath to a definite value and keeping the level of mercury at the same mark, drop sufficient liquid, if necessary, to saturate the space. Note down h again. Repeat the above operations several times for higher and lower temperatures. The plot will be a curve as shown in Fig. 5.2. This shows that saturated vapour does not obey Charles' law.

5.5. Dalton's Law of Vapour Pressure

So far we have considered the formation of vapour in vacuum and the pressure on the mercury surface is entirely due to vapour only. When the space contains some gas or vapour of any other liquid, the pressure of vapour was investigated by Sir John Dalton who formulated two laws from his experimental observations. These are known as Dalton's laws of vapour pressure.

DALTONS' LAWS OF VAPOUR PRESSURE

There are two laws—the law for saturated vapour pressure and law of partial pressures.

(i) *Law of saturated vapour pressure.* The saturation vapour pressure of a liquid is a *function of temperature alone* and is independent of the volume or the pressure of other gases and vapours with which it does not react chemically.

(ii) *Law of partial pressures.* The total pressure of a mixture of several vapours or vapours and gases, which do not react chemically with each other, is equal to the sum of their partial pressures, i.e., the pressures they would exert individually had each of them occupied the entire space.

Experimental verification of Dalton's laws of vapour pressure. Take the apparatus as in fig. 5.4. Open S_1 and S_2 and raise R till

the level of mercury is at the middle of the tube T. Close S_2 and take some water in the funnel. Now close S_1 as well. Mark the level of mercury in the tube T. Introduce water into the space above mercury (now not vacuous but containing some air) by opening S_2. The addition of water is continued till a thin layer of water collects over the surface of mercury. Raise R to bring the level of mercury in T to the same mark so as to keep the volume fixed. Note down the difference (h) of mercury levels in T and R. Note that the level in R stands higher than the level in T. If h be the difference then, $P_{mixture}=H+h$. But $P_{mixture}=H+P_{saturated\ vapour}$

or $$P_{saturated\ vapour}=H+h-H=h.$$

This gives the saturated vapour pressure of water in the presence of air.

Next proceed to find the saturated vapour pressure in vacuum. To begin with, open S_1 and S_2 and raise R until the mercury level in T reaches S_1. Close S_1. Lower R till the level goes down well below S_2 creating some vacuous space. Note that the level of mercury in R will be **below** the level in T by a distance equal to the barometer height at the time of performance of the experiment. Close S_2 and open S_1. Pour the same liquid (here water) in the funnel and close S_1. Now drop water into the vacuous space by opening S_2. The process of dropping of liquid is continued till some of it collects over the surface of mercury in T. Maintain the level of mercury at the same mark to keep the volume fixed at the same value. Note down the difference of mercury levels. Now the mercury level in R stands lower than the level in T. If h be the difference in mercury levels then $P_{vapour}=(H-h)$. If the experiment is correctly done it will be found that the saturated vapour pressure in vacuum is the same as that found when mixed with air. Thus saturated vapour pressure is independent of the presence of air. The experiment may be repeated at other temperatures and volume mixing the vapour with other vapours or gases and the truth of the law may be established.

To verify the second law of vapour pressure (i.e., the law of partial pressures), first obtain data for P and V **with air as the enclosed gas** in the tube T and draw $P-V$ graph. Next obtain data

for P and V **with unsatururated vapour in vacuum** as the enclosed gas and draw $P-V$ diagram and finally obtain data for P and V with the mixture of the same mass of air and vapour as the enclosed gas and draw a $P-V$ graph. It is found that if the pressures at a fixed volume obtained from the first two graphs be added, they become equal to the pressure of the mixture at the same volume.

5.6. Measurement of Saturation Vapour Pressure of Water at Different Temperatures (Regnault's Methods)

Regnault determined the saturation vapour pressures of water over a wide range of temperatures and incorporated them in a chart known as Regnault's table. This is available in all physical constant books.

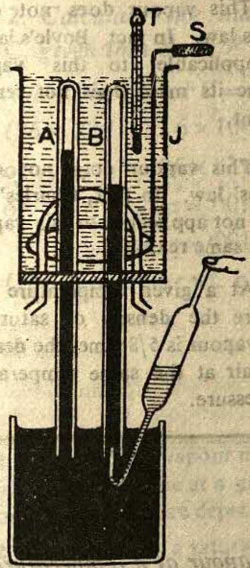


Fig. 5.5

Method I (*from* 0°C *to* 50°C). Two barometer tubes A and B are set as usual, i.e., first filled with mercury and then inverted over a mercury vessel D. Almost the entire upper half of each tube is enclosed in a water bath. The temperature of the bath is regulated by passing steam through a copper tube or by adding ice to the water of the bath. The uniformity of temperature of the bath is maintained by constant stirring of the water of the bath. When the desired temperature is attained, water is introduced into the vacuous space of one of the tubes, say, of B by means of a bent pipette until the space is saturated with water vapour, i.e., until a little water collects on the surface of the mercury. The difference of levels in the two barometer tubes is read at this stage, which gives directly the saturated vapour pressure of water at that temperature.

Method II (*from* 50°C *to* 250°C). The principle of this method is that a liquid boils when its saturated vapour pressure is equal to the superincumbent pressure. For experimental details and figure, see Art. 4.8 of Chapter IV.

5.7. Distinction between Saturated and Unsaturated Vapour

Unsaturated Vapour	Saturated Vapour
1. It is a vapour which cannot exist in equilibrium with its liquid. When it is brought in contact with its liquid, evaporation takes place till it becomes saturated.	1. It is a vapour which can exist in equilibrium with its liquid. When it is brought in contact with its liquid, no evaporation takes place from the liquid.
2. The pressure of this vapour depends on volume and temperature.	2. The pressure of this depends upon temperature alone.
3. For a given volume and temperature the pressure depends upon the mass of the vapour.	3. For a given volume and temperature the pressure of this vapour is a unique function of temperature.
4. This vapour obeys Boyle's law.	4. This vapour does not obey Boyle's law. In fact, Boyle's law is not applicable to this vapour because its mass does not remain constant.
5 This vapour obeys Charles' law.	5. This vapour does not obey Charles' law. In fact, Charles' law also is not applicable to this vapour for the same reason.
6. At a given temperature and pressure the density of unsaturated water vapour does not bear any definite relation with the density of dry air.	6. At a given temperature and pressure the density of saturated water vapour is 5/8 times the density of dry air at the same temperature and pressure.

Examples

1. *A mass of air is saturated with water vapour at a temperature of* 100°C. *On raising the temperature to* 200°C *without change of volume, the pressure is raised to two atmospheres. Find the pressure of this volume of dry air alone at* 0°C.

Sol. Let P be the pressure of dry air at 100°C. Then by Dalton's law of partial pressures, the pressure of the moist air (a mixture of dry air and water vapour) $=(P+1)$ atm.

[$\because$ S. V. P. at 100°C = 1 atmosphere (atm)]

For the mixture we may apply $\frac{PV}{T}$ = a constant (perfect gas equation).

$$\frac{P+1}{273+100}=\frac{2}{273+200}$$

or $$P=\frac{2\times 373}{473}-1=\cdot 577 \text{ atm.}$$

For the dry air also we have $\frac{PV}{T}$ = a const.

$$\frac{P_0}{273}=\frac{\cdot 577}{273+100}, \quad \text{or} \quad P_0=\frac{\cdot 577\times 273}{373}=\cdot 4223 \text{ atm.}$$

or $$P_0=\cdot 4223\times 76=32\cdot 1 \text{ cm of mercury. Ans.}$$

2. *Calculate the mass of* 1 *litre of saturated water vapour at* 27°C, *given that S. V. P at* 27°C *is* 26 *mm of mercury.*

Sol. We have $PV=mRT$ where m is mass of the gas in gm-mol.

$$(26\times 10^{-3}\times 13\cdot 6\times 1000\times 9\cdot 8)\times 1000\times 10^{-6}=m\times 8\cdot 3\times(273+27)$$

or $$26\times 13\cdot 6\times 9\cdot 8\times 10^{-3}=m\times 8\cdot 3\times 300$$

or $$m=\cdot 00139 \text{ gm-mol.}$$

$$\text{mass}=\cdot 00139\times 18=\cdot 025 \text{ gm.} \quad \textbf{Ans.}$$

($\because$ molecular wt. of water vapour = 18)

QUESTIONS

(A)

1. The saturated vapour exerts (a) maximum pressure at a given temperature, (b) minimum pressure at a given temperature, (c) neither maximum nor minimum, (d) the pressure depends on the volume.

2. The volume of a saturated vapour is doubled in the presence of its liquid. Its vapour pressure is (a) doubled, (b) halved, (c) quadrupled, (d) remains same.

3. The volume of a saturated vapour out of contact of its liquid is doubled. Its pressure is (a) doubled, (b) halved, (c) remains same, (d) quadrupled.

4. The volume of a saturated vapour out of contact of its liquid is halved. Its pressure is (a) doubled, (b) quadrupled, (c) halved, (d) remains same.

5. It the density of dry air at a given temperature and pressure is 1·28 kgm^{-3}, the density of the saturated water vapour at the same temperature and pressure is (a) 1·28 kgm^{-3}, (b) ·8 kgm^{-3}, (c) 1 kgm^{-3}, (d) none of these.

6. If the temperature of a saturated vapour is doubled in the presence of its liquid, its pressure is (a) doubled, (b) halved, (c) quadrupled, (d) none of these.

7. The critical temperature of a gas is that temperature (a) above which it can be liquefied by exerting pressure, (b) below which it can be liquefied by exerting pressure, (c) at which the gas is at the critical stage of being liquefied, (d) at which the gas ceases to obey Boyle's law.

[Ans. 1. (a), 2. (d), 3. (b), 4. (d), 5. (b), 6. (d), 7. (b).]

(B)

1. What is the critical temperature ? Explain its significance.

2. Explain 'partial pressure'. State Dalton's law of partial pressure.

3. Distinguish between vapour and gas.

(C)

1. Distinguish between saturated and unsaturated vapours and discuss their properties as regards changes contemplated by Boyle's and Charles' laws.

2. Enunciate Dalton's laws of vapour pressure. Describe experiments to verify them.

3. Describe experiments to show that a saturated vapour does not obey Boyle's law and Charles' law, whereas an unsaturated vapour does.

4. Describe experiments to determine the saturated vapour pressure of water at different temperatures below and above 100°C.

(D)

1. The volume of a room is 160 m^3 and it is saturated with water vapour at 27°C. Calculate the mass of vapour that will condense when the temperature falls to 20°C. The saturated vapour pressure of water at 20°C = 17·4 mm of mercury and that at 27°C = 26 mm of mercury. The molecular weight of water vapour = 18 and the universal gas constant = 8·3 $Jmol^{-1}\,K^{-1}$. (Ans. 138·74 gm)

2. A vessel of water is put in a dry sealed room 76 m^3 at a temperature of 17°C. The saturated vapour pressure of water at 17°C is 15 mm of mercury. How much water will evaporate before the water is in equilibrium with its vapour ?

Assume that the vapour obeys the perfect gas laws. (I. I. T. 1978)
(Ans. 1·136 kg)

3. Two vessels containing m_1 and m_2 kg of a gas at pressures P_1 and P_2 are put into communication. What will be the pressure of the mixture ?

(Hint. Use $pV=mRT$) (Ans $\frac{(m_1+m_2)P_1P_2}{m_1P_2+m_2P_1}$)

(E)

1. The gaseous state of a substance below its critical temperature is called ....(gas, vapour) state.

2. The saturated vapour pressure of a liquid is a function of...................... (temperature, volume, mass) only.

3. The saturated vapour does not obey.......................

4. A saturated vapour is always in.......................with its liquid at a given temperature.

(Ans. 1. vapour, 2. temperature, 3. Boyle's and Charles' laws, 4. equilibrium.)

CHAPTER 6

HYGROMETRY

6.1. Hygrometry

Our atmosphere contains water vapour mostly in the unsaturated state; for evaporation is taking place from the surface of water in wells, lakes, rivers, seas, the vegetation etc. The formation of clouds, mist, dew etc. proves that water vapour is present in the atmosphere. A tumbler of cold water becomes covered with fine grains of water due to condensation of water vapour from the air. There are numerous other examples which show the existence of water vapour in the atmosphere. Hygrometry is the science of measuring this water vapour present in a certain volume of air.

6.2. The Dew-point and Relative Humidity

Ordinarily, the water vapour present in the atmosphere remains in the unsturated state. If, however, the temperature of the atmosphere goes down, at one stage it will be saturated with the water vapour already present. The temperature at which the atmosphere would be saturated with water vapour already present in it is called the *dew-point*. The name of this temperature as the dew-point is quite justified, because if the temperature falls a little more, some vapour will condense as dew.

Absolute Humidity and Relative humidity. The amount of water vapour actually present in a given volume of air is called the absolute humidity and it is expressed in 'kg per cubic metre'.

For meteorological work, the degree of saturation of the atmosphere is more important than the actual amount of water vapour in the air. This is known as Relative humidity and is defined as *the ratio of the mass of water vapour actually present in any volume of air at t°C to the mass of water vapour required to saturate the same volume of air at the same temperature t°C.*

If m' is the mass of water vapour actually present in a certain volume of air at t°C and m is the mass of water vapour required to saturate the same volume of air at the same temperature t°C, then

$$\text{Relative humidity (R. H.)} = \frac{m'}{m}. \qquad \text{(6.1)}$$

Relative humidity is generally expressed as a percentage and hence

$$\text{Relative humidity (R.H.)} = \frac{m'}{m} \times 100. \qquad (6.1a)$$

Since at constant volume and temperature, the pressure is proportional to the mass, we may write

Relative humidity

$$= \frac{\text{unsaturated vapour pressure (U.V.P.) at room temperature}}{\text{saturated vapour pressure (S.V.P.) at room temperature}}$$

The present unsaturated vapour will exert saturated vapour pressure at the corresponding dew-point because the present vapour will become saturated at the dew-point. Hence

$$\text{Relative humidity (R.H.)} = \frac{\text{S.V.P. at the dew point}}{\text{S.V.P. at t°C}} \times 100. \qquad (6.1b)$$

The reason for attaching so much importance to the relative humidity is that our sense of dryness and dampness does not depend on the absolute humidity but on relative humidity and it affords information as to the likelihood of rain. When the relative humidity s low, we feel dry and when it is high we feel damp. We feel comfortable neither in a hot and highly humid atmosphere nor in a cold and dry atmosphere. The optimum temperature is 24-25°C and the optimum relative humidity is 60-65%. The air-conditioning of auditoriums, theatres, cinema halls etc. means the artificial maintenance of temperature and relative humidity at the above specified values. A very high percentage of relative humidity is an indication of the likelihood of rain in near future.

The record of the relative humidity is also useful in the public Health Department to forecast the spread of some diseases as they thrive in damp atmosphere. In cotton mills the artificial maintenance of relative humidity is necessary because weaving and spinning can be conducted satisfactorily only when the air is comparatively damp. For this reason the natural damp climate of Lancashire in England and Bombay in India have been found suitable for the development of the cotton industry.

6.3. The Hygrometers and Determination of Relative Humidity

Hygrometers are instruments to measure the hygrometric state defined by the relative humidity of atmosphere. There are

four types of hygrometers :

1. *Dew-point hygrometers* : (i) Daniell's hygrometer,
(ii) Regnault's hygrometer.
2. *Wet and Dry Bulb hygrometer.*
3. *Chemical hygrometer*
4. *Hair hygrometer.*

1. DEW-POINT HYGROMETERS

(i) *Daniell's hygrometer.* This hygrometer consists of two glass bulbs A and B joined by a wide tube bent twice at right angles. The bulb A contains ether and the bulb B contains only vapour of ether. There is complete evacuation inside the bulbs. Before sealing the apparatus a sensitive thermometer t is left in A. The bulb B is covered with muslin. The thermometer T, fixed on the stand of the apparatus, records the temperature of the atmosphere at the time of performing the experiment. The bulb A is silvered or gilt within to ascertain the appearance or disappearance of dew. It becomes dull when dew is formed on it and becomes bright when dew disappears.

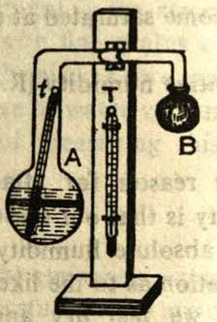


Fig. 6.1

To determine the dew-doint and hence the relative humidity, some ether is poured on the muslin. Evaporation of ether cools the bulb and consequently the ether vapour inside it is condensed causing a drop of vapour pressure over the surface of ether in A. This causes the ether in A to evaporate and A becomes colder. The temperature of A is reduced this way until dew is formed when the silver lining over A becomes dull. The temperature recorded by t is noted. Now the process of cooling is discontinued and the temperature when the dew disappears, i.e., the silver lining regains its original brightness is again noted. The mean of the two temperatures is the dew-point. The reading of the thermometer T is also noted. This gives the temperature of the atmosphere. The relative humidity is given by,

$$\text{Relative humidity} = \frac{\text{S.V.P. at the dew-point}}{\text{S.V.P. at the temperature of air}} \times 100.$$

This hygrometer has the following inherent defects : (i) ether evaporating from B changes the hygrometric state of the air, (ii) it is very difficult to ascertain when dew exactly appears and disappears as there is no provision for comparison, (iii) inside A ether evaporates mostly at the surface of ether and hence the temperature throughout is not the same as there is no provision for stirring the liquid, (iv) glass being a bad conductor, the temperature outside is not the same as that inside.

(ii) *Regnault*'s *hygrometer*. This hygrometer consists of a test tube A, the lower part of which is made of silver. This tube contains ether and it is connected to an aspirator D by a long rubber tube. A glass tube T goes down to the bottom of the tube. A' is a replica of A which serves as standard. Thermometers t' and t are inserted in A and A' to record the dew-point and the temperature of the atmosphere.

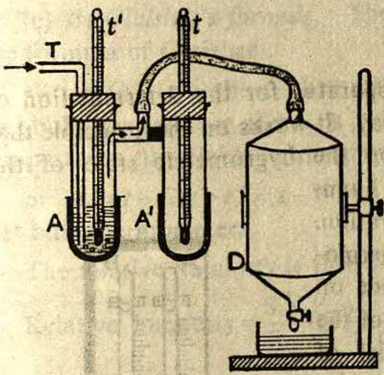


Fig. 6.2

To proceed with the actual experiment, first the aspirator is filled with water and ether is poured into A. The aspirator is then opened. A current of air is drawn through the ether which causes rapid evaporation. The temperature of the ether, therefore, falls rapidly as evaporation causes cooling. From a distance a telescope is focussed on the lower part of A and A' and their shining is keenly observed. When dew is formed on A, the silver surface becomes comparatively dull with respect to A'. The moment its shining is lost, the reading of t' is noted and the aspirator is shut off at once. When dew disappears, i.e , shining is restored, the reading of t' is again noted. The mean of the two readings is the dew-point. The reading of the thermometer t is also noted. This gives the temperature of air. The relative humidity is then given by

$$\text{Relative humidity} = \frac{\text{S.V.P. at dew-point}}{\text{S.V.P. at air temperature}} \times 100.$$

ADVANTAGES OF REGNAULT'S HYGROMETER

(i) Silver being a good conductor of heat, the temperature inside and outside is almost the same; (ii) due to bubbling of air through ether there is automatic stirring of the liquid; (iii) the dummy tube facilitates the observation of the appearance and disapearance of dew more confidently; (iv) the rate of evaporation can be controlled more effectively by controlling the flow of water in the aspirator; (v) observations being taken from a distance by a telescope, the hygrometric state of air is not affected by the breath of the experimenter.

2. DRY AND WET BULB HYGROMETER

This hygrometer is a reliable apparatus for the determination of the relative humidity in an easier way. It works on the principle that the rate of evaporation depends on the hygrometric state of the atmosphere. The less the relative humidity, the greater is the rate of evaporation.

This consists of two mercury thermometers, placed vertically side by side on a wooden board. The bulb of one of the thermometers is covered with a wick of muslin, which is always kept wet by dipping it in water contained in a small bottle. This is the wet thermometer. The bulb of the other thermometer is kept dry. Due to continuous evaporation, the wet thermometer always records a temperature lower than the dry thermometer. The difference of temperatures depends on the hygrometric state of the atmosphere. The drier the air, the quicker is the evaporation and so the greater is the difference between the readings of the two thermometers.

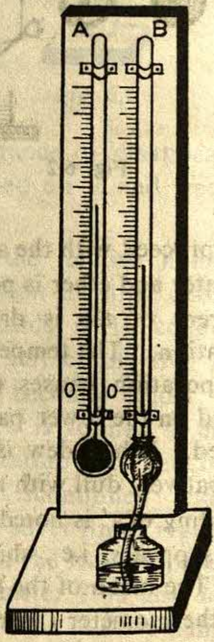


Fig. 6.3

To find the relative humidity, the readings of the dry and wet thermometers are noted and the relative humidity is calculated in one of the following ways.

(a) *By tables.* Every physical data book provides a table giving

directly the relative humidity for various values of the dry bulb temperature and the depression of the wet bulb temperature.

(b) *By formula.* If t is the temperature of the dry bulb and t' that of the wet bulb in degree celsius, then the S.V.P at the dew-point is given by

$f'=f-\cdot 00077(t-t')\,H,$ where f is the S.V.P. at the dry bulb temperature t°C and H is the atmospheric pressure in mm. The relative humidity is then given by

$$\text{Relative humidity}=\frac{f'}{f}\times 100.$$

(c) *By Glaisher's formula.* The dew-point (t_d) is calculated from the formula of Glaisher

$t-t_d=G(t-t')$ where t is the dry bulb temperature and t' is the wet bulb temperature and G is a factor (dependent on the dry bulb temperature t) available in any physical data book.

or $t_d=t-G(t-t')=t-G\,\triangle t$ where $\triangle t$ is the depression of the wet bulb thermometer.

The relative humidity is then calculated from the formula

$$\text{Relative humidity}=\frac{\text{S.V.P. at } t_d\text{°C}}{\text{S.V.P. at } t\text{°C}}\times 100.$$

3. CHEMICAL HYGROMETER

This hygrometer measures the mass of water vapour present in any volume of air and hence the relative humidity is calculated directly from the basic definition.

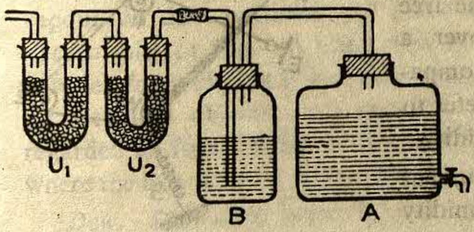


Fig. 6.4

The hygrometer consists of two U-tubes containing phosphorus pentoxide or anhydrous calcium chloride. It is connected to a bottle B containing conc. H_2SO_4 and this bottle in its turn, is connected to an aspirator A.

First the aspirator is filled with water and the U-tubes are detached from the bottle B and weighed on a balance. The U-tubes are then attached to B and water is run out from the aspirator

whereon a slow current of air is drawn through the U-tubes. During its passage through the tubes the water vapour present is absorbed by the chemicals in them. When the entire water runs out of the aspirator, the tubes are detached and weighed again. The difference in the two weights gives the mass of water vapour present in the volume of air equal to the volume of water run out from the aspirator. Next the same experiment is repeated, but this time air is bubbled through water contained in a bottle and then passed through the U-tubes. The same volume of water is run out from the aspirator slowly so that the same volume of air is drawn through the U-tubes. This time air is thoroughly saturated due to slow bubbling through the water. Finally, the tubes are detached and are again weighed. The difference between the second and third weights gives the mass of water vapour required to saturate the same volume of air. Hence

$$\text{Relative humidity}=\frac{W_2-W_1}{W_3-W_2}\times 100.$$

4. HAIR HYGROMETER

The principle of this hygrometer is the fact that hair slightly increases in length when moist.

A clean dry hair is tied to a fixed support E at one end and the other end is attached to a fixed light spring S after passing over a grooved wheel W. A light pointer is attached to the wheel. The free end of the pointer moves over a scale which is graduated by comparison with a standard hygrometer to read directly the relative humidity.

The advantage of this instrument is that it reads the relative humidity directly but its sensitivity is poor.

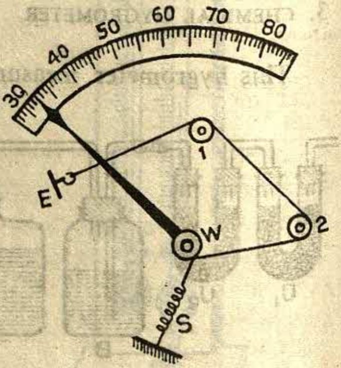


Fig. 6.5

6 4. Natural Phenomena (Clouds, Rain, Snow, Hail, Frost, Fog or Mist)

The condensation of water vapour in the atmosphere gives rise to natural phenomena such as mist or fog, dew, clouds and rains. Water may also condense into the solid form giving rise to such phenomena as hoar-frost. snow, and hailstorms.

Clouds. Moist air is lighter than dry air. As such it tends to go higher up. In doing so it is cooled, firstly due to fall in temperature with height, and secondly due to expansion because of fall of pressure with height. The latter is usually the more dominant factor. When the moist air is chilled sufficiently it becomes saturated with water vapour. A little more cooling leads to formation of tiny droplets. Clouds are nothing but these droplets suspended high up in the atmosphere.

Clouds are of different types. The dense dark rain-clouds called *nimbus clouds* are clouds formed at comparatively **lower heights.** The white clouds known as *cirrus clouds* are due to formation of minute ice particles at **great heights.** When an ascending column of moist air condenses at the top at **moderate heights** the cloud formed is called *cumulus cloud*.

Rain. Dense clouds at lower levels are responsible for rain. If cooling of the atmosphere takes place much below the saturation limit, the droplets of the clouds (nimbus clouds) may tend to merge with each other and form drops of water. Due to gravity the drops fall downward with increasing velocity. During fall water vapour of the successive layers condenses on the cold drop which thus grows in size as it falls. The ultimate size is however limited by the gravitational pull and the air resistance. The drops ultimately fall with a limiting velocity which is larger for drops of bigger sizes.

Mist or fog. If cooling of the atmosphere **near the surface** of the earth due to loss of heat by radiation be sufficient to cross the limit of saturation, the situation favours the condensation of water vapour on dust particles and other centres of condensation giving rise to mist or fog. A thick mist is called fog. So the distinction between fog and mist is one of degree and not of kind. Mists which do not allow objects at a distance of 1 km or less to be seen are regarded as fogs. Mist or fog is better formed in industrial areas where the atmosphere is charged with smoke and dust particles.

Dew. Dew is the water formed by condensation of water vapour directly on the objects such as grass, green leaves of trees, stones etc. due to their cooling below the dew-point. When the sky is clear coloured objects such as grass, green leaves etc. which are good radiators of heat lose heat by radiation below the dew-point. Water vapour is, therefore, condensed on them. Dew formation is favoured further if there is no wind and the object is a bad conductor of heat

Hoar-frost, Snow ana Hail-storm. If the temperature of the earth's surface and of the objects on it rapidly falls below 0°C before air reaches the dew-point, the water vapour in contact with the surfaces directly turns into ice and the surfaces are fast covered with white ice called hoar-frost or simply frost. This is an extreme state of the condition that favours formation of dew. Thus hoar-frost is the counterpart of dew when the cold is so extreme that water vapour is directly converted into ice crystals on coming in contact with cold objects on the surface of the earth.

Similarly, snow is the counterpart of rain when the cold is so intense that minute particles freeze into ice particles which fall on the earth under gravitational pull.

If the raindrops already formed are frozen into ice balls, the result is hail.

Examples

1. *A cubic metre of air at 30°C and 80% relative humidity is cooled to 5°C. Find the quantity of vapour which will be condensed into water. The maximum pressure of aqueous vapour at 30°C = 31·6 mm and at 5°C = 6·5 mm. The density of saturated water vapour at any temperature and pressure = 5/8 times density of dry air at the same temperature and pressure. Density of dry air at STP = 1·293* kg m^{-3}.

Sol Let m be the mass of 1 cubic metre of saturated vapour at 30°C and 31·6 mm of mercury. The volume of the same mass of vapour at STP is $\frac{273}{303} \times \frac{31{\cdot}6}{760}$ $\left(\text{on using } \frac{pV}{T} = \text{a constant}\right)$.

$$\therefore \quad m = \text{volume} \times \text{density} = \frac{273}{303} \times \frac{31{\cdot}6}{760} \times \left(\frac{5}{8} \times 1{\cdot}293\right) = {\cdot}03027 \text{ kg.}$$

Let m' be the mass of vapour present in cubic metre of air at 30°C.

Then $80 = \frac{m'}{m} \times 100$, or $m' = {\cdot}8 \times 03027 = {\cdot}02422$ kg.

Let m'' be the mass of saturated vapour at 5°C and 6·5 mm of mercury. Then $m'' = \frac{273}{278} \times \frac{6{\cdot}5}{760} \times \left(\frac{5}{8} \times 1{\cdot}293\right) = {\cdot}00679$ kg.

$$\begin{aligned} \therefore \quad \text{mass of vapour condensed} &= {\cdot}02422 - {\cdot}00679 \\ &= {\cdot}01743 \text{ kg} \\ &= 17{\cdot}43 \text{ gm. } \textbf{Ans.} \end{aligned}$$

2. *Water of mass 200 gm is sprayed into a dry room of 50 m^3 at 30°C and 760 mm. What will be the relative humidity of the air in the room? (S.V.P. at 30°C = 31·6 mm and density of dry air at STP = 1·293 kg m^{-3})*

Sol. Let m be the mass of vapour required to saturate the room at 30°C. Let V be the volume of 50 m³ of saturated vapour at 30°C and 31·6 mm when reduced to STP. Then $\dfrac{760\times V}{273}=\dfrac{50\times 31{\cdot}6}{273+30}$

or
$$V=\frac{50\times 31{\cdot}6\times 273}{760\times 303}$$

$$\therefore \quad m=\frac{50\times 31{\cdot}6\times 273}{760\times 303}\times\left(\frac{5}{8}\times 1{\cdot}293\right)=1{\cdot}5137 \text{ kg.}$$

$\therefore$ Relative humidity $=\dfrac{\cdot 2}{1{\cdot}5137}\times 100=13{\cdot}2\%$. **Ans.**

3. *The temperature of the air in a closed space is 15°C and the dew-point 8°C. If the temperature falls to 10°C, how will the dew-point be affected? (S.V.P at 7°C = 7·49 mm and at 8°C = 8·02 mm)*

Sol. Since it is a closed space, volume is a constant.

$\therefore$ pressure $\propto$ temperature ($\because$ $p\propto T$ when V is a constant)

$$\therefore \quad \frac{\text{U.V.P at 15°C}}{\text{U.V.P. at 10°C}}=\frac{273+15}{273+10}.$$

But U.V.P. at 15°C = S.V.P. at 8°C ($\because$ 8°C is the dew-point)
= 8·02 mm (given)

$\therefore$ U.V.P. at 10°C $=\dfrac{8{\cdot}02\times 283}{288}=7{\cdot}88$ mm.

The temperature where S V.P. is 7·88 mm will be the new dew-point. 7·88 lies between 7·49 mm and 8·02 mm and therefore new dew-point lies between 7°C and 8°C.

Change of S.V P. by $(8{\cdot}02-7{\cdot}49)\equiv$ change in temperature by 1°C

$\therefore$,, ,, ,, by $(8{\cdot}02-7{\cdot}88)\equiv$,, ,, ,, $\dfrac{8{\cdot}02-7{\cdot}88}{8{\cdot}02-7{\cdot}49}$

$$=\frac{\cdot 14}{\cdot 53}=\cdot 264$$

$\therefore$ the new dew-point $=8-{\cdot}264=7{\cdot}736$°C. **Ans.**

QUESTIONS

(A)

1. The density of saturated vapour at a given temperature and pressure bears a constant ratio to the density of dry air at the same temperature and pressure. This constant ratio is (a) 3 : 5, (b) 5 : 8, (c) 8 : 5, (d) 5 : 3.

2. (a) Cloudless windy nights, (b) cloudy windy nights, (c) cloudless calm nights, (d) cloudy calm nights are more favourable for the formation of dew.

3. The conditions favouring the formation of dew are (a) clear sky, (b) an absence of wind, (c) good radiating capacity, (d) poor conductivity, (e) height of the object above the earth. On account of which one of the above conditions is dew formed on blades of grass and not on the leaves of trees ?

4. The optimum relative humidity for comforts is between (a) 30–35%, (b) 40–45%, (c) 50–55%, (d) 60–65%.

5. The optimum temperature at which human being feels comfortable is between (a) 0–4°C, (b) 15°–16°C, (c) 24–25°C, (d) 30–31°C.

[Ans 1. (b), 2. (c), 3. (e), 4. (d), 5. (c).]

(B)

1. Explain dew, hoar-frost and hail.

2. Explain clouds, fog and dew.

3. Describe a hair hygrometer.

4. Define (a) Dew-point and (b) Relative humidity. Discuss the importance of relative humidity.

5. Explain dampness and dryness.

(C)

1. What is a hygrometer ? Describe Regnault's hygrometer and show how you would use it to determine the relative humidity in the laboratory.

2. Define dew-point. Describe an accurate method of determining the dew-point in the laboratory.

3. Describe the wet and dry bulb hygrometer and explain the method of finding the relative humidity with it.

4. Describe a chemical hygrometer and show how you would use it to determine the relative humidity in the laboratory. What use is made of this instrument in weather forecasting ?

(D)

1. On a day with temperature 27°C an experiment with a chemical hygrometer showed that saturated air contained ·0253 gm of water vapour per litre of air. Calculate the saturated vapour pressure of water at 27°C. (The density of dry air at STP=1·293 kg m^{-3}) (Ans. 26·13 mm)

2. Determine the dew-point when air at 15°C is saturated 2/3 with water vapour. The boiling points of water at the pressures of 7, 9, 11, 13 mm are 6°C, 10°C, 13°C and 15°C respectively. **(Ans. 9·34°C)**

3. Calculate what fraction of the mass of water vapour contained in air would condense if the temperature of the air originally at 30°C suddenly falls to 10·2°C, the relative humidity at 30°C being 60%. (S.V.P of water at 30°C=31·17 mm of mercury and at 10·2°C=9·35 mm of mercury) **(Ans. ·5)**

4. Calculate the mass of one litre of moist air at 32°C and 758·2 mm, the dew-point being 15°C. (S.V.P. at 15°C=12·7 mm, Density of dry air at S.T.P.=1·293 kg m^{-3}. The ratio of density of saturated aqueous vapour to that of dry air at any temperature and pressure=5 : 8) **(Ans. $1·1473 \times 10^{-3}$ kg)**

5. Calculate the temperature at which dew will be deposited when the hygrometric state of the air at 20°C is 50% relative humidity. Consult any constant data book for S.V.P's. **(Ans. 10·1°C)**

6. At present the relative humidity is 60% and temperature is 30°C. If the temperature falls to 25°C, calculate the dew-point. You are free to consult Regnault table from any constant data book. **(Ans. 21°C)**

7. If the temperature of air whose relative humidity is 60% falls from 20°C to 5°C, calculate the fraction of the mass of water vapour contained in air which will condense into drops. (Saturated vapour pressure of water at 20°C=17·5 mm, and at 5°C=6·5 mm) **(Ans. ·381)**

(E)

1. Why a glass tumbler cloud over on the outside when 'ice cold' water is poured into it ?

2. A hot day in Puri causes greater discomfort than equally hot day in Delhi. Why ?

3. Wet clothes are usually seen to dry sooner in cold weather than in the rainy season, though the temperatutre in the latter case is higher. Why ?

4. Dew is formed on blades of grass and not on the leaves of trees. Why ?

5. Just after a fall of snow, the atmosphere feels warmer. Why ?

Ans. 1. formation of dew round the tumbler. 2. because of high relative humidity in Puri. 3. low relative humidity in winter. 4. because of the height of the leaves above the earth. 5. When the water vapour in the atmosphere changes into snow, it gives out its latent heat which causes a temporary rise of temperature of the atmosphere.

CHAPTER 7

TRANSMISSION OF HEAT CONDUCTION : CONVECTION : RADIATION

7.1. Modes of Transmission of Heat

There are three different ways of transmission of heat from one body to another or from one part to another of the same body. These are *conduction*, *convection* and *radiation*.

1. *Conduction.* Conduction is the process of transmission of heat from the hotter part to the colder part of a body without any bodily movement of the constituent atoms or molecules of the body. A solid is heated by this process.

2. *Convection.* Convection is the process of transference of heat from the hotter to the colder parts of a body by the actual bodily movement of the constituent atoms or molecules of the body. A liquid or gas is mostly heated by this process.

3. *Radiation.* Radiation is the process of transference of energy (heat) from one body to another irrespective of their temperature through vacuum. The heat of the sun is received on the earth's surface by this process.

7.2. Conduction : Thermal Conductivity and Thermometric Conductivity

A solid consists of vibrating atoms at fixed positions called lattice sites. When one end of solid is put in contact with a hot body (source of heat), the agitated atoms of the source force the atoms of the solid to vibrate more vigorously, and these atoms force the neighbouring atoms to vibrate vigorously and so on. This way energy is transmitted from one end to the other. If the solid contains free electrons, then these electrons also carry the heat energy. In pure metals electrons carry most of the energy and the transport through atoms is very small This is why all good electrical conductors are also good thermal conductors. In electrical insulators where there are no free electrons, the only process of transmission of heat is via vibrations of atoms. A solid

having definite crystalline structure, i.e., where atoms are arranged orderly, heat is conducted in a better way. This is why bricks, wood, glass etc. are bad conductors of heat. Crystalline solids conduct more heat than amorphous solids.

Consider a slab of material of cross-sectional area A and thickness $\triangle x$. Let $\triangle Q$ be the heat that flows perpendicularly to the faces for a time $\triangle t$. Then it is found experimentally that

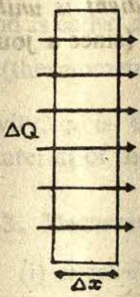


Fig 7.1

$\triangle Q \propto A$, area of the slab

$\propto \triangle\theta$, difference in temperature between the faces

$\propto \triangle t$

$\propto \dfrac{1}{\triangle x}$

$$\therefore \quad \triangle Q = \frac{\lambda A \triangle\theta \triangle t}{\triangle x}$$

where λ is a constant depending on the material of the slab. This constant is called the *thermal conductivity* of the material of the slab. Here $\dfrac{\triangle Q}{\triangle t}$ is the rate of flow of heat across the area A and hence we may appropriately call it the heat current (Q) in analogy with electric current which is the rate of flow of electricity across a surface. $\dfrac{\triangle\theta}{\triangle x}$ is the '*temperature gradient*'

$$Q = \lambda A \frac{\triangle\theta}{\triangle x}.$$

If we choose the direction of heat flow to be the direction in which x increases and wish current to be positive then we must introduce a minus sign before $\dfrac{\triangle\theta}{\triangle x}$ because heat flows in the direction of decreasing θ.

$$Q = -\lambda A \frac{\triangle\theta}{\triangle x} \qquad \text{.. (7.1)}$$

In the limit of a slab of infinitesimally small thickness,

$$Q = -\lambda A \frac{d\theta}{dx} \qquad \text{.. (7.1 a)}$$

or $$Q=-\lambda A\frac{dT}{dx} \quad \cdots \quad (7.1\ b)$$

$$(\because \quad \theta=T-273{\cdot}15)$$

If $A=1\ m^2$, $\frac{d\theta}{dx}=1°C\ m^{-1}$, then $\lambda=Q$ (numerically).

Thus thermal conductivity may be defined as *the rate of flow of heat across a surface of unit area where temperature gradient is unity.* Its unit is $Js^{-1}m^{-1}\,°C^{-1}$ or $Js^{-1}m^{-1}K^{-1}$ or watt $m^{-1}K^{-1}$ (since a joule per second is a watt).

We can rewrite Eq. 7.1 as

$$Q=\frac{-\Delta\theta}{\frac{1}{\lambda}\left(\frac{\Delta x}{A}\right)}.$$

Comparing this equation with Ohm's law $\left(I=\frac{V}{R}\right)$, we have $\frac{\Delta x}{\lambda A}$ in place of R. This is called the thermal resistance of the slab. Thus

$$R_{thermal}=\frac{1}{\lambda}\cdot\frac{\Delta x}{A} \quad \left(\text{cf. } R_{electrical}=\frac{1}{\sigma}\cdot\frac{\Delta x}{A}\right).$$

Because of this similarity all the electrical laws, viz., Ohm's law, Kirchoff's point rule and loop rule can be extended to combination of conductors in the 'steady state'.

Let us now consider a metal bar whose one end has just been placed in contact with a source of heat. In the beginning the heat reaching each elementary layer in a short interval Δt due to conduction is partly absorbed by the layer itself on account of which its temperature θ rises by $\Delta\theta$, partly lost to the environment by radiation at a rate proportional to the excess of its temperature above that of the environment and the balance is passed on to the next layer. Thus in the beginning each layer records a rise in temperature. This condition of the body when the temperature of its different parts record a rise in temperature is called *variable state.* After some time a stage is reached when the rate of supply of heat by conduction to

any layer becomes equal to the rate of loss of heat by radiation to the environment and by conduction to the next layer. The temperature of the layer then becomes stationary. This is called the *steady state*. Mathematically, in the steady state $\frac{d\theta}{dt}=0$, i.e.. the rate of rise of temperature is zero.

The rapidity with which temperature changes take place in a given rod is called *diffusivity* or *thermometric conductivity*. This is given by h (thermometric conductivity) $=\frac{\lambda}{\rho s}$ where λ is the thermal conductivity, ρ is the density and s is the specific heat capacity of the material of the rod.

7.3. Measurement of Thermal Conductivity

(i) *Ingen-Hauz's experiment* (*A comparative method*). A simple method of comparing the conductivity and thermometric conductivity of different substances was devised by Ingen-Hauz.

A number of metal rods of the same diameter are introduced into the side holes of a trough. All the rods are previously coated with a uniform coating of wax. Boiling water is then poured into the trough. The wax melts to different lengths along different bars. In bismuth the temperature wave is observed to travel faster than in iron but in the long run wax melts over a longer distance in iron than in bismuth. This means that the thermometric conductivity of bismuth is greater than that of iron but thermal conductivity of iron is greater than that of bismuth. The thermal conductivities are proportional to the square of the lengths up to which the wax melts.

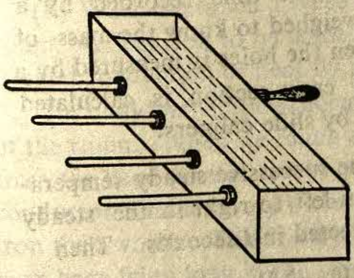


Fig. 7.3

or
$$\frac{l_1^2}{\lambda_1}=\frac{l_2^2}{\lambda_2}=\frac{l_3^2}{\lambda_3} \quad .. \quad (7.3).$$

(ii) *Searle's method* (*an absolute method*). In this method, devised by G. F. C. Searle, a thick bar of the test material is taken and one

end is placed inside a steam chest. A copper tube is wound round the bar at the other end. A steady flow of water is passed through it. The temperatures of the in-coming and out-going water are measured by sensitive thermometers introduced at the entrance and the exit of the tube. Two holes are drilled into the bar and are filled with mercury such that the thermometers inserted into them may be in good thermal contact. The whole apparatus is well lagged with layers of wool or felt.

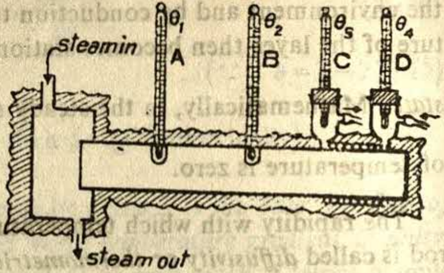


Fig. 7.4

In the actual experiment steam is passed through the steam chest and a slow and steady flow of water from a constant 'pressure head' arrangement is made through the copper tube. All the four thermometers show a gradual rise in temperature. After some time, the temperatures become stationary when the *steady state* is reached. The readings of all the four thermometers are noted after the steady state is attained. A dry empty beaker is weighed and the outgoing hot water is collected in it in a certain interval of time recorded by a stop-watch. Later the beaker is again weighed to know the mass of water collected. The distance (d) between the holes is measured by a divider and a metre scale. The area of cross-section is calculated after measuring the diameter of the rod by slide calipers.

Calculation. Let θ_1, θ_2, θ_3 and θ_4 be the successive steady temperatures recorded by the thermometers from left to right in the steady state and let m be the mass of water collected in t seconds. Then

$$\text{Heat conducted through the rod} = \frac{\lambda A(\theta_1-\theta_2)t}{d}.$$

Heat absorbed by water $= ms\ (\theta_3-\theta_4)$ where s is the sp. heat capacity of water. In the 'steady state', the heat conducted is absorbed by flowing water.

$$\therefore \qquad \frac{\lambda A(\theta_1-\theta_2)t}{d} = ms\ (\theta_3-\theta_4) \qquad .. \quad (7.4).$$

This equation gives λ.

7.4. Some Interesting Facts on Conductivity of Metals and Liquids

(a) Place a piece of wire gauge upon the flame of a Bunsen burner. The flame burns only **below** the burner and does not pass through the gauge. Now turn off the gas supply and place the gauge a little above the burner. When the gauge is cold, turn on the gas supply and light the gas **above** the gauge. The flame burns above and it does not travel down the gauge. The reason in either case is the easy conduction of heat by the wire-gauge. A substance will burn only when it attains a particular temperature called *ignition temperature*. In the above cases the metal wire conducts away the heat so rapidly that the temperature of the gas on the other side does not rise high enough to ignite the gas. This fact is utilised in the construction of Davy's Safety lamp used in mines as a safety device against the emanation of explosive gases. It consists of an oil lamp, the flame of which is surrounded by a cylindrical wire-gauge. The lamp is lighted with a small flame and taken inside a mine. In case there is emanation of some explosive gas, it enters into the lamp through the gauge and the flame becomes bigger in size and its colour also changes, but the heat of combustion of gas is conducted away by the wire-gauge and so the explosion of the gas outside is prevented. The miners get the danger signal from the character of the flame.

(b) When we touch a piece of iron and wood placed in the same room, we feel iron to be colder than the wood. Here also the reason is good conduction of heat by iron.

Our body temperature (normally 37°C) is generally greater than that of the room. When we touch an iron piece, heat is rapidly conducted from the hand (hotter body) to the iron, and wood being a bad conductor conducts very little heat. So we lose more heat by touching iron than wood. This is why iron appears colder. But when both have been lying long in the sun and then they are touched, the iron piece appears hotter than the wooden piece. This can be explained exactly in the same way.

(c) Take a compound rod one-half of which consists of brass and the other half of wood. Wrap a piece of paper tightly round the rod and hold the middle portion over a Bunsen burner. It is observed that the paper over the wooden portion is scorched long before any effect is produced on the other portion which is on the brass. Brass conducts away heat and the paper is not affected, but wood being a non-conductor is not able to conduct away heat and so

the paper on it is scorched.

(d) Liquids and gases (except hydrogen and helium) are bad conductors of heat. Poor conductivity of l'quids can be demonstrated by the following simple experiment.

Take a test tube and fill it with water. Dip a piece of ice with the help of a sinker. Now heat the upper part of water with a flame. It will be found that water boils at the upper part but ice does not melt appreciably. This shows that water is a bad conductor of heat.

Our woollen clothes are warmer than cotton clothes due to poor conductivity of the air enclosed in fabric.

7.5. Convection

This is a process in which heat is transferred from the hotter part to the colder part by the actual transference of atoms or molecules of the substance. Liquids and gases are heated by this process. When a column of liquid is heated at the bottom, the lower layers suffer thermal expansion and hence there appears a density gradient with the denser layers lying above. Gravity tends to set up an opposite type of distribution. As such, the hotter portion of the liquid migrates to the upper part carrying along with it the heat that was supplied, and the colder part comes down. Thus currents are set up in the liquid called convection currents. These convection currents can be easily demonstrated by heating some water in a flask in which some potassium permanganate crystals are kept at the bottom of the flask. It is clear from the above that convection is a gravity affair. This is why in the previous experiment water was heated on top so that there might not be transfer of heat to the ice by convection.

7.6. Some Practical Applications of Convection

(*a*) *Ventilation.* The ventilation of a room is the process of expulsion of warm and impure air and the introduction of cold and fresh air into the room. For proper ventilation an outlet is necessary near the top of the room, and an inlet near the bottom of the room. The hot and impure air being light escapes through the top outlet and fresh air enters into the room through the inlet at the bottom of the room.

(*b*) *Chimney.* Smoke issuing from tall chimneys is a common sight in factories. A chimney acts through convection. Hot air and

smoke being lighter move up by the chimney while cold and heavier air is continually drawn in at the bottom. Thus a convection current is set up. The taller the chimney, the greater will be difference in density between the top and the bottom. A tall chimney therefore promotes a greater **draught**. But too tall a chimney will be of no advantage, unless there is enough fire at the bottom to keep the gas hot all the way up the chimney. Narrow chimneys are preferable to the wide ones because they can prevent downward currents of air more effectively.

(*c*) *Hot-water heating system.* In cold countries rooms are heated by convection. The principle of the heating system is illustrated by the Fig. 7.5. A pipe *D* rises from the upper part of the boiler (*B*) up to the upper part of the reservoir *R* of water at the top of the building. Another pipe *C* descends from the bottom of the reservoir up to the bottom of the boiler after passing through the rooms in a zig-zag way. Hot water from the boiler rises up the pipe *C* because its mouth is nearer to the heating system and is cooled on the way to the reservoir giving its heat to the rooms. Cold water from the reservoir descends down the pipe *D* to the boiler. This way a convection current is set up in the system.

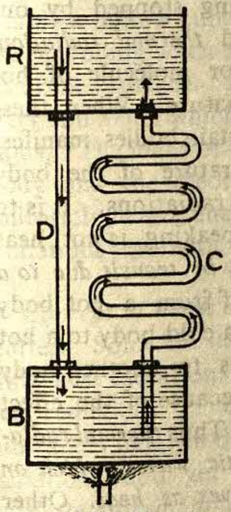


Fig. 7.5.

(*d*) *Winds and breezes.* Trade winds, land breeze and sea breeze are consequences of convection in nature. The equatorial belt of the earth is hotter than other regions. Air of this belt, therefore, becomes lighter and rises up. Cold air from North and South rushes to take up the place of the hot air. Due to the rotation of the earth from West to East this natural current of air actually flows from the North-Eastern direction in the Northern hemisphere and from South-Western direction in the Southern hemisphere. These are called trade winds.

Land is a better absorber of heat and has a higher thermometric conductivity than water. As a result of this, land attains a higher temperature during the day time than water. Air over land, therefore, becomes lighter and rises up. Cold air from the seas, oceans or lakes, whatever they be near the land, rushes to the land. This is called sea-

breeze. A body which is a good absorber of heat is also a good radiator of heat. At night, therefore, the temperature over land goes down much earlier than that over water. Air over water being hotter rises up and cold air from land flows to the sea. This is called land breeze.

7.7. Radiation

The word radiation' is used in two senses : in one sense it is a process of transmission of energy from one body to another through vacuum and in the other sense it is used to indicate the energy that is transmitted. When we sit beside a fire, we feel hot. It is obvious that we do not get heat from the fire by conduction nor by convection. So we receive something from the fire, which being stopped by our body, makes us feel hot. This something is called *thermal radiation.* Actually, what is going on is that excited atoms or molecule of hot bodies emit waves, that are electromagnetic in nature. When these electromagnetic waves, on being absorbed by certain bodies, manifest themselves as heat in the body, i.e., the temperature of the body increases, they are called radiant heat or thermal radiations. It is to be noted that radiant heat or radiaton, strictly speaking, is not heat in the sense we understand it, i.e., it is not *energy in transit due to a difference in temperature.* Heat can flow by itself from a hot body to a cold body but can never flow by itself from a cold body to a hot body. But radiation emitted by a body flows to another body whatever be its temperature. It is called 'heat' because of the effect that it produces when absorbed by certain bodies. Thus *thermal radiations (or simply radiations) are the electromagnetic waves which on being absorbed by certain bodies manifest themselves as heat.* Other electromagnetic waves are also emitted and they are given names according to the effect they produce on being stopped. The electromagnetic waves that excite the sensation of vision are called *luminous radiations* while those producing chemical changes are called actinic radiations. So *heat radiation and light are both forms of the same radiant energy and the difference between them is a difference of frequency or wavelength, but not of kind.* In fact, there is a long chain of electromagentic waves—as long as 500 m and as short as 10^{-13} m or even less. They are not produced by the same source nor can they be detected by the same detector. The following is the chart of the entire family of electromagnetic waves. The scale of the chart reads log of wave-length of radiation. The anti-log of the scale reading

gives the wave-length in metre.

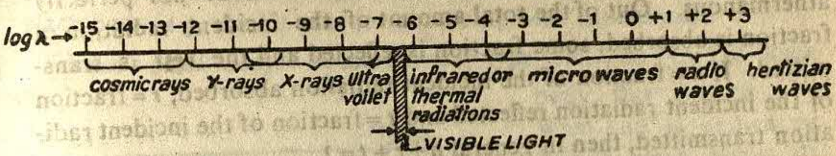


Fig. 7.6

7.8. Sources of Thermal Radiations : Concept of Black Body

Any hot body maintained at a constant temperature may serve as a source of thermal radiations. The earliest source of thermal radiation suitable for simple experiments on radiation is a Leslie cube which is simply a hollow cube of metal blackened on one side with lamp-soot. When boiling water is poured into it radiations are emitted from the blackened surface. But two special devices have been designed by Fery and Wien which are usually used as the source of thermal radiations. The ideas of these radiations came from the concept of black body and the theoretical fact that the quality of radiations inside a uniformly heated enclosure is exactly the same as that of a black body radiations (radiations from a hot black body).

Concept of black body. Consider the passage of radiation through matter. Some substance like dry air, sulphur dioxide, rock salt etc. allow heat radiations to pass through them, i.e , they do not absorb heat radiations. Such substances are called *dia-thermanous* while substances like wood, glass, metal, flesh, cloud which absorb heat radiations and get heated are called *athermanous.* Moist air is partially athermanous Thus, the moisture of the air prevents the earth from being heated too much during day time by the sun and also prevents too much cooling of the earth at night Dense clouds are completely athermanous. This is why a clear night is colder than a cloudy night. Glass is athermanous when the source is below 100°C, but if the source is very hot such as the sun it transmits about fifty per cent of the incident radiations. The glass windows of rooms thus allow heat rays from the sun to enter into the rooms, but heat rays from the objects inside the room which are at a temperature below 100°C are prevented from passing out. Glass windows thus serve as a trap to the sun's rays. This fact is utilised in 'green houses' where green plants are nursed. In fact, none of the substances

mentioned above are neither perfectly diathermanous nor perfectly athermanous. Out of the total amount of the incident radiation a fraction is absorbed, some fraction is reflected and the rest is transmitted. If a = fraction of the incident radiation absorbed, r = fraction of the incident radiation reflected and t = fraction of the incident radiation transmitted, then in general $a+r+t=1$.

Now, if for a body $r=0$, $t=0$, that is. *the body neither reflects nor transmits, rather absorbs the whole of the incident radiation, it is called a perfect black body or simply a black body.* Such a body absorbs hundred per cent of the incident radiation when cold, but when heated such a body emits heat radiations of all wave-lengths. Radiations of all wave-lengths are termed as *full or total radiation.* Thus a black body is also a full radiator. Lamp black is the nearest approach to such a body. In contrast to black bodies white bodies are defined as those for which $r=1$, $t=0$, $a=0$. Such bodies neither absorb nor transmit, rather return the whole of the incident radiation. When such bodies are heated, a very poor emission of radiations takes place from them.

It was theoretically proved by Kirchoff that the radiations inside a uniformly heated enclosure have the same characteristics as the radiations from a black body. This fact has been utilised in the design of a practical black body.

Fery's black body. It consists of metallic shell M having a small opening 'O' and a small conical projection C opposite to O. The inside of the wall of the shell is painted with platinum black and it is surrounded by a coating of fire-clay. This makes the enclosure quite impervious to heat. The projection C is necessitated to make the enclosure a perfect black body. Owing to the presence of the opening O, rays entering directly into the enclosure might be returned directly into the external space and as such the enclosure might cease to behave like a black body. The projection will reflect such rays on to the wall of the shell. After successive reflection the radiations are completely absorbed in the enclosure. Although when this enclosure will be heated to a high temperature

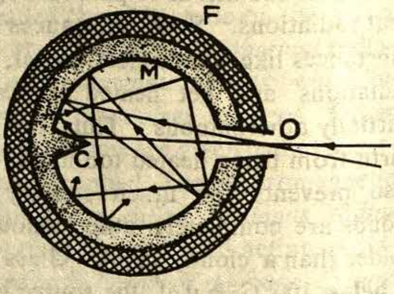


Fig. 7.7 a

there will be a current of heat radiation through the opening *O*, this black body is generally used for absorbing heat radiations.

Wien's black body. A more useful type (emission type) of black body was devised by Wien and is known as Wien's black body. It consists of a cylindrical brass chamber *A* heated electrically by means of a platinum wire shown by dots in the Fig. 7.7 b. *A* is surrounded by a porcelain tube and this tube in its turn is surrounded by a thick layer of magnesium oxide. The inner wall of the chamber is painted black to ensure quick attainment of the equilibrium state. A series of diaphragms is mounted coaxially inside the inner walls of the chamber. The aperture of the diaphragms gradually decreases and finally ends at the opening *O*. The radiation emerges out of the opening *O*. A thermocouple introduced through the other end and held in a position by a number of fixtures measures the temperature of the enclosure.

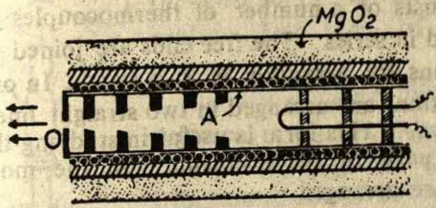


Fig. 7.7 b

7.9. Detection of Thermal Radiation

Thermal radiations may be detected by the following instruments : (i) Ether thermoscope, (ii) Thermopile, (iii) Radiometer, (iv) Bolometer, (v) Radiomicrometer, (vi) Thermistor.

(*i*) *Ether thermoscope.* This consists of two glass bulbs *A* and *B* connected by U-tube of unequal limbs. The bulb *B* attached to the shorter limbs contains ether and ether vapour and the other bulb *A* contains only vapour of the ether. The bulb *B* is blackened so that it may absorb heat quickly and soon attain the equilibrium state. When thermal radiation falls on *B*, its temperature and consequently that of ether vapour contained in it increases. This increases the pressure of vapour in *B*. As a result, the level of ether in *B* is pushed down and that in the other limb is pushed up.

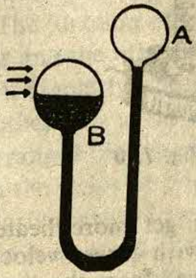


Fig. 7.8

(*ii*) *Thermopile* This is a very sensitive electrical instrument to measure and detect

thermal radiations. It works on the 'thermo-electric effect', that is, production of electrical current when junctions of two dissimilar metallic wires forming a closed circuit are maintained at different temperatures. For a given difference of temperatures between the junctions the thermo-electric current is maximum for antimony and bismuth. A thermopile consists of a number of thermocouples of antimony and bismuth joined in series The free ends are joined to a galvanometer. The junctions are arranged in two ways. In one type the hot and cold junctions are arranged in two straight lines. This is called a *linear thermopile.* This form is useful in studying the heating effect of the different parts of the spectrum In the more useful form the junctions are arranged in vertical and horizontal rows so that the hot junctions of all the rows lie on one and the same face and the cold junctions on the other face. Such thermopile is called a *surface thermopile.* The face of hot junctions is blackened and exposed to radiations and the other face is protected from radiations by means of a metallic cover. A metallic cone is generally connected to the hot face. This concentrates radiation on the pile and also protects the face from stray radiations. The pile is fitted on a stand of adjustable height (Fig. 7.10).

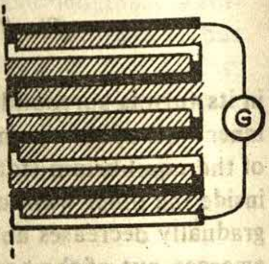


Fig. 7.9

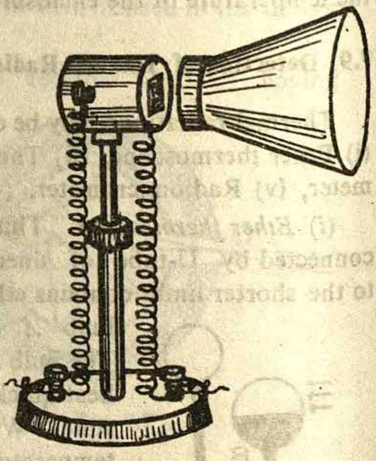


Fig. 7.10

(*iii*) *Radiometer.* This is also a very sensitive instrument for detecting thermal radiations. It consists of a glass bulb almost completely evacuated. A light wheel consisting of four vanes is capable of rotating freely about the vertical axis. One face of each of these vanes is blackened in the same way round. The pressure inside being low the mean free paths of the gas molecules are very large. When the instrument is exposed to radiation, the blackened faces of the vanes get more heated. A molecule colliding with these faces rebounds with larger velocity —the greater reaction of which pushes the vanes away from the direction of radiation (Fig. 7.11).

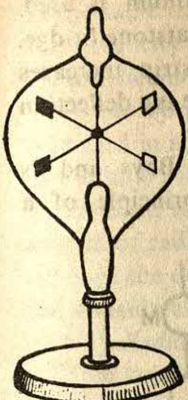


Fig. 7.11.

(*iv*) *The Bolometer.* The Bolometer though extensively used as a detector of thermal radiations by earlier scientists, is now much less used, being superseded by thermopiles. It depends for its action on the change of resistance of platinum wires when heated. Two types of bolometers have been in common use.

(*a*) *The surface bolometer* for total radiation measurement

(*b*) *The linear bolometer* for measurement of distribution of energy in the spectrum of a black body.

The surface bolometer consists of a number of thin strips of platinum joined in series. They are mounted parallel to each other so as to form a 'grid'

Surface bolometer
Fig. 7.12 a

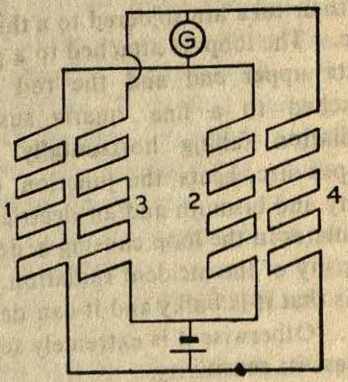


Fig. 7.12 b

For experimental purpose four such bolometers, similar in all respects, are connected in the form of a wheatstone bridge. Grid 1 is joined to grid 2, grid 2 to grid 3, grid 3 to grid 4 and grid 4 to grid 1. The junction of 1 and 2 is joined to one terminal of a galvanometer and the other terminal of the galvanometer to the junction of 3 and 4. Similarly, the junction of 2 and 3 is joined to a pole of a cell and the other pole of the cell to the junction 4 and 1. The grids 1 and 3 are so arranged that the strips in grid 3 receive the radiation passing between the strips of grid 1 and so the effect is doubled; 2 and 4 are similarly arranged and are protected from radiation (Fig. 7.12 b). The whole is enclosed in a box. In the absence of radiation the galvanometer shows no deflection. When the radiation is incident on grids 1 and 3, the galvanometer shows a deflection in proportion to the intensity of radiation.

In the linear bolometer a single thin strip of platinum is used. This is placed in the unknown resistance arm of a wheatstone bridge. When radiation falls on the strip the resistance of the strip increases and so the bridge goes out of balance. As a result there is a deflection in the galvanometer of the bridge.

(*v*) *Radiomicrometer.* This was invented by C. V. Boys and is known as Boys' radiomicrometer. It works on the principle of a thermopile in conjunction with the principle of a moving coil galvanometer. It consists of a single loop of a fine copper or silver wire suspended between the poles of a strong permanent magnet *NS* as in the suspended coil galvanometer. To the lower end of the loop two thin rods of antimony and bismuth are attached and these two in their turn are soldered to a thin copper disc. The loop is attached to a glass rod at its upper end and the rod itself is attached to a fine quartz suspension. Radiation falling horizontally on the copper disc, heats the junction of antimony and bismuth and an electric current circulates in the loop causing a deflection which depends upon the intensity of the incident radiation. The disadvantage of this detector is that it is bulky and it can detect radiation travelling horizontally. Otherwise it is extremely sensitive and none of the above can surpass its sensitivity.

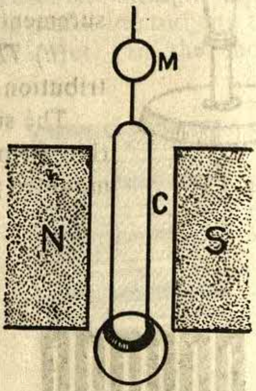


Fig. 7.13

(*vi*) *Thermistor.* Thermistor is a bead of semi-conducting materials such as oxides of cobalt, manganese etc. It works on characteristic property of semi-conductors that their resistance decreases considerably with rise of temperature. When exposed to radiation the resistance of a thermistor decreases and consequently the current in the circuit increases which can be detected by a galvanometer or microammeter. Being small in size such element can be used to detect and measure the intensity of radiation at a point.

7.10. Radiant Heat and Light Compared

(A) SIMILARITY

(*i*) *Radiant heat and light travel in vacuum with the same velocity:* At the time of an eclipse of the sun, heat and light are cut off

at the same instant, showing that the heat and light travel with the same velocity (3×10^8 ms^{-1}) in vacuum.

(*ii*) *Radiant heat and light travel in straight lines* :

Three wooden screens having narrow holes at the centre are arranged in a straight line. A red hot metal ball is placed in front of the hole of the first screen and a thermopile in front of the hole of the third screen. The thermopile records a deflection showing the arrival of radiation. Then the middle screen is displaced a little. At once the deflection in the galvanometer of the thermopile is reduced considerably. This proves the above fact.

(*iii*) *Radiant heat is reflected and refracted as light* :

Two tubes are supported horizontally at the same inclination to a vertical polished tin plate.

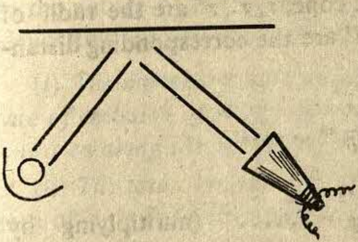


Fig. 7.14

Now a hot metal ball is placed near the end of one tube and thermopile at the end of the other tube. The thermopile galvanometer records deflection. If the first tube together with the ball is turned in any direction the galvanometer deflection falls to almost zero.

A piece of paper placed at the optical focus of a convex lens is burnt when sun's rays are collected by it. This shows that heat rays and light rays are refracted according to the same laws of refraction.

(*iv*) *The intensity of heat radiation falls off inversely as the square of the distance as the intensity of light does.*

Experiment. A rectangular tin-plate box *B*, one face of which is coated with lamp black, is filled with hot water and it serves as the source of heat radiation. A thermopile is placed at a small distance on the axis of the source. The galvanometer deflection is noted. It is now moved to a greater distance, but not s great that the projection of the cone of the thermopile on the radiating face becomes bigger than the area of the radiating surface.

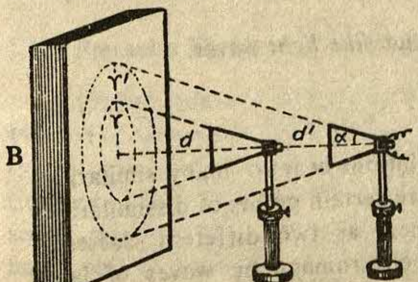


Fig. 7.15

It is observed that there is **no change** in the deflection of the galvanometer. At a given position of the detector, the deflection (θ) is proportional to the intensity of radiation due to a point source, that is, the amount of radiation passing normally through unit area in unit time and the number of point sources radiating to the receiver of the thermopile. The greater the area of the projection of the cone of the receiver on the radiating surface, the greater is the number of the point sources radiating to the thermopile. This is shown by dotted circle in Fig. 7.15.

i.e. $\theta \propto IS$ where I = intensity of radiation due to a point source and S = area of projection. Since deflection remains constant,

$$IS = \text{constant or } IS = I'S' = I''S'' \qquad \text{.. (i).}$$

If α be the semivertical angle of the cone; r, r', r'' are the radii of the projections of the cone and d, d', d'' are the corresponding distances of the detector along the axes then

$$\tan\alpha = \frac{r}{d} = \frac{r'}{d'} = \frac{r''}{d''} = \ldots\ldots$$

or, $$\frac{\pi r^2}{d^2} = \frac{\pi r'^2}{d'^2} = \frac{\pi r''^2}{d''^2} = \ldots\ldots \quad \text{(multiplying by } \pi \text{ throughout)}$$

or, $$\frac{S}{d^2} = \frac{S'}{d'^2} = \frac{S''}{d''^2} = \ldots\ldots \qquad \text{.. (ii).}$$

Combining (i) and (ii) we have

$$Id^2 = I'^2 d'^2 = I''^2 d''^2 = \ldots\ldots$$

or $I \propto \frac{1}{d^2}$. So no deflection in the galvanometer proves that the intensity decreases as the square of the distance.

(*v*) *Heat waves interfere and diffract exactly in the same way as light.*

(*vi*) *Heat waves exhibit polarisation like light waves.*

(B) DISSIMILARITIES

Though thermal and optical radiations bear so many similarities with each other, even then there are certain points of dissimilarities which clearly distinguish between them as two different classes of radiations. Thermal radiations are electromagnetic waves of large wavelengths whereas optical radiations are eloctromagnetic waves of comparatively shorter wavelengths. The photons of heat are less

energetic than photons of light waves, because energy packets (photons) contain an energy '$h\nu$' where h is the universal Planck's constant and ν is the frequency of the wave. Light waves can affect photographic plates, liberate electrons from photo-sensitive materials like sodium, potassium and cesium etc. which thermal radiations cannot easily do.

7.11. Emissive Power and Absorptive Power

EMISSIVE POWER. Any hot body emits radiations. Its emissive power is defined as *the rate of emission of energy from unit area along the normal to the surface through unit solid angle.* It depends on the nature of the surface and on the wavelength of radiation. Hence two emissive powers are considered.

(*i*) *The monochromatic emissive power* (e_λ). This is defined as the *rate of emission of radiation in the wavelength range* λ *and* $\lambda+d\lambda$ *from unit area along the normal to the surface through unit solid angle.*

(*ii*) *The total emissive power.* The total emissive power is the sum of e_λ 's for radiations of all possible values of λ.

[That is, $$E=\int_0^\infty e_\lambda d\lambda]$$

ABSORPTIVE POWER. Like emissive powers two absorptive powers are defined.

(*i*) *The monochromatic absorptive power* (a_λ). This is defined as the fraction of incident radiation in the wavelength range λ and $\lambda+d\lambda$ that is absorbed by the surface.

(*ii*) *The total absorptive power* (a). This is defined as the fraction of the total incident radiation, that is absorbed by the surface.

7.12. Prevost's Theory of Exchanges

A fundamental concept about radiation processes was given by Prevost of Geneva in 1792. Before Prevost, people had the idea that there are two radiations—hot and cold. Cold bodies emit cold radiations and hot bodies emit hot radiations. A block of ice was supposed to emit cold radiations, becasue it produced a sensation of cold and a red-hot iron ball was supposed to emit hot radiations, as it produced the sensation of warmth. If this is a fact then there must be a basis to define hot and cold bodies. We have no such basis.

This way of thinking had, therefore, no scientific background. The correct explanation was given by Prevost. He taught us for the first time that all bodies irrespective of their temperature emit only one kind of radiation, the temperature being the only decisive factor regarding its rate of emission. At low temperatures, the total emission (radiation of all wavelengths) is poor and at high temperatures the emission of radiation is very rich. The rise or fall of temperature, which is observed in a body, is due to its exchange of radiant energy with surrounding bodies, which goes on uninterrupted in equal amount even after the attainment of thermal equilibrium. This is known as Prevost's Theory of Exchanges.

When a hot body is left in an environment of lower temperature, it not only gives away energy to the environment, but also receives some radiant energy from the environment. Because of its higher temperature it emits more than what it receives from the environment and hence its own temperature falls and that of its surrounding rises till they attain a common temperature. However, the exchange of radiant energy does not cease, rather goes on uninterrupted in equal amounts. Thus according to Theory of Exchanges the temperature equilibrium between a body and its environment is not a static but a **dynamic one**. When we stand near a fire, we have the sensation of warmth, because our body, which is also a radiator, is receiving more energy from the fire than it is losing radiation. When we stand near a block of ice, we feel a sensation of cold because our body emits more radiant energy than what it receives from the ice. These considerations are quite general and may be applied to all similar phenomena.

Consequences of Prevost's theory. There are immediately certain obvious consequences which are found to be facts and these facts support Prevost's theory of exchanges.

(i) If a hot body not only emits radiation, but also receives the same from its environment it can easily be inferred that the rate of loss of radiant heat must be a function of both its own temperature and that of its environment. This inference is amply supported by the two laws of radiation—Newton's law of cooling and Stefan-Boltzmann law.

(ii) If a body is placed inside a uniformly heated enclosure, it will bring about an equalisation of temperature and, in equilibrium state it must absorb as much of the radiation as it acutally emits. The power of emission must be directly related to the power of absorption—a fact supported by Kirchhoff's law of radiation.

7.13. Laws of Radiation

(i) KIRCHHOFF'S LAW : *At any temperature, the ratio of emissive power of any body to its absorptive power is a constant and is equal to the emissive power of a black body of the same temperature.*

If e is the emissive power of a body and a is its absorptive power, then according to Kirchhoff's law

$$\frac{e}{a} = a \text{ constant} = e_{black\ body} \quad \quad (7.5)$$

$$\therefore \quad e \propto a.$$

That is, the emissive power is proportional to the absorptive power. This is why Kirchhoff's law is also stated as—'*good emitters are also good absorbers*'.

Experimental verification (*Ritchie's experiment*). The empty cylindrical metal vessels C and D are joined by a glass U-tube containing some coloured liquid. A third cylindrical vessel A is supported in between C and D. The surface of C facing A is blackened with lamp-black and also the surface of A facing D. The surfaces opposite to the blackened faces are polished. When the vessel A is filled with boiling water, the level of the coloured liquid is found to remain the same. Let e and a be the emissive and absorptive powers of the polished surfaces respectively and E is the emissive power of a black-body then

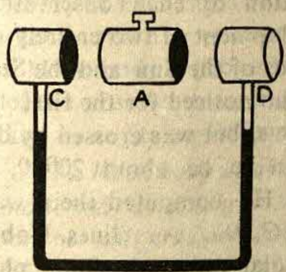


Fig. 7.16

energy absorbed by C in 1 second = whole of the energy emitted by polished surface of A in 1 second $= e$

energy absorbed by D in 1 sec $= a \times$ energy emitted by blackened surface of $A = aE$.

Since level remains same,

$$e = aE \text{ or } \frac{e}{a} = E.$$

Applications of Kirchhoff's law. The law embodies in itself two distinct relations, a qualitative and a quantitative one. Qualitatively it implies that if a body is capable of strongly emitting certain radia-

tions, it is also capable of absorbing them when cold. Quantitatively it says that the ratio of the emissive power to the absorptive power of all bodies is a constant and is equal to the emissive power of a black body.

There are numerous observations in support of the qualitative aspect of the law. When a polished metal ball having a dark spot on it is heated to about 1000°C and taken suddenly to a dark room, it is found that the black spot is shining much more brilliantly than the polished surfacc A green glass heated in a furnace is found to glow with red light when taken out and a red glass is found to glow with green light. We know that a red glass looks red because it absorbs green light strongly and reflects or transmits the red (red and green are complementary colours). Hence when heated strongly, it must emit green light strongly in compliance with Kirchhoff's law. Similarly, by Kirchhoff's law, a green glass must emit red light strongly. Besides being a basis for the explanation of such observations, Kirchoff's law is responsible for the development of two entirely new branches of science—**Astrophysics** (Physics of the Sun and the Stars) and **Spectroscopy**. It was Fraunhofer who noticed for the first time that the solar spectrum was not continuous, but was crossed by dark lines. Their number is at present known to be about 20000, but Fraunhofer noticed only 500 of them. He computed their wavelengths and designated them as A, B, C, D........lines. Nobody knew the reason of the dark lines in the solar spectrum. Some physicists, notably Fizeau, observed that if the solar spectrum is examined side by side with the spectrum of a sodium flame, the yellow lines appear in the same place as the D band of the Fraunhofer spectrum. Similar is the case with the hydrogen spectrum. The explanation of the lines was put forward by Kirchhoff on the basis of his law for radiation. According to Kirchhoff, the core of the sun is a glowing mass called *photosphere* giving off radiations of all wavelengths. This is surrounded by a relatively cool envelope called *Chromosphere* which, Kirchhoff presumed, contains elements like H_2, N_2, O_2, Na, Cu etc. Light from the photosphere passes through the chromosphere and then it reaches our spectrometer. So the elements in the chromosphere absorb selectively the radiations which they would themselves emit when heated strongly. This explains the presence of dark lines in the solar spectrum. This success gave impetus to scientists to study the spectrum of stars and a new branch of physics—Astrophysics—was opened up.

In the spectrum analysis an atom is recognised by the particular lines it emits. Nearly forty elements were added by the spectrum analysis of atoms to the list already known.

(ii) STEFAN-BOLTZMANN (TOTAL RADIATION LAW) :

In 1879 J. Stefan showed empirically that the absolute rate of emission from unit area of a black body called '*radiancy*' is proportional to the fourth power of its absolute temperature. That is, R (radiancy)$=\sigma T^4$ where σ is a universal constant called Stefan's constant. For any other surface

R (radiancy)$=e\sigma T^4$ where e, called the emissivity, depends on the nature of the surface and the temperature. For black body $e=1$. For any other surface $e<1$.

In 1884 Boltzmann gave a theoretical proof of the law from thermodynamical considerations. He showed that the law is strictly applicable to the emission from a black body. By applying Prevost's theory of Exchanges of heat radiation between two black bodies, he showed that the amount of energy lost per unit area per unit time by a black body is proportional to the difference of the fourth power of the absolute temperature of the body and its environment. This is known as Stefan-Boltzmann law The law is strictly stated as

If a black body at absolute temperature T *be surrounded by another black body at absolute temperature* T_0, *the amount of radiant energy loss per unit area per unit time* is $E=\sigma\,(T^4-T_0^4)$ where σ is a constant called Stefan's constant and its value is $5{\cdot}6697\times10^{-8}$ $\mathbf{Wm^{-2}K^{-4}}$ (watt per square metre per fourth power of temperature in kelvin).

Newton's law of cooling from Stefan-Boltzmann law. Newton's law of cooling is an approximation of Stefan-Boltzmann law for small differences of temperature between a body and its surroundings.

According to Stefan-Boltzmann law

$$E=\sigma\,(T^4-T_0^4)=\sigma\,(T-T_0)\,(T^3+T^2T_0+TT_0^2+T_0^3).$$

When the difference is small, $T\simeq T_0$

$$E=\sigma\,(T-T_0)\,4\,T_0^3 \text{ (replacing } T \text{ by } T_0)$$
$$=4\sigma\,T_0^3\,(T-T_0).$$

$\therefore\ E\propto(T-T_0)$ which is Newton's law of cooling.

**7.14. Energy Distribution of Black Body Radiation

Nothing can be more fascinating in the whole realm of physics than the study of the energy distribution of black body radiation. In

an attempt to develop a theory to explain the energy distribution curve of a black body, Planck propounded the revolutionary concept of a 'quantisation' of energy of oscillators. In this problem one sees how scientists put forward theory after theory to fit the experimental facts and finally reach a sound theory on the topic. We will now see in brief how this problem was first unsuccessfully attempted by Wien, Rayleigh and Jeans and finally Planck provided the correct explanation introducing the concept of quantisation of energy of oscillators.

The radiation from a black body was experimentally studied by a number of authors—Paschen, Lummer, Pringsheim and Rubens. Their black body was an **electrically heated carbon tube**. The radiation was dispersed by **fluor-spar prism**. A **linear bolometer** was used by them to measure the energy of the different parts of the spectrum. The curves obtained by them are shown in Fig. 7.17. The characteristics of the curves are :

(i) Every energy curve shows a peak value (E_m) corresponding to a particular wavelength (λ) which (the peak) is displaced to the longer wavelength side as the temperature decreases. (ii) Any curve corresponding to a temperature always lies outside the one corresponding to a lower temperature.

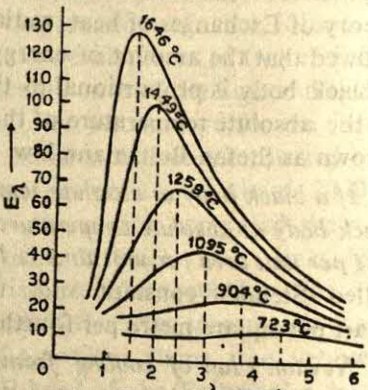


Fig 7.17

From purely thermodynamical considerations Wien obtained $\lambda\ T$=constant (b). This is called Wien's displacement law and the constant (b) is called Wien's displacement constant. Its value is $2{\cdot}898\times10^{-3}$ mK (metre kelvin). This furnishes us with a simple method of calculating the temperature of all radiating bodies including those in the heavens. For the moon $\lambda_m=14\times10^{-6}$ m and hence the temperature of the moon is $T=\dfrac{2{\cdot}898\times10^{-3}}{14\times10^{-6}}=207°\text{K}=-66°\text{C}$.

Assuming that the radiation is emitted by resonators of molecular size and that the frequency of the emitted radiation is proportional

to the kinetic energy of the resonators, Wiens obtained

$$E_\lambda = \frac{c_1}{\lambda^5} e^{-\frac{c_2}{\lambda T}}$$

where c_1 and c_2 are two constants. This is called Wien's distribution law. The plots of this law along with the experimental curves are shown in Fig. 7.18. The full line curves are experimental curves and the dotted curves represent Wien's energy distribution curves. The two curves coincide at shorter wave-lengths but deviate from each other as wave length increases. Thus the theory fails to explain black body radiation.

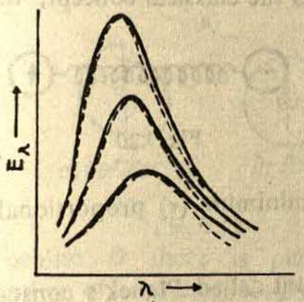


Fig. 7.18

The next attempt was made by Lord Rayleigh. But before he could finish his work, he died, and his work was completed by J. H. Jeans. Rayleigh considered heat radiation as electromagnetic waves. He computed modes of formation of stationary waves in a uniformly heated enclosure (being equivalent to a black body) and applied the principle of equipartition of energy to each mode. The result obtained by Rayleigh and Jeans is $E_\lambda = \frac{8\pi kT}{\lambda^4}$ where k is the universal Boltzmann constant. This is called the Rayleigh-Jeans law.

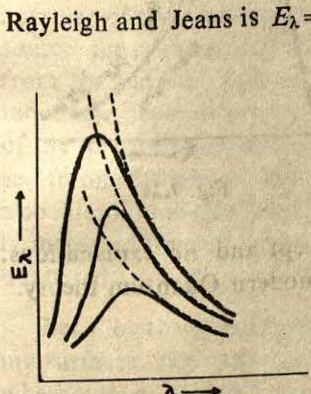


Fig. 7.19

When compared with the real curves (the experimental curves) Rayleigh-Jeans law is found to hold good for longer wave-lengths and at shorter wave-length side, what to speak of quantitative, there is not even qualitative agreement. The law requires that the radiation intensity should increase as the wave length decreases, whereas in reality the radiation intensity rises to a maximum and then falls sharply.

The correct black body radiation formula was discovered by Max Planck (1900) Planck considered the wall of an uniformly heated enclosure to contain electrical oscillators of all frequencies. Taking a clue from the result of electrodynamics that an accelerated charged particle emits radiation, Planck thought that emission of radiation inside a uniformly heated enclosure was due to accelerated charged particles of the oscillators (Fig. 7.20). In the classical concept, these oscillators could emit energy continuously. Planck put restriction on this continuous emission of radiation by the oscillators and assumed that they emit energy only when their energy increases in bundles of certain minimum (ϵ) proportional to the frequency (ν) of the oscillator.

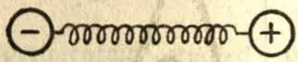


Fig. 7.20

$\therefore$ $\epsilon = h\nu$ where h is a universal constant called Planck's constant. Its value is $6{\cdot}67 \times 10^{-34}$ Js (joule second). Planck obtained

$$E_\lambda = \frac{8\pi hc}{\lambda^5} \quad \frac{1}{e^{\frac{hc}{\lambda kT}} - 1}$$

where c is the velocity of electromagnetic waves in vacuum. This is called *Planck's law*. This law fits the experimental curves remarkably well. See this remarkable resemblance in Fig. 7.21 where all the three laws are shown along with the experimental curve represented by circles.

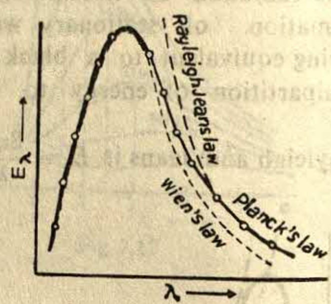


Fig. 7.21

Planck's quantum concept brought a revolution in physics. It enabled physicists to explain many experimental results for which the older concept had no explanations. Planck's law was the starting point of the modern Quantum theory.

**7.15. Radiation Pyrometer

Temperature measuring devices making use of the radiation laws are called pyrometers. These are meant for measuring very high temperatures.

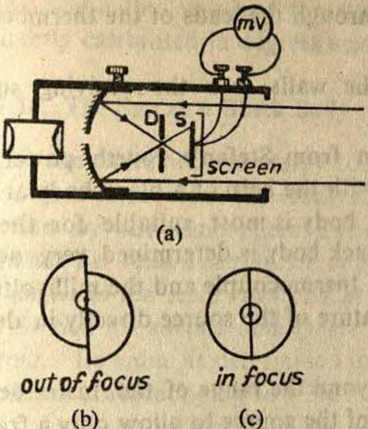


Fig. 7.22

(i) *Fery's total pyrometer*. The basic principle of this pyrometer is that total radiation emitted by a black body is proportional to the fourth power of its absolute temperature (Stefan's law).

Rays from the source are made to fall on a concave mirror (Fig. 7.22 a) which focuses them on a diaphragm D. Immediately behind D there is placed a small blackened surface. One junction of a thermocouple is soldered to the back of this surface. The mirror can be slided inside the main tube so as to focus the image of the source exactly at the aperture D. In order to enable an observer to make accurate focusing, the diaphragm D is highly polished to form a circular mirror with a central hole. The diaphragm is cut in two halves and reset at inclination of 10°. D is viewed through an eyepiece. When the image of the source is formed exactly at the surface of D, the two halves of the image of the source in the eyepiece appear undisplaced relative to each other as shown in Fig. 7.22 (c), otherwise they appear displaced as in Fig. 7.22 (b). If the aperture of the diaphragm is completely covered by the image of the source, the reading of the thermocouple is independent of the distance from the furnace. The reason for this is that the reading of the thermocouple depends only on the intensity of the image. If the distance of the furnace from the pyrometer is doubled, the intensity is decreased to one-fourth on this account, but the area of the image is also simultaneously reduced to one-fourth and hence the intensity is increased four times on this account. Thus the intensity remains unaltered.

Let T be the temperature of the source and T_0 that of the receiving surfaces. Then the voltmeter reading V is given by $V=a(T^c-T^c_0)$ where a is a constant and c is another constant ranging from 3·8 to 4·2. In fact, according to Stefan's law c should be exactly 4 but in practice it differs from 4 on account of the following :

(a) Thermo—emf is not proportional to the temperature difference.

(b) Conduction takes place through the leads of the thermocouple to the cold junction.

(c) Stray reflections from the walls into the receiving surface produce disturbances.

On account of this deviation from Stefan's fourth power law, this pyrometer is calibrated with the help of a black body at different temperatures. Wien's black body is most suitable for the purpose. The temperature of the black body is determined very accurately with the help of standard thermocouple and the millivoltmeter is calibrated to read the temperature of the source directly in degrees celsius.

To measure temperatures beyond the range of the instrument, a sectored disc is held in front of the source to allow only a fraction of the incident radiation to enter the pyrometer. If the cut off portion of the disc makes an angle θ then

$$\frac{T'^4}{T^4}=\frac{\theta}{2\pi} \quad \text{or} \quad T'=T\sqrt[4]{\frac{\theta}{2\pi}}.$$

If $\theta=4°$, then an instrument having a normal range of 1500°C can be used up to about 4600°C with the help of the sector.

(ii) *The Disappearing Filament Pyrometer.* It is essentially a telescope having a lamp at the position of the cross-wires. The objective of the telescope focuses the radiations from the source on the lamp. The lamp is viewed through the eye-piece of the telescope A red filter is used to select radiation of a particular colour. The diaphragms D_1 and D_2 limit the radiation reaching the eye. The filament of the lamp is heated by the battery and the current can be adjusted to any value by varying the rheostat R.

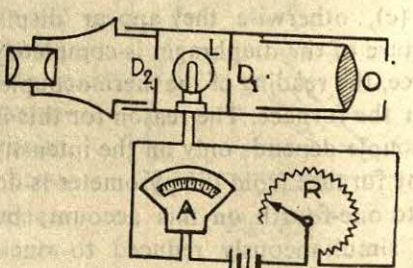


Fig. 7.23

In actual practice the heating current of the filament is varied till the filament becomes invisible against the image of the source If the filament is overheated, it shines out brightly while if the current is too low it looks black. The instrument is calibrated by direct

comparison with a standard thermocouple. The associated ammeter is directly calibrated in degrees celsius.

** 7.16. The Sun as a Black Body : The Solar Constant

The sun emits radiant energy continuously in space of which only a small fraction reaches the earth, and that too after absorption and reflection by the atmosphere. *The amount of solar radiation received per minute per square centimetre of a perfectly black surface held at right angles to the sun's rays and at the mean distance of the earth from the sun, the earth's atmosphere being absent, is called the solar constant.* Its value as determined in 1920 is 1946 calorie per sq. cm. per minute corresponding to $1{\cdot}362\times10^3$ Jm^{-2} s^{-1} or $1{\cdot}362$ kWm^{-2}.

Determination of solar constant by Angstrom's compensation pyrheliometer. This pyrheliometer consists of two thin metal strips A and B, identical in every way—one is blackened with lampblack and is exposed to the sun's rays and the other is shielded from the sun's rays by a screen. The junctions of a thermocouple are soldered to the back of these strips. The strip shielded from the sun's rays is heated electrically and the strength of the current is so regulated that there is no deflection in the galvanometer. The energy of the incident radiation is now equal to the electrical energy supplied. If V is the voltmeter reading and I is the ammeter reading then the rate of supply of electrical energy $= VI$ Js^{-1}.

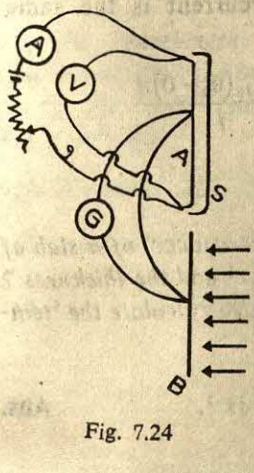


Fig. 7.24

$\therefore$ Rate of reception of solar energy $= VI$ Js^{-1}.

If A is the area of either strip then

$$S \text{ (solar constant)} = \frac{VI}{A} \text{ Jm}^{-2}\text{s}^{-1}$$

or

$$\frac{60\,VI}{4{\cdot}2\,A}\times10^{-4} \text{ cal cm}^{-2} \text{ min}^{-1}.$$

But this is not the real solar constant as there is absorption and reflection of solar energy by the atmosphere. If S is the observed value at the zenith distance Z of the sun then

$S=S_0\, a^{\sec Z}$ where S_0 is the real solar constant, a is average coefficient of transparency of the atmosphere. Then taking logarithms

$$\log S=\log S_0+\log a \sec Z.$$

Plotting the value of $\log S$ as ordinate and corresponding values of $\sec Z$ as abscissa a straight line graph is obtained whose intercept on the ordinate axis gives $\log S_0$, whence S_0 is found.

Examples :

Three metal rods, made of copper, aluminium, and brass, are each of the same length and diameter. These rods are placed end-to-end, with the aluminium between the other two. The free ends of the copper and brass rods are maintained at 100 *and* 0°C, *respectively. Find the equilibrium temperature of the copper-aluminium junction and the aluminium-brass junction.* ($\lambda_{Cu} : \lambda_{Al} : \lambda_{brass}=4 : 2 : 1$)

Sol. In the equilibrium state, the heat current is the same through all the rods.

$$\therefore \quad \frac{\lambda_{Cu}(100-\theta_1)A}{l}=\frac{\lambda_{Al}(\theta_1-\theta_2)A}{l}=\frac{\lambda_{brass}(\theta_2-0)A}{l}$$

or $\quad 4(100-\theta_1)=2(\theta_1-\theta_2)=\theta_2.$

Hence $\quad \theta_1=85{\cdot}7°C$ and $\theta_2=58{\cdot}3°C$. **Ans.**

2. *Calculate the 'heat current' and 'thermal resistance' of a slab of area* 1 m^2, *the thermal conductivity* 210 $Js^{-1}m^{-1}K^{-1}$ *and the thickness* 2 *cm, when the temperature difference is* 80°C. *Also calculate the 'temperature gradient' in the slab.*

Sol. $$Q=\frac{210\times 80\times 1}{2\times 10^{-2}}=84\times 10^{4}\ Js^{-1}. \qquad \textbf{Ans.}$$

$$R_{thermal}=\frac{1}{\lambda}\cdot\frac{l}{A}=\frac{1}{210}\cdot\frac{2\times 10^{-2}}{1}=9{\cdot}5\times 10^{-5}\ \text{SI units.} \qquad \textbf{Ans.}$$

$$\text{Temperature gradient}=\frac{80}{2\times 10^{-2}}=4\times 10^{3}\ Km^{-1} \qquad \textbf{Ans.}$$

3. *Calculate the temperature of the sun from the following data :*

earth-to sun distance $=1{\cdot}5\times 10^{11}$ m
radius of the sun $=7\times 10^{8}$ m
solar constant $=1{\cdot}36\times 10^{3}\ Jm^{-2}s^{-1}$
Stefan's constant $=5{\cdot}67\times 10^{-3}\ Wm^{-2}K^{-4}$.

Sol. Consider the sun as a black body.

Radiant energy emitted by it is $=4\pi R^2\times\sigma T^4$.

This energy falls uniformly on the inner surface of a sphere of radius d where d is the distance of the earth from the sun.

Energy falling on unit area in 1 s $= \dfrac{4\pi R^2 \sigma T^4}{4\pi d^2} = \sigma\left(\dfrac{R}{d}\right)^2 T^4$.

This is the solar constant.

$$\therefore \qquad S=\sigma\left(\frac{R}{d}\right)^2 T^4$$

$$\text{or} \qquad T^4=\frac{S}{\sigma}\left(\frac{d}{R}\right)^2=\frac{1{\cdot}36\times10^3}{5{\cdot}67\times10^{-8}}\left(\frac{1{\cdot}5\times10^{11}}{7\times10^8}\right)^2$$

$$\text{or} \qquad T^4=\frac{1{\cdot}36}{5{\cdot}67}\times10^{11}\times\left(\frac{1{\cdot}5}{7}\right)^2\times10^6=\frac{1{\cdot}36\times1{\cdot}5^2}{5{\cdot}67\times7^2}\times10^{17}$$

or $\qquad T=5760$ K. **Ans.**

4. *The wavelength corresponding to the maximum emission by the sun as observed by Wilsing is* 4820 Å. *Calculate the temperature of the sun. Given that Wien's displacement constant is* $2{\cdot}898\times10^{-3}$ *mK.*

Sol. We have $\lambda_m T=b$ by Wien's displacement law of radiation.

$$\therefore \qquad 4820\times10^{-10}\times T=2{\cdot}898\times10^{-3} \ (1 \text{ Å}=10^{-10} \text{ m})$$

$$\text{or} \qquad T=\frac{2{\cdot}898}{4820}\times10^7=6012 \text{ K.} \quad \textbf{Ans.}$$

QUESTIONS

(A)

1. In the Ingen-Hauz's experiment the length up to which wax melts is (a) proportional to the thermal conductivity, (b) proportional to the square of thermal conductivity, (c) proportional to the square root of thermal conductivity, (d) inversely proportional to the thermal conductivity.

2. A perfect black body has (a) $r=0$, $t=0$, $a=1$, (b) $r=1$, $t=0$, $a=0$, (c) $r=0$, $t=1$, $a=0$, (d) $r=0$, $t=0$, $a=0$, where a, r and t are the absorption coefficient, the reflection coefficient and the transmission coefficient respectively.

3. Of all bodies, a perfectly black body has (a) the maximum emissive power, (b) the minimum emissive power, (c) zero emissive power, (d) any emissive power.

4. The function of holes below the chimney of an oil lamp is (a) to increase the luminosity of the flame, (b) to provide an outlet to the smoke, (c) to provide

oxygen for the burning of the oil, (d) to maintain convection currents when the lamp burns.

5. The loss of heat due to radiation of a hot body is dependent on (a) the temperature of the body only, (b) the temperature of the surrounding only, (c) the difference of temperatures of the body and the surrounding, (d) the average temperature of the body and the surrounding.

6. Four identical copper cubes are painted with different types of paint. If all of them are heated to the same temperature and left in vacuum, which one would lose heat most rapidly ? The one that is (a) painted white, (b) painted black, (c) painted red, (d) painted yellow.

7. A thermopile is an instrument to measure (a) temperature, (b) temperature difference, (c) radiation, (d) conduction.

8. The instrument suitable for measuring the temperature of the sun is (a) a thermopile, (b) a thermocouple, (c) a platinum resistance thermometer, (d) a pyrometer.

9. Land breeze blows (a) from land to sea during day-time, (b) from land to sea during night, (c) from sea to land during day-time, (d) from land to sea during night.

10. Sea breeze blows (a) from land to sea during day-time, (b) from land to sea during night, (c) from sea to land during day-time, (d) from land to sea during night.

11. Two spheres having radii r_1 and r_2, but made of the same material and having identical surfaces, are heated equally. Then they are kept in similar conditions to cool. The ratio of the loss of heat by them is given by (a) r_1/r_2, (b) r_1^2/r_2^2, (c) r_1^3/r_2^3, (d) r_2^2/r_1^2.

12. Metals are good conductors of heat because (a) their atoms are very close to each other, (b) they contain free electrons, (c) their atoms sometimes collide with each other, (d) they have a definite crystalline structure.

13. A green glass heated strongly will appear (a) red, (b) green, (c) yellow, (d) white.

14. A red glass heated strongly will appear (a) red, (b) green, (c) yellow, (d) white.

[**Ans.** 1. c, 2. a, 3. a, 4. d, 5. c, 6. b, 7. c, 8. d, 9. b, 10. c
11. b, 12. b+d, 13. a, 14. b.]

(B)

1. Distinguish between 'variable' and 'steady' state.

2. Explain Prevost's theory of exchange. Mention and discuss some of the important conclusions drawn from the theory.

3. Describe a bolometer and explain its action.

4. Describe a radiomicrometer and discuss its merits and demerits.

5. What is a radiometer ?

****6.** What is pyrometry ? Describe the disappearing filament pyrometer.

**7. Write a note on Fery's total pyrometer.

**8. Explain the energy distribution in black body radiation.

**9. Discuss Wien's, Rayleigh-Jeans and Planck's theory of black body radiation.

10. Describe an experiment to show that the intensity of radiation at a point is inversely proportional to the square of the distance of the point from the source.

11. Explain a perfect black body.

(C)

1. What are the different modes of transmission of heat ?

2. Define thermal conductivity. Give its unit and obtain its dimensions. Describe with theory Searle's method of determining the thermal conductivity of a metal rod.

3. Describe the experiment you will perform to compare the thermal conductivities of metals.

4. Describe an experiment to show that good absorbers are also good emitters of radiant heat.

5. State and explain Kirchhoff's law of radiation. Discuss the importance of the law.

6. What is convection ? Explain the action of ventilation, chimney and hot water heating system.

(D)

1. If the walls of a room be 42 cm thick, the thermal conductivity of the materials of the walls be $1{\cdot}6\ Js^{-1}m^{-1}K^{-1}$ and the temperature inside the room be $10°C$ higher than outside, calculate the loss of heat per second through each square metre of the wall.

(**Ans.** $38{\cdot}1\ Js^{-1}m^{-2}$)

2. The opposite faces of an iron cube of cross-section $4 \times 10^{-4} m^2$ are kept in contact with steam and melting ice. How much ice will melt at the end of 5 minutes ? (Thermal conductivity of iron $= 84\ Wm^{-1}K^{-1}$ and the specific latent heat of fusion of ice $= 336 \times 10^3\ Jkg^{-1}$).

(**Ans.** $\cdot15$ kg.)

3. A hollow glass sphere whose thickness is 2 mm and external radius 10 cm is filled with ice and is placed in a bath containing boiling water at $100°C$. Calculate the rate at which ice melts. (Thermal conductivity of glass $= {\cdot}252\ Js^{-1}m^{-1}K^{-1}$ and L of ice $= 336 \times 10^3 J\ kg^{-1}$).

(**Ans.** $4{\cdot}7 \times 10^{-3}$kg/s)

4. If 10 cm thick ice is formed on a pond and the air temperature is $-50°C$, how long will it take for the next 1 mm of ice to form ? (Thermal conductivity of ice $= 2{\cdot}1\ Wm^{-1}K^{-1}$ and the specific latent heat of ice $= 336 \times 10^3 J\ kg^{-1}$)

(**Ans.** 5·33 min.)

5. The top of a steam chamber is a concrete slab 300 cm$\times$200 cm$\times$6 cm. A block of ice of 50 kg is melted in 15 minutes. What is the thermal conductivity of concrete ? (The latent heat of fusion of ice at 0°C is 336×10^3 Jkg^{-1}.

(Ans. 1·87 $Wm^{-1}K^{-1}$**)**

6. A 1000-watt electric iron has base area 300 cm^2 and thickness 2 cm. Calculate the difference of temperature between the inner and outer surfaces of the iron. Thermal conductivity of iron=60 $Wm^{-1}K^{-1}$). **(Ans.** 11·11°C**)**

7 Calculate the temperature of the filament of a 60-watt lamp whose length is 10 cm and radius ·1 mm. Stefan's constant is $5{\cdot}7\times10^{-8}$ $Wm^{-2}K^{-4}$.

(Ans. 2023°K**)**

(E)

1. Why is felt rather than air used for thermal insulation, even though the thermal conductivity of air is less than that of felt ? **(I. I. T. 1978)**

2. On a winter night you feel warmer when clouds cover the sky than when the sky is clear. Why ? **(I. I. T. 1974)**

3. Two thermometers are constructed in the same way except that one has a spherical bulb and the other an elongated cylindrical bulb. Which one will respond quickly to temperature changes ? **(I. I. T. 1973)**

4. Two ordinary glass thermometers are placed side by side and the bulb of one of them is coated with lamp black. How will their readings differ in the beginning when they are exposed (i) on a damp cloudy night, (ii) on a clear dry night in the cold weather, (iii) in the sun.

5. If you touch a piece of iron and a piece of wood lying exposed to the sun, which feels hotter and why ?

6. If you touch a piece of iron and a piece of wood lying in the same room, which feels colder and why ?

7. A piece of paper wrapped tightly on wooden rod is found to get charred quickly when held over a flame compared to a similar piece of paper when wrapped on a brass rod. Why ? **(I. I. T. 1974)**

8. Can you boil water in a paper vessel ?

9. Why a person prefers to wear loosely woven clothes during cold seasons ?

10. You are to serve hot tea to your guest. Would you prepare the tea with hot liquor and cold milk before he gets ready or after ?

11. Why a highly polished concave mirror is preferred to a converging lens to collect solar energy ?

12. The radiation from a body............(increases or decreases) with roughness of its surface.

13. 'Pockets' formed by the coals in a coal fire seem brighter than the coals themselves. Why ?

14. It is found that if we look into a cavity whose walls are maintained at a constant temperature no details of the interior are visible. Why ?

15. A pyrometer works on............(Boyle's law, Stefan-Boltzmann law, Charles' law).

16. A pyrheliometer is an instrument to measure..................(temperature, pressure, solar constant).

17. A harmonic electrical oscillator of frequency ν emits radiation in multiples of............($h\nu$, $k\nu$).

Ans. **1.** Air cannot prevent transmission of heat by convection. **2.** The cloud reflects the radiation back to the earth. **3.** The thermometer with elongated cylindrical bulb because it has larger area. **4.** (i) The blackened one will record higher temperature as it will absorb more heat radiation, (ii) the blackened one will show less temperature as it will radiate more, (iii) the blackened one will show higher temperature as it will absorb more. **5.** Iron because it will conduct a greater amount of heat to your body. **6.** Iron because more heat will be conducted away from your body to the iron piece. **7.** Because of bad conduction of heat by wood. **8.** Yes, all the heat will be conducted to the water in the vessel and so the paper will never reach its ignition point. **9.** Due to bad conduction of air trapped between fibres. **10.** Before he gets ready. Due to mixing of hot liquor with cold milk before he gets ready the temperature will come down and the loss of heat in a given interval of time will be less. If hot liquor is not mixed with cold milk, during the period the guest gets ready, it will lose much more heat during the same period because of its greater excess of temperature over that of its surroundings. **11.** Solar energy contains infrared and ultraviolet radiations. Ultraviolet and infrared radiations are strongly absorbed by glass. Hence highly polished concave mirror is preferable to a convex lens for collecting solar energy. Moreover in a lens there will be dispersion but in a mirror there will be no dispersion. **12.** Increases due to larger surface area. **13.** Pockets are equivalent to black bodies. Radiancy of black bodies is greater than that of any other body. This is why 'pockets' appear brighter. **14.** The walls of cavities behave like black bodies irrespective of their shape and size and hence they appear identical and they become indistinguishable from each other. **15.** Stefan-Boltmann law. **16.** Solar constant **17.** $h\nu$.

CHAPTER 8

WORK AND HEAT : MECHANICAL EQUIVALENT OF HEAT : FIRST LAW OF THERMODYNAMICS : SIMPLE APPLICATIONS OF FIRST LAW : ADIABATIC PROCESS

8.1. Heat and Work

When two bodies at different temperatures are placed in contact, some energy flows from the hot body to the cold body. *This energy in transit due to temperature difference is called Heat.* All material bodies possess energy in the mechanical form. A solid possesses energy due to rapid vibrations of its atoms at the lattice sites, a fluid possesses energy due to the random motion of its molecules or atoms and their vibratory motion. The higher the temperature of a solid or fluid the more vigorous are the vibrations of its atoms or the more chaotic is the motion of the atoms or molecules of the fluid. This energy in flow due to temperature difference is called heat. The moment it ceases to flow it is no more called heat. Internal energy differs from heat in the same way as 'rain water' differs from 'water in a lake'. Water in the form of droplets in motion is called rain. After it (rain water) mixes with lake water it loses its identity as rain water and becomes lake water. Exactly in the same way heat differs from internal energy. Heat is analogous to 'rain' and 'water in a lake' is the analog of internal energy. When the internal energy is in flow due to temperature difference it is called heat. The moment it ceases to flow, it loses its identity as heat.

The unit of heat is the kilo calorie (kcal) or the 'joule.' When we take one kilogram of water at 14·5° C (a cold body) and place it over a hot plate (a hot body), energy (internal energy of the hot plate) will flow to water due to difference in temperature between water and the plate. Here we say that heat is flowing from the hot plate to water. Due to the addition of heat the molecules of water

will be agitated more and so its temperature will rise. Let it go up by 1°C. Then the amount of energy that flows to water till its temperature rises by 1°C from 14·5°C to 15·5°C is called a kilocalorie. This is called the 15°C kilocalorie. The reason for selecting this particular temperature is that specific heat capacity of water is not a constant rather it varies with temperature. At 15°C the specific heat capacity of $H_2{}^1O^{16}$ is ·99999 of that of normal water.

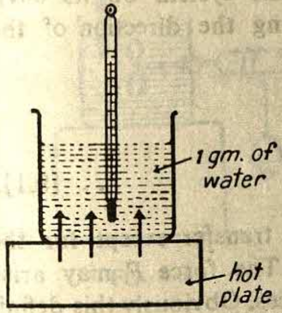


Fig. 8.1

Definition of kilocalorie. *The amount of energy that will flow from a hot body to a kilogram of water due to difference of its temperature with that of the hot body till its temperature rises by* 1°C *from* 14·5°C *to* 15·5°C *is called a kilocalorie.*

One thousandth of a kilocalorie is a **calorie**. A calorie is the amount of energy that will flow from a hot body to a gram of water due to difference of its temperature with that of the hot body till its temperature rises by 1°C from 14·5°C to 15·5°C.

In the same way 'work' is defined *as 'energy in transit' not by virtue of the temperature difference but by virtue of the change of state or configuration.*

Suppose that there is a gaseous system enclosed in a cylinder fitted with a frictionless piston. All the walls of the cylinder are impervious to heat. The gas exerts a force on the piston tending to push it out. Therefore, an external agent must apply a force of the same magnitude, but in the opposite direction to keep it in position. Suppose you yourself do this job. Let F be the force exerted by the gas on the piston and F' be the force exerted by you on the piston. Then by Newton's third law of motion $F=F'$. Now push the piston a little. In this process there is a flow of energy from your body to the gas. This energy flow is not due to a difference in temperature, but due to a change of state of the gas. This 'energy in flow' not due to a temperature difference, but due to change of state or configuration, is called *work*. Similarly when you let the piston move out-

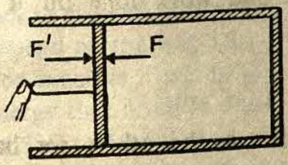


Fig. 8.2

ward, some energy will flow from the gas to your body and it will be said that there is 'work flow' from the gas to the environment (here your body). Work is always done through the action of some force. In the above examples 'work flow' involves displacement in favour of one force and opposite to the other. The work done by a system is measured by the force exerted by the system on its environment multiplied by the displacement along the direction of the force. More precisely

$$dW = \vec{F} \cdot \vec{ds}$$

and $$W = \int \vec{F} \cdot \vec{ds} \quad \dots (8.1).$$

The term 'work' includes all kinds of energy transfer except for the one arising from temperature differences. The force F may arise from electrical, magnetic, gravitational sources. Obviously this definition is consistent with our previous definition of the term, namely, work in mechanics.

In the above example, $F = pA$ where p is the pressure of the gas and A is the area of the piston. Let the piston be allowed to move a little by dx. Then

$$dW = pA\ dx = p(Adx) = pdV$$

$$\therefore \quad W = \int pdV \quad \dots (8.1\text{ a}).$$

This is the most useful form for work done when a change of volume is involved.

The unit of work in SI is joule (J). *When the point of application of one newton force is displaced through one metre, one joule of work is said to flow from one system to the other.* In thermodynamics work done by a system is considered *positive* and work done on it as *negative* work for the system.

8.2. Equivalence of Work and Heat

When something produces the same effect as the other, we have no difficulty in accepting one as equivalent to the other. Suppose that 1 gm of water at 14·5°C is placed in contact with a hot plate (Fig. 8.1). Energy will flow from the hot plate to water due to the difference in temperature. This energy in transit is called heat and the amount of heat energy that will flow by the time the temperature of water rises by 1°C is called a *calorie*.

Now let us remove that water from the hot plate and dip a paddle wheel in it (Fig. 8.3). Now rotate this wheel with the help of falling weight. Here work flow will take place from the weight to the water. A similar effect, namely, rise in temperature will be observed. Thus whatever effect 'heat flow' produces in water, exactly the same effect is produced by 'work flow'. Hence we say that heat and work are equivalent to each other. It is seen that water in the container is heated through 1°C when the falling weight will do 4·2 joule of work, that is 'work flow' by 4·2 joule will take place from the falling weight to the water in the container. Hence we say that 1 calorie of heat is equivalent to 4·2 joule of work. In general if W is the work flow and H is the amount of heat produced, then

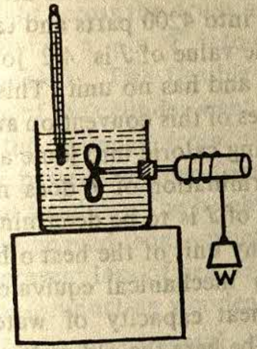


Fig. 8.3

$$J=\frac{W}{H} \quad \text{.. (8.2).}$$

where J is a universal constant called Mechanical equivalent of heat.

DEFINITION OF J. Clearly if $H=1$, $W=J$. Thus mechanical equivalent of heat is the *amount of 'work flow' needed to produce 1 calorie of heat*

The standard value of J is 4·2 joule* per calorie. In SI it is proposed to use this equivalent value of calorie in place of 'calorie'. This does not mean that the use of calorie is forbidden. It is simply desired in SI that whenever one uses calorie as unit of heat, the conversion factor to joule be stated.

IMPORTANT NOTE. The equivalence of heat to work is a basic fact in science, but not the value and unit of J. They will solely depend on the way one wishes to fix up the size of the unit for heat. For example if we fix the size of the unit of heat as the amount of energy that flows to 1 gm. of water when its temperature rises from 14·5°C to 15·5°C and call it a calorie, the paddle wheel experiment shows that it is equivalent to 4·2 joule of work. So the value of J is 4·2 joule per calorie. If on the other hand we take the size of the unit of heat as the amount of energy that flows to 1 kg of water when its temperature rises from 14·5°C to 15·5°C and is called a kilo-

*More accurately it is 4·1855.

calorie, the paddle wheel cxperiment shows that it is equivalent to 4200 joule of work and so J is 4200 joule per kilo calorie. Now if we divide one calorie into 4·2 parts or one kilo calorie into 4200 parts and take each part as the size of the unit heat then the value of J is 4·2 joule of work per 4·2 joule of heat. That is, $\mathbf{J=1}$ and has no unit. This is exactly what is proposed in SI. The advantages of this convention are: firstly, one gets rid of the trouble of converting calorie into joule and vice versa and secondly the question of determination of J does not arise as it is fixed at '1'. However, the value of J is to be determined by the paddle wheel experiment so long as any unit of the heat other than the joule is used. Thus in SI there is no mechanical equivalent of heat, but instead there is the specific heat capacity of water, expressed in J $kg^{-1}K^{-1}$**. However, we describe here the old methods of determining J to cover up the courses of studies.

8.3. Experimental Determination of J

(A) JOULE'S METHOD.

It was Joule who first carefully measured the mechanical energy equivalent to heat energy, that is, the number of Joules equivalent to one calorie.

This apparatus consisted of a specially constructed copper calorimeter C divided into four quadrants by four partition walls at right angles to each other. Two such partition walls are shown in Fig. 8.4. A vertical spindle passing centrally down the calorimeter carried eight vanes which could pass in slots cut in the partition walls exactly in the same way as a key turning in the wards of a lock. This arrangement prevented water taken in the calorimeter to gain motion, i.e., it prevented conversion of 'work flow' into kinetic energy of water and ensured complete conversion of 'work flow' into heat. To prevent the conduction of heat along the

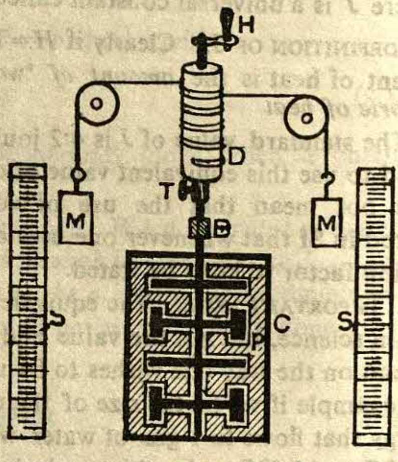


Fig. 8.4

***Heat and Thermodynamics* by Mark W. Zemansky, 5th edition, page 87

spindle it was interrupted by a piece of wooden cylinder B. The upper end of the spindle was introduced axially into a wide wooden cylinder D right across the drum and the spindle so that by introducing or removing a pin T into the hole the two could be coupled or decoupled. Two flexible cords tied at one end to a peg fixed in the body of the cylinder went several times round the cylinder and finally left the cylinder from its diametrically opposite ends. After this they passed over two frictionless pulleys and ultimately ended in two equal weights. The heights of these weights from the ground below could be read from vertical scales placed by the side of the weights. To prevent loss of heat from the calorimeter it is placed as usual in a non-conducting wooden box packed with cotton or wool.

In the experimental procedure a known mass of water was taken in the calorimeter and its temperature was recorded by a mercury thermometer. By removing the pin T, the hanging weights were raised to a maximum height by turning the handle. The pin was then introduced and the weights were allowed to fall. Work flow took place from the falling weights to the calorimeter and its contents. The operation was repeated several times till the temperature rose by several degrees.

To determine the velocity acquired by the weights on reaching the ground, the time of fall was recorded by a stop-watch.

Calculation : Let m = mass of water taken

w = water equivalent of the calorimeter

θ_1 = initial temperature

θ_2 = final temperature

M = mass of each hanging weight

h = height through which each weight falls and

v = velocity acquired by each and

n = number of falls

'Work flow' per fall = Loss in energy of the weights

$$= 2(Mgh - \tfrac{1}{2}Mv^2)$$
$$= 2Mgh - Mv^2.$$

W, total 'work flow' to water and calorimeter

$$= n(2Mgh - Mv^2)$$

H, Heat produced $= (m+w) \times s \times (\theta_2 - \theta_1)$ where s is the specific heat capacity of water in calorie per kg per kelvin.

$$\therefore \quad J = \frac{W}{H} = \frac{n(2Mgh - Mv^2)}{(m+w)s(\theta_2 - \theta_1)} \text{ joule per calorie.}$$

Let t be the time of fall. Then

$v=ft$ where f is the acceleration of fall

and $$h=\frac{1}{2}ft^2=\frac{1}{2}\frac{v}{t}t^2=\frac{vt}{2} \quad \text{or} \quad v=\frac{2h}{t}.$$

This gives v. Knowing v in this way and other quantities in the expression for J being known, it can be calculated out.

Defects of Joule's experiment. The defects of Joule's experiment are : (i) rise of temperature was very small, (ii) the mercury thermometer was not standardised by comparison with a standard thermometer, (iii) the specific heat capacity of water was assumed to be the same at all temperatures which is not a fact.

(B) SEARLE'S FRICTION CONE METHOD (A LABORATORY METHOD)

The apparatus consists of two truncated brass cones A and B, one of which fits exactly into the other. The outer cone B is fixed to a non-conducting base which in its turn is fixed on the top of a vertical spindle S. A large circular wooden disc is attached to inner cone A. The disc is grooved round the edge. A flexible string tied to a peg in the groove at one end is wound round the circumference of the disc, passes over a frictionless pulley P and finally carries a pan at the other end. The spindle is provided with a pulley at the lower end and this pulley is coupled to a wheel W by an endless cord. The spindle can be rotated by turning the wheel manually or electrically and its number of rotation is recorded by a speed counter C driven by the threaded portion of the spindle itself. When the cone B is rotated slowly, A also tends to rotate along with it, but the tension of the string acting tangentially to the disc prevents it from doing so. In the experimental procedure the value of the weight on the pan is so adjusted that this tendency is just overcome and the cone A just shows the tendency to slip inside B. In this case 'work

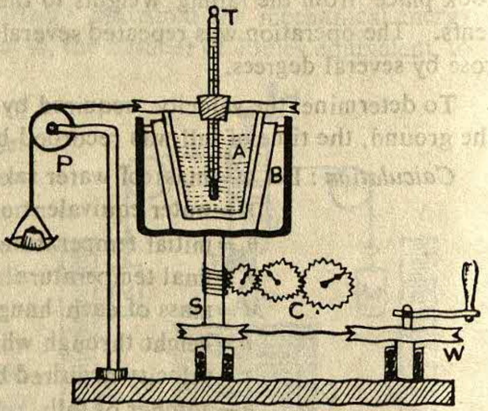


Fig. 8.5

flow' takes place to the cones and their contents from the agent who drives the wheel through the action of the limiting frictional force between the cones. The moment of the limiting frictional force between the cones is balanced by the moment of the tension of the string.

In the experimental procedure a known mass of water is taken in the inner cone and its temperature is recorded. The outer cone is rotated at a constant speed and weights on the pans are so adjusted that the inner cone and therefore the weight as well just remain **stationary**. The rotation of the cone is continued till the temperature of water rises by several degrees. The difference of the initial and final readings of the speed counter gives the number of rotations during the experiment.

Calculation. Let M = mass suspended; D = diameter of the wooden disc; F = average limiting frictional force between the cones; r = mean radius of the cones; n = number of revolutions; m = mass of water taken; w = water equivalent of cones; θ_1 = initial temperature; θ_2 = final temperature. 'Work Flow' to the cones and water per revolution $= F \times 2\pi r$.

W, total 'work flow' to the cones and water $= n \times 2\pi r F$.

Considering the equilibrium of the inner cone we have

$$T.\frac{D}{2} = F.r \text{ where } T \text{ is the tension of the string.}$$

Again considering the equilibrium of the weight, we have

$$T - Mg = 0 \quad \text{or} \quad T = Mg$$

$$\therefore \quad Mg\frac{D}{2} = Fr.$$

$$\therefore \quad W = n2\pi r.\frac{MgD}{2r} = n\pi MgD.$$

H, heat produced $= (m+w)s\,(\theta_2 - \theta_1)$ when s is the specific heat capacity of water in calorie per kg per kelvin.

$$\therefore \quad J = \frac{W}{H} = \frac{n\pi MgD}{(m+w)s\,(\theta_2 - \theta_1)} \text{ joule per calorie.}$$

(C) BY CALLENDAR AND BARNES' CONTINUOUS FLOW CALORIMETER

See Art. 3.5 of chapter II in Heat for description of apparatus and procedure.

Calculation. In the steady state, suppose V is the potential difference across the heating wire and I is the current through it. Take a weighed beaker and collect the out-flowing water in it for t seconds. Let m be the mass of water collected and $(\theta_2-\theta_1)=\triangle\theta$ be the difference in temperature.

Then the electrical energy supplied to the heating coil $=VIt$ joules.

Heat absorbed $=ms\triangle\theta$ where s is the specific heat capacity of water in **calorie per kg** per kelvin and loss of heat by radiation $=ht$.

$$\therefore \quad \text{total heat produced} = ms\triangle\theta + ht$$

$$\therefore \quad J=\frac{W}{H}=\frac{VIt}{ms\triangle\theta+ht} \text{ joule per calorie}$$

or
$$\frac{VIt}{J}=ms\triangle\theta+ht.$$

To eliminate h, two sets of observations are taken by suitably adjusting the strength of electric current and the rate of flow of water so as to secure the same temperature difference. Thus for the two rates of flow we have,

$$\frac{V_1I_1t}{J}=m_1s\triangle\theta+ht$$

$$\frac{V_2I_2t}{J}=m_2s\triangle\theta+ht$$

$$\frac{(V_1I_1-V_2I_2)t}{J}=(m_1-m_2)s\triangle\theta$$

or
$$J=\frac{(V_1I_1-V_2I_2)t}{(m_1-m_2)s\triangle\theta} \text{ joule per calorie.}$$

(D) A SIMPLE ELECTRICAL METHOD

See Art. 5.5 of chapter 5 in current electricity.

Examples :

1. *An electric motor ·5 HP is used to drill hole in a ·5 kg brass block. It takes 2 minutes time. (a) How much heat is generated? (b) What is the rise in temperature if 75% of the heat generated warms the brass? (specific heat capacity of brass = 88 calorie per kg per kelvin and 1 HP = 746 watt)*

Sol. Power of motor $= \cdot 5 \times 746$ watt.

Work flow $=$ Power $\times$ time $= \cdot 5 \times 746 \times 2 \times 60$ joule

$$= \frac{\cdot 5 \times 746 \times 2 \times 60}{4 \cdot 2} \text{ calorie}$$

$$= 10657 \cdot 1 \text{ cal. Ans.}$$

Let θ be the rise in temperature.

$$\frac{75}{100} \times 10657 \cdot 1 = \cdot 5 \times 88 \times \theta.$$

or $$\theta = \frac{75 \times 10657 \cdot 1}{100 \times \cdot 5 \times 88} = 181 \cdot 6°\text{C. Ans.}$$

2. *A 2 gm lead bullet moving at a speed of 200 ms^{-1} becomes embedded in a 2 kg wooden block of a ballistic pendulum. Calculate the rise in temperature of the bullet, assuming all the heat generated raises the bullet's temperature. (Specific heat capacity of lead $= 30 \cdot 5$ cal per kg per kelvin; $J = 4 \cdot 2$ joule per calorie)*

Sol. 'Work flow' to the bullet $=$ change in kinetic energy

$$= \tfrac{1}{2}(2 \times 10^{-3}) \cdot 200^2 - \tfrac{1}{2}(2 + 2 \times 10^{-3})v^2$$

$$= 40 - \tfrac{1}{2} \times 2 \cdot 002 v^2.$$

By conservation of momentum

$$2 \times 10^{-3} \times 200 = 2 \cdot 002\, v \text{ or } v = \frac{\cdot 4}{2 \cdot 002}.$$

$$\therefore \quad \text{Work flow} = 40 - \tfrac{1}{2} \times 2 \cdot 002 \times \frac{\cdot 4^2}{2 \cdot 002^2}$$

$$= 40 - \tfrac{1}{2} \times \frac{\cdot 4^2}{2 \cdot 002}$$

$$= 40 - \cdot 04 = 39 \cdot 96 \text{ joule}$$

$$= \frac{39 \cdot 9}{4 \cdot 2} = 9 \cdot 5 \text{ cal.}$$

$$\therefore \quad 9 \cdot 5 = 2 \times 10^{-3} \times 30 \cdot 5 \times \theta$$

or $$\theta = \frac{9 \cdot 5}{61 \times 10^{-3}} = 155 \cdot 7°\text{C. Ans.}$$

3. *Calculate the difference in temperatures between the water at the top and that at the bottom of a water fall which is 50 m high, given $J = 4 \cdot 2$ joule per calorie.* Specific heat capacity of water $= 1 \text{kcal kg}^{-1} \text{K}^{-1}$.

Sol. Work flow = change in potential energy
$= mgh$ joule
$= ms\theta$ joule where s is the specific heat capacity of water in $Jkg^{-1}K^{-1} = 1000 \times 4{\cdot}2$

$$\therefore \quad \theta = \frac{gh}{s} = \frac{9{\cdot}8 \times 50}{4200} = {\cdot}1167°C. \textbf{ Ans.}$$

8.4. Definitions of Terms used in Thermodynamics

(i) *System*. A definite quantity of matter bounded by some closed surface is known as a *system*. Matter and space external to the system is called its environment. A system and its environment constitute a 'universe'. The boundaries of a system may be real or imaginary, but the existence of the boundaries is of utmost importance in order to visualise the system under consideration distinctly from the rest of the universe. Water in a beaker is a simple example of a thermodynamical system and the beaker, the atmosphere and the source of heat constitute its environment. A quantity of a gas inside a baloon is a thermodynamic system with definite boundaries.

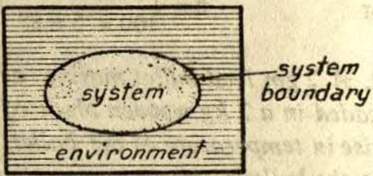


Fig. 8.6

A system is said to be *closed* when only *energy, but not mass* may cross its boundaries.

A system is said to be *isolated*, when it can exchange *neither mass nor energy* with its environment. The universe itself is an *isolated* system.

(ii) *State of a system*. The state of a system is its condition or position determined by the three fundamental properties volume, pressure and temperature.

(iii) *Property of a system*. In thermodynamics 'property' is not a loose term. It is applied in a strict sense in thermodynamics. It is also called *state variable* or *state function*.

Any observable quantity that depends upon the state of the system and not upon how that state have been attained is called a property of the system. In this sense volume, temperature, pressure and internal energy are properties of a system. To see whether any other observable is a property of the system, examine whether a

change in that quantity between two states depends on the path along which it is carried from one state to the other. If it is independent of the path, it is a property. If not, the observable is not a property of the system.

Intensive property. Property that does not depend on the extent of the system is called *intensive property of the system.* For example, pressure and temperature.

Extensive property. Property that depends on the extent of the system is called *extensive property of the system.* For example, volume, internal energy, mass etc.

(iv) *Thermodynamic Process.* A thermodynamic process is said to occur when a system undergoes a change of state. When water in a beaker is heated, pressure and volume remain practically same, but its temperature increases. We say a thermodynamic process has ocoured in the system (water in the beaker).

Processes are named after the quantity that remains constant e.g. isothermal process (temperature constant), isochoric process (constant volume), isobaric (constant pressure), isentropic (constant entropy), adiabatic process (no exchange of heat between the system and its environment).

(v) *Cyclic process.* Any process whose end states are indentical is called a *cyclic process* (Fig.8.7).

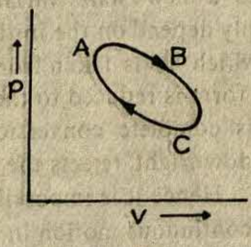


Fig. 8.7

8.5. The First law of Thermodynamics

The first law of thermodynamics is simply the principle of conservation of energy applied to a thermodynamic system undergoing some change. It has been stated in three different ways. All the statements are equivalent to each other. In the simplest form it is stated as follows :

"Whenever 'work flow' is completely converted into heat or vice versa, one is proportional to the other."

If W is the work flow to a system and H is the heat produced,

$$W \propto H$$

or

$$W = JH.$$

In the more useful form it is stated as follows :

"In all transformations, the in-flow of energy to a system in the form of 'heat' must be equal to the out-flow of energy in the form of work plus the increase in internal energy of the system.

If $\triangle Q$ is the heat flow to a system, $\triangle W$ is the work done *by* the system and $\triangle U$ is the increase in internal energy of the system and all the energies are taken in the same unit, namely, either joule or calorie, then

$$\triangle Q = \triangle U + \triangle W \quad .. \quad (8.3).$$

If our system undergoes an infinitesimal change in state, only an infinitesimal amount of heat dQ is absorbed and only an infinitesimal amount of work dW is done, so that the internal energy change dU is also infinitesimal, we can re-write

$$dQ = dU + dW \quad .. \quad (8.3a).$$

In the third form, it is stated as follows : '*Perpetual motion of the first kind is impossible.*

In the simplest form this law simply asserts that there exists a definite relationship between **heat** and **work**. In the more elaborate form (Eq. 8.3) it introduces a new state variable called internal energy whose change will simply depend on the initial and final state and not on the process by which it is taken from the former to the later. Note that this second form is reduced to the first form when $\triangle U = 0$, that is, when there is complete conversion work into heat and vice versa. This law downright rejects the possibility of creation of energy out of nothing. Hence it is impossible to think of a machine which would produce continuous motion in the different parts of our machineries without consuming energy from some source. A hypothetical machine which could do this is termed as *perpetual motion machine of the first kind.* The first law rules out the possibility of such machines. Thus all the three statements converge to the same verdict.

8.6. Simple Applications of the First Law of Thermodynamics

(i) *To calculate change in internal energy of water when converted into steam at constant pressure.* We have seen that when a system expands the work done *by* it on its environment is

$$\triangle W = \int pdV.$$

If V_v is the volume of certain mass of steam and V_l is the volume of the same mass of water then

$$\triangle W = \int_{V_l}^{V_v} pdV = p\int_{V_l}^{V_v} dV \quad (\because \ p \text{ is a constant})$$

$$= p[V]_{V_l}^{V_v} = p(V_v - V_l)$$

If L is the specific latent heat capacity of water, then

$$\triangle Q = mL.$$

By the first law of thermodynamics we have

$$\triangle Q = \triangle U + \triangle W$$

or $$\triangle U = mL - p(V_v - V_l).$$

(ii) *To deduce $p-V$ relation in an adiabatic process.* Let us consider one mole of a **perfect gas**. Let its initial pressure, volume and temperature be p, V and T respectively. Consider an infinitesimal flow of heat to the system. Due to this addition of heat, let its pressure change by an infinitesimal amount dp, its volume by dV and its internal energy by dU. Then dW (infinitesimal work done by the gas) $= pdV$.

By the first law of thermodynamics we have

$$dU = dQ - dW = dQ - pdV.$$

By definition $dQ = C_v dT$ at constant volume

and $dV = 0$ at constant volume.

Since U is a property whatever is the value of dU in a constant volume process that is also the value of dU in any other process occuring between the same two states.

$$\therefore \ dU = C_v dT.$$

Thus by the first law of thermodynamics we have

$$dQ = C_v dT + pdV.$$

In an adiabatic process $dQ = 0$.

Therefore we have $C_v dT + pdV = 0$ for an adiabatic transformation .. (i).

For one mole of a perfect gas we have

$$pV = RT.$$

Differentiating this we have

$$pdV + Vdp = RdT \quad \text{.. (ii).}$$

Substituting for dT from (ii) in (i) we get

$$C_v\left(\frac{pdV + Vdp}{R}\right) + pdV = 0$$

or $$C_v\, pdV + C_v\, Vdp + RpdV = 0.$$
But $$C_p - C_v = R.$$
$\therefore$ $$C_v\, pdV + C_v\, Vdp + (C_p - C_v)\, pdV = 0$$
or $$C_v\, Vdp + C_p\, pdV = 0.$$

Dividing throughout by pV,

$$C_v \frac{dp}{p} + C_p \frac{dV}{V} = 0 \quad \text{or} \quad \frac{dp}{p} + \frac{C_p}{C_v} \frac{dV}{V} = 0.$$

For a given gas $C_p/C_v = \gamma$ is a constant.

$$\therefore \quad \frac{dp}{p} + \gamma \frac{dV}{V} = 0.$$

Integrating, $\log p + \gamma \log V = a$ constant

or $\log pV^\gamma = a$ constant

or $pV^\gamma = a$ constant .. (8.4).

Eliminating p with the help of the relation $pV = RT$,

we have $TV^{\gamma-1} =$ a constant .. (8.4a).

Eliminating V with the help of the same relation

$p^{1-\gamma}\, T^\gamma =$ a constant .. (8.4b).

(iii) *To calculate the change in internal energy of a gas in an isochoric process.* By the first law of thermodynamics we have

$$dQ = dU + dW.$$

The infinitesimally small work done is $dW = pdV$

$$\therefore \quad dQ = dU + pdV.$$

In an isochoric process $dV = 0$ ($\because$ V is a constant)

$\therefore$ $dQ = dU$ at constant volume.

Now, $$C_v = \left(\frac{\triangle Q}{\triangle T}\right)_v.$$

In the differential form

$$C_v = \left(\frac{dQ}{dT}\right)_v \quad \text{or } dQ = C_v\, dT \text{ at constant volume}$$

$\therefore$ $dU = C_v dT$ at constant volume

$$\therefore \quad C_v = \left(\frac{dU}{dT}\right)_v.$$

(iv) *To show that $C_p - C_v = R$ for a perfect gas.* We have by the first law of thermodynamics $dQ = C_v dT + pdV$. Let heat be added at constant pressure. Then $dQ = C_p dT$ and $pdV = d(pV)$, since pressure remains constant. But for a perfect gas $pV = RT$ (always).

$$\therefore \quad C_p dT = C_v dT + d(RT) \text{ or } (C_p - C_v)dT = RdT \text{ or } C_p - C_v = R.$$

Proved.

Examples :

1. *Calculate the internal work done in overcoming the strong attraction of H_2O molecules for one another in the liquid state when 1 gm of water is vaporised. (Density of water = 1000 kgm^{-3}; density of steam = ·5984 kgm^{-3}; $L = 539 \times 10^3$ cal kg^{-1}; $J = 4{\cdot}186$ joule per calorie; 1 atmosphere = $1{\cdot}013 \times 10^5 Nm^{-2}$).*

Sol. $\triangle Q = 10^{-3} \times 539 \times 10^3 = 539$ cal $= 539 \times 4{\cdot}186 = 2256{\cdot}25$ joule

V_l, volume of 1 gm of water $= 10^{-6} m^3$

V_v, volume of 1 gm of steam $= \dfrac{10^{-3}}{\cdot 5984} = 1671 \times 10^{-6} m^3$

$$\begin{aligned}\therefore \quad \triangle W = p(V_v - V_l) &= 1{\cdot}013 \times 10^5 (1671 \times 10^{-6} - 10^{-6}) \\ &= 1{\cdot}013 \times 10^5 \times 1670 \times 10^{-6} \\ &= 1{\cdot}013 \times 167 \\ &= 169{\cdot}2 \text{ joule.}\end{aligned}$$

By the first law of thermodynamics

$$\triangle Q = \triangle U + \triangle W.$$

$\therefore \quad \triangle U = 2256{\cdot}25 - 169{\cdot}2 = 2087$ joule. **Ans.**

2. *When a system is taken from state i to state f along the path iaf, it is found that $\triangle Q = 50$ cal and $\triangle W = 20$ cal. Along the path ibf $\triangle Q = 36$ cal. (a) What is $\triangle W$ along the path ibf? (b) if $\triangle W = -13$ cal for the curved return path fi; what is $\triangle Q$ for this path? (c) Take $U_i = 10$ cal, what is U_f? (d) If $U_b = 22$ cal, what is $\triangle Q$ for the path ib?*

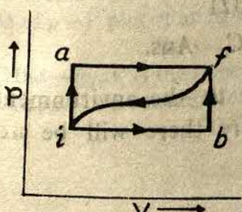


Fig. 8.8

Sol. Since the internal energy is a state variable (property), its change between two states is the same whatever be the path.

(a) Along *iaf*, $\triangle W = 20$ cal; $\triangle Q = 50$ cal.

By the first law of thermodynamics

$$\triangle Q = \triangle W + \triangle U$$

$$\triangle U = \triangle Q - \triangle W = 50 - 20 = 30 \text{ cal.}$$

Along the path *ibf*, $\triangle Q=36$ cal. $\triangle U=30$ cal because it is independent of the path.

$\therefore\quad \triangle W=\triangle Q-\triangle U=36-30=6$ cal $=6\times 4{\cdot}2=25{\cdot}2$ joule. **Ans.**

(b) $\triangle Q=\triangle W+\triangle U=-13+(-30)=-43$ cal. **Ans.**

(c) $\triangle U=U_f-U_i$ or $30=U_f-10$ or $U_f=40$ cal. **Ans.**

(d) For the path *ib*, $\triangle U=U_b-U_i=22-10=12$ cal.

$$\triangle W_{ibf}=\triangle W_{ib}+\triangle W_{bf}$$

or $$6=\triangle W_{ib}+0$$

($\because$ volume remains constant along if, work done is zero)

$$\therefore\quad \triangle Q_{ib}=\triangle U_{ib}+\triangle W_{ib}=12+6=18 \text{ cal. } \textbf{Ans.}$$

3. *A monatomic ideal gas initially at 27°C is suddenly compressed to one-tenth of its original volume. Calculate its temperature after compression. What will be the temperature if the gas is compressed slowly by the same amount ?*

Sol. When compressed suddenly, heat can neither leave the gas nor enter into the gas for environment and hence the process is adiabatic. For adiabatic process

$$TV^{\gamma-1}=\text{a constant}$$

$$\therefore\quad (273+27)V^{1\cdot66-1}=T\left(\frac{V}{10}\right)^{1\cdot66-1}$$

($\because\ \gamma=1{\cdot}66$ for monatomic gas)

or $$300=\frac{T}{10^{\cdot66}} \quad\text{or}\quad T=300\times10^{\cdot66}$$

or $\log T=\log 300+{\cdot}66=2{\cdot}4771+66=3{\cdot}1371$

$\therefore\quad T=\text{antilog } 3{\cdot}1371=1371°\text{K}=1098°\text{C}.$ **Ans.**

When compressed slowly, heat will flow out to the environment and hence the process is isothermal. Therefore there will be no change in temperature.

QUESTIONS

(A)

1. In Searle's friction cone method (a) 'work' flows to the cones from the hanging weight, (b) 'work' flows from air, (c) 'work' flows from the agent who rotates the wheel, (d) no 'work' flows from anywhere.

2. A state variable or property of a thermodynamic system is that observable whose change from one state to another (a) depends on the path along which it is taken, (b) does not depend on the path, (c) the change is always zero, (d) the change is simple harmonic.

3. When a gas undergoes an adiabatic change, the relations between p and V is given by (a) pV=constant, (b) pV^{γ}=constant, (c) $p^{\gamma}V^{\gamma-1}$=constant, (d) pV^{-1} =constant.

4. In an adiabatic change, the relation between temperature and pressure of a perfect gas is given by (a) $T^{1-\gamma}p$=constant, (b) $Tp^{1-\gamma}$=constant, (c) $T^{1-\gamma/\gamma}p$ =constant, (d) $Tp^{1-\gamma/\gamma}$=constant.

5. For a perfect gas undergoing adiabatic change, the relation between temperature and volume is (a) $TV^{\gamma-1}$=constant, (b) $T^{\gamma-1}V$=constant, (c) $T^{\gamma-1}V^{\gamma-1}$ =constant, (d) $TV^{1-\gamma}$=constant.

6. If heat is expressed in joule, the mechanical equivalent of heat is (a) 1, (b) 4·2, (c) 4·2 joule, (d) 1 joule.

7. In an isochoric process (a) temperature remains constant, (b) pressure remains constant, (c) volume remains constant, (d) mass remains constant.

(Ans. : 1. c, 2. b, 3. b, 4. d, 5. a, 6. a, 7. c.)

(B)

1. Define *kilocalorie* and *joule*.

2. Explain what is meant by property (or state variable) of a system in thermodynamics ? What are extensive and intensive properties of a system.

3. Explain thermodynamic process and a cyclic process. Give examples.

4. Explain isothermal, isobaric, isochoric, isentropic, and adiabatic processes. Show that at a point the slope of $p-V$ plot of an adiabatic process is γ times the slope of $p-V$ plot of an isothermal process at the same point.

[**Hint**—dp/dV is the slope of $p-V$ curve at a point. pV=a constant for thermal process; pV^{γ}=a constant for adiabatic process.]

(C)

1. Explain 'Heat' and 'Work' ? What do you mean by mechanical equivalent of heat. Describe Joule's method of determining mechanical equivalent of heat. Does mechanical equivalent of heat exist if joule is also used to express heat energy ?

2. Is 'heat' stored energy in a body ? Explain.

Describe Searle's friction cone method of determining J.

[**Hint.** No; if it were a stored energy, it would not be possible to remove heat indefinitely from a body. Yet Rumford showed that this was possible.]

3. Explain the terms (i) system, (ii) environment, (iii) universe, (iv) open closed and isolated system, (v) state variable of a system. State and explain the first law of thermodynamics.

4. What is an adiabatic process ? Apply the first law of thermodynamics to obtain the $p-V$ relation in an adiabatic process.

5. Describe Callender and Barnes' continuous flow method of determining J.

6. Distinguish between isothermal and adiabatic changes.

(D)

1. A block of ice at 0°C whose mass is initially 50 kgm slides along a horizontal surface, starting at a speed of 5·38 ms^{-1} and finally coming to rest after travelling 28·3 m. Compute the mass of ice melted as a result of the friction between the block and the surface. (L of ice = 80 kilo calorie per kg and $J = 4{\cdot}186$ joule per calorie) (Ans. 2 gm.)

2. A motor car tyre has a pressure of 2 atmospheres at a room temperature of 27°C. If the tyre suddenly bursts, find the resulting temperature. (γ for air $= 1{\cdot}4$) (Ans. $-26{\cdot}9$°C)

3. A quantity of air ($\gamma = 1{\cdot}4$) at 27°C is compressed (a) slowly and (b) suddenly to one-third of its volume. Find the change in temperature in each case. (Ans. No change; 165·5°C)

4. How would you expect the 80 kilocalorie that are needed to melt 1 kg of ice to water (at 0°C 1 atm pressure) to be shared by external work and the internal energy ? (Density of water $= 1000\ kgm^{-3}$; density of ice $= 931\ kgm^{-3}$; 1 atm $= 1{\cdot}013 \times 10^5\ Nm^{-2}$) (Ans. 1·79 cal; 79998·2 cal)

5. Calculate the work done when one mole of a perfect gas is compressed adiabatically. The initial pressure and volume of the gas are $10^5\ Nm^{-2}$ and 6 litres respectively. The final volume of the gas is 2 litres. Molar specific heat of the gas at constant volume is $3R/2$. (I. I. T. 1982)

[Hint—In an adiabatic change, $PV^\gamma = k$.

$$W = \int_{V_1}^{V_2} pdV = \int_{V_1}^{V_2} kV^{-\gamma} dV = \frac{p_1V_1 - p_2V_2}{\gamma - 1}.$$ Here $\gamma = 5/3$.] (Ans. 971·4 J)

6. A gas at 27°C is compressed adiabatically to 1/4th of its volume. Calculate the resulting temperature. (γ for the gas $= 1{\cdot}5$) (Ans. 327°C)

7. Dry air at 15°C and 10 atmospheric pressure is suddenly released to atmospheric pressures, find the temperature of the air. (γ of air $= 1{\cdot}41$) (Ans. $-125{\cdot}6$°C)

8. Calculate the energy required in joule to convert 50 gm of ice at 0°C into steam at 100°C. (The specific latent heat of ice $=336\times10^3$ Jkg^{-1}. The specific latent heat of steam $=2{\cdot}25\times10^6$ Jkg^{-1}. The specific heat capacity of water $=4200$ $Jkg^{-1}K^{-1}$) (Ans. $1{\cdot}5\times10^5$ J)

9. If the height of a water fall is 300 m find the difference in temperature of the water above and below the fall. (The specific heat capacity of water $=4200$ $Jkg^{-1}K^{-1}$) (Ans. ·7°C)

(E)

1. In an adiabatic process............($\triangle Q$ or $\triangle U$) is zero.

2. In SI the numerical value of J is............and its unit............

3. When 'work flow' is expressed in joule and 'heat flow' in kilocalorie, the numerical value of J is............and its unit............

4. In an isochoric process..................(volume or temperature or pressure) remains constant.

5. In an isobaric process..................(volume or temperature or pressure) remains constant.

6. In an isothermal process...............(volume or temperature or pressure) remains constant.

7. In an isochoric process...............($\triangle V$, or $\triangle p$ or $\triangle T$) is zero.

8. A gas is suddenly compressed. Is the process...........(adiabatic, isobaric) ?

9. A gas contained in a vessel of non-conducting wall is compressed. Is the process...............(adiabatic or isothermal) ?

(Ans. 1. $\triangle Q$, 2. 1 and no unit. 3. 4200; joule per kilocalorie. 4. volume. 5. pressure. 6. temperature. 7. $\triangle V$. 8. adiabatic. 9. adiabatic.)

CHAPTER 9

REVERSIBLE AND IRREVERSIBLE PROCESSES : *CARNOT ENGINE : PRACTICAL ENGINES : *SECOND LAW OF THERMODYNAMICS : *THERMODYNAMIC SCALE OF TEMPERATURE : *REFRIGERATORS : *ENTROPY

9.1. Reversible and Irreversible Processes

A thermodynamic process which can be traced back with the system passing through all stages in the reverse process as in the direct process and at the conclusion of direct and reverse processes, both the system and the local surroundings may be restored to their initial states without producing any change in the rest of the universe is called a reversible process.

The processes which cannot be traced back without leaving some change somewhere in the universe are called irreversible processes.

As an example of a reversibie process consider a gas enclosed in a cylinder fitted with a non-conducting piston and immersed in a constant temperature bath. We assume that the walls of the cylinder are made of perfectly heat conducting material. To begin with let p, V and T be pressure, volume and temperature of the gas respectively. For equilibrium of the gas an external pressure of the same magnitude must be applied on the piston. In practice this pressure is constituted by the atmospheric pressure as well as the pressure due to weight on the piston. Here gas is our thermodynamical system. The bath and weight (atmosphere also) are its local surroundings. Let us reduce the external pressure by an infinitisimal amount $\triangle p$ so

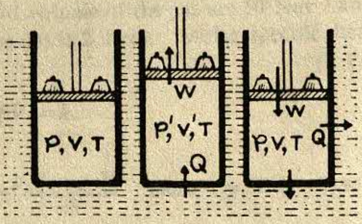


Fig 9.1

that the system continues to be virtually in equilibrium with the environment. The gas will expand. 'Heat flow' will take place from the bath to the gas and 'work flow' will take place from the gas to the weights (+atmosphere). Let us give sufficient 'pause' for heat to pass on to the gas so that equalisation of temperature is fully ensured. Now, again release the pressure by an infinitesimal amount and give sufficient pause. Repeat the process till the gas has expanded to a volume V', when Q units of total heat have flown from the bath to the system and W units of total work have flown out from the system to the weight (+atmosphere). This process by which the state of our system has been changed from p, V, T to p', V', T is an example of a reversible isothermal process. Since the process is carried out extremely slowly and that too through virtual equilibrium (static) states of the system with the surroundings, it is also called *quasi static process* (quasi means 'almost'). We say that the above process is reversible, because if we like we can restore the system to its initial state by the same quasi-static process by increasing the pressure by infinitesimal amounts in every step and giving sufficient 'pauses' in between the steps, when the same amount of heat will flow out from the system to the bath and the same amount of work will flow from the weights (+atmosphere) to the system. At the conclusion of the forward and reverse processes, the system is restored to its initial condition with no change anywhere in the universe. It is clear that slower the process the greater is the degree of its reversibility. *In fact, thermodynamical processes are reversible, when carried out very slowly.*

Examples of irreversible processes. A reversible process is an ideal concept. It can never be realised in practice. All the processes going on around us are irreversible. Conduction of heat from a hot body to a cold one is an irreversible process. Suppose water in a vessel (system) is heated from 20°C to 30°C by heating over a bunsen burner. This process (isobaric-isochoric) is irreversible because to bring it back to 20°C, it is to be kept in contact with a cold body. So it can be brought back to the initial condition only at the cost of a change of the internal energy of another body. Hence it is irreversible.

If the pressure over the piston of the cylinder (Fig. 9.1) is withdrawn, the gas expands immediately and no work is done by it on the environment as the external pressure on it is zero. But if the gas has to be compressed to the original condition, considerable

work would have to be performed on it. That is, W (work flow) in the two opposite directions is different and so the process is irreversible.

Reversible and Irreversible Processes Compared

Reversible process	Irreversible process
1. At the conclusion of the direct and reverse process, the system is brought to the initial state without any change anywhere in the rest of the universe.	1. At the conclusion of the direct and reverse process the system is brought to the initial condition only with some change somewhere in the universe.
2. At the conclusion of the direct and reverse process, the change in internal energy of the system is zero, i.e., $\triangle U=0$.	2. At the conclusion of the direct and reverse process the change in internal energy is zero, i.e., $\triangle U=0$.
3. At the conclusion of the direct and reverse process the work flow is zero, i.e., $\triangle W=0$. or $\triangle W=0=\triangle W_{direct}+\triangle W_{reverse}$ or $\triangle W_{direct}=-\triangle W_{reverse}$	3. At the conclusion of the direct and reverse process, the work flow is not zero, i.e., $\triangle W\neq 0$. or $\triangle W_{direct}\neq -\triangle W_{reverse}$
4. At the conclusion of the direct and reverse process, the net heat flow is zero, i.e., $\triangle Q=0$ or $\triangle Q_{direct}=-\triangle Q_{reverse}$	4. At the conclusion of the direct and reverse process, the net heat flow is not zero, i.e., $\triangle Q\neq 0$ or $\triangle Q_{direct}\neq -\triangle Q_{reverse}$
5. This process occurs only when the system passes quasi-statically through small stages of virtual equilibrium, so that the change can be effected in either way by slight changes of the external conditions.	5. This process occurs spontaneously and unidirectionally, violently disturbing the equilibrium.
6. This process takes place in an infinite time.	6. This process takes place in a short time.
7. This is an ideal concept.	7. This is the most practical one.
8. This process must be free from dissipative forces, such as friction, viscosity, electrical resistance.	8. A process is irreversible only when there is some dissipative force.

9.2. Heat Engines

A contrivance to convert 'heat' into useful work is called a *Heat engine.* They must work in a cyclic process in order that they may

continuously convert heat into work. A cycle will involve 'in-flow of heat' and 'out-flow of work' in the direct process and 'out-flow of heat' and 'in-flow of work' in the reverse process along a different path. So all engines theoretically must work between two heat reservoirs at different temperatures T and T'. The reservoir of higher temperature is called the *Source* and the one at lower temperature is called the *Sink*.

If $\quad Q =$ heat-flow to the system from the source

$\quad Q' =$ heat-flow from the system to the sink

and $\quad W =$ net work done by the system.

Then the ratio $\frac{W}{Q}$ is called '*thermal efficiency*' of the engine.

Thus
$$\eta = \frac{W}{Q}$$

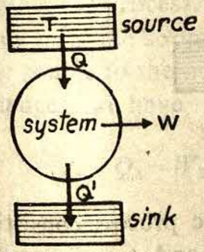


Fig. 9.2

By the first law of thermodynamics

$$\Delta Q = \Delta U + \Delta W$$

In a cyclic process $\Delta U = 0$

Here $\quad \Delta Q = Q - Q', \quad \Delta W = W$

$\therefore$ In a cyclic process, $Q - Q' = W$

$$\therefore \quad \eta = \frac{Q - Q'}{Q} = 1 - \frac{Q'}{Q}$$

$$\eta = 1 - \frac{Q'}{Q} \qquad .. \quad (9.1)$$

**9.3. Carnot's engine and Carnot cycle

In the beginning of the nineteenth century, quite a number of heat engines (steam engine, petrol engine, diesel engine) were in use, but their efficiency was very low. People thought that it was all due to bad design and enormous friction on the moving parts of the engine. A French engineer, **Sadi Carnot** in 1824 conceived a theoretical engine perfect in design and working and obtained the upper limit of the efficiency of practical engines. This is called the Carnot reversible engine. It is never possible to realise this engine in practice. Even then its concept is very useful to find the upper limit of efficiency of practical engines. The other importance of this engine is that it is the work on this theoretical engine that led to the formulation of the second law of thermodynamics.

The plan of Carnot's ideal engine is as shown in Fig. 9.3. The essential parts of it are :

(i) A cylinder '*C*' with **perfectly non-conducting walls** but perfectly conducting base containing some perfect gas as working substance and fitted with a perfectly insulating and frictionless piston.

(ii) A source '*S*' of heat of infinite capacity at temperature T_1 on perfect gas scale.

(iii) A sink of infinite thermal capacity at temperature T_2 on perfect gas scale.

(iv) A perfectly insulating platform *G* serving as a **'stand'** for the cylinder.

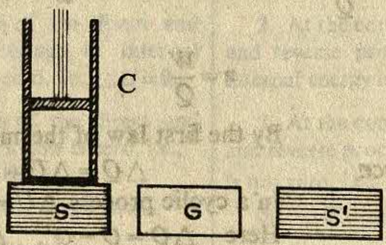


Fig. 9.3a : Plan of Carnot's engine

CARNOT CYCLE

The working substance is subjected to a cyclic process consisting of four operations—two reversible isothermals and two reversible

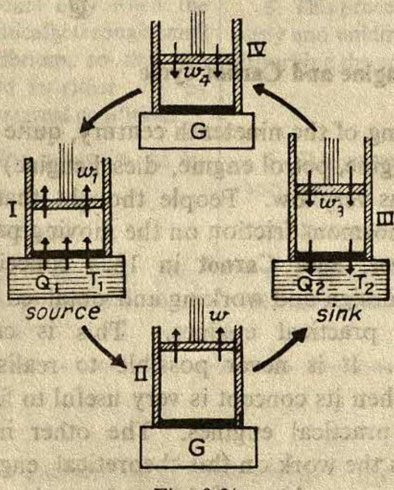


Fig. 9.3b

adiabatics. The cyclic process in which a Carnot engine is worked is called Carnot's cycle.

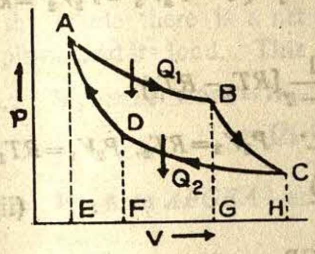


Fig. 9.4

Operation I. The cylinder is placed on the source and the gas is allowed to expand quasi-statically, when there is flow of heat from the source to the system and work-flow from the system to the piston and its load. The volume and pressure of the gas change slowly, but its temperature remains constant. So the first operation is an isothermal process. This is represented by *AB* in the indicator diagram, (i.e., $p-V$ plot). By the first law of thermodynamics $\triangle Q = \triangle U + \triangle W$.

The internal energy of an ideal gas remains constant in an isothermal process. Hence $\triangle U = O$ and here $\Sigma\triangle Q = Q_1$, the heat flow from the source to the gas and $\Sigma\triangle W = W_1$, the work flow from the source to the piston and its load. Taking infinitesimally small changes, we have

$$\therefore \quad Q_1 = W_1 = \int_{V_1}^{V_2} pdV = \int_{V_1}^{V_2} RT_1 \frac{dV}{V} \quad (\because \quad pV = RT)$$

or $$Q_1 = RT_1 \log \frac{V_2}{V_1} \qquad \text{.. (i)}$$

$$= \text{area } ABGEA \qquad \text{.. (i a)}$$

Operation II. The cylinder is removed, put on the non-conducting stand *G* and allowed to expand quasi-statically till its temperature falls to that of the sink. This is an adiabatic change and is represented by *BC* in the indicator diagram. There is no heat flow and there is work-flow from the gas to the piston and its load.

$$W_2 = \int_{V_2}^{V_3} pdV = \int_{V_2}^{V_3} kV^{-\gamma}\, dV \quad (\because \quad pV^{\gamma} = k, \text{ a constant})$$

or $$W_2 = k\int_{V_2}^{V_3} V^{-\gamma}\; dV = k\left[\frac{V^{1-\gamma}}{1-\gamma}\right]_{V_2}^{V_3}$$

or $W_2 = \frac{k}{1-\gamma}[V_3^{1-\gamma} - V_2^{1-\gamma}] = \frac{1}{1-\gamma}[kV_3^{1-\gamma} - kV_2^{1-\gamma}]$

$$= \frac{1}{1-\gamma}[P_3V^{\gamma}_3V_3^{1-\gamma} - P_2V^{\gamma}_2V_2^{1-\gamma}] \quad (\because \; P_3V_3^{\gamma} = P_2V_2^{\gamma} = k)$$

or $$W_2 = \frac{1}{1-\gamma}[P_3V_3 - P_2V_2] = \frac{1}{1-\gamma}[RT_2 - RT_1]$$

$$(\because \; P_3V_3 = RT_2; \; P_2V_2 = RT_1)$$

or $$W_2 = \frac{R(T_1-T_2)}{\gamma-1} \quad .. \quad \text{(ii)}$$

$$= \text{area } BCHGB$$

$$(\because \int_{V_2}^{V_3} pdV = \text{area}) \quad .. \quad \text{(ii a)}$$

Operation III. The cylinder is removed and placed in contact with the sink and the gas is compressed quasi-statically till it reaches a point on the adiabatic through A. This is an isothermal change at temperature T_2 on the perfect gas scale. There is a heat flow from the gas to the sink and a work-flow from the piston and its load to the gas. This operation is represented by CD in the indicator diagram.

As in operation I,

Q_2, heat rejected to the sink $= W_3$, work done **on** the gas

or $$Q_2 = RT_2 \log \frac{V_3}{V_4} \quad .. \quad \text{(iii)}$$

$$= \text{area } DCHFD \quad .. \quad \text{(iii a)}$$

Operation IV. The cylinder is transferred to the insulating stand G again and the gas is compressed quasi-statically till its pressure, volume and temperature are restored to the point A. This is an adiabatic process. There is a flow of work from the piston and its load to the gas.

As in operation II,

W_4, work done **on** the gas $= \dfrac{R(T_1-T_2)}{(\gamma-1)}$ .. (iv)

$$= \text{area } ADFEA \quad .. \quad \text{(iv a)}$$

Calculation of efficiency. In the operation of a Carnot engine there is 'work-flow' from the system (gas in the cylinder) to the piston and its load in the direct process and in the reverse process there is a 'work-flow' from the piston and its load to the system, but on the whole there is a net work-flow from the gas (the system) to the piston and its load. This is the useful work.

$$\therefore \quad W, \text{ useful work} = W_1 + W_2 + (-W_3) + (-W_4)$$

$$= Q_1 + W_2 - Q_2 - W_4 = Q_1 - Q_2 \qquad (\because \; W_2 = W_4)$$

or $W = \text{area } ABGEA + \text{area } BCHGB - \text{area } DCHED - \text{area } ADFEA$

$$= \text{area } ABCD$$

$\therefore \; \eta$, efficiency of Carnot engine $= \dfrac{W}{Q_1}$ (by definition of efficiency)

or
$$\eta = 1 - \frac{Q_2}{Q_1} = 1 - \frac{RT_2 \log \dfrac{V_3}{V_4}}{RT_1 \log \dfrac{V_2}{V_1}}$$

or
$$\eta = 1 - \frac{T_2}{T_1} \frac{\log \dfrac{V_3}{V_4}}{\log \dfrac{V_2}{V_1}}$$

For an adiabatic change, we have $TV^{\gamma-1} =$ a constant

Since B and C are points on the same adiabatic

$$\therefore \quad T_2 V_3^{\gamma-1} = T_1 V_2^{\gamma-1}.$$

Also A and D are points on the same adiabatic.

$$\therefore \quad T_1 V_1^{\gamma-1} = T_2 V_4^{\gamma-1}$$

Dividing we have

$$\frac{V_2}{V_1} = \frac{V_3}{V_4}$$

$$\therefore \quad \frac{Q_2}{Q_1} = \frac{RT_2 \log \dfrac{V_3}{V_4}}{RT_1 \log \dfrac{V_2}{V_1}} = \frac{T_2}{T_1}$$

or
$$\frac{Q_1}{T_1} = \frac{Q_2}{T_2} \qquad \ldots \quad (9.2)$$

Thus we see that in an isothermal expansion of a perfect gas, the ratio of the final volume to the initial volume is a constant called the isothermal expansion ratio (r)

$$\therefore \quad \frac{V_2}{V_1}=\frac{V_3}{V_4}=r$$

This can be re-written as

$$\frac{V_3}{V_2}=\frac{V_4}{V_1}=\rho$$

called the adiabatic expansion ratio

$$\therefore \quad \eta=1-\frac{T_2}{T_1}\frac{\log r}{\log r}=1-\frac{T_2}{T_1}$$

$$\boxed{\eta=1-\frac{T_2}{T_1}} \qquad .. \qquad (9.3)$$

In terms of ρ, $\dfrac{T_2}{T_1}=\left(\dfrac{V_2}{V_3}\right)^{\gamma-1}=\left(\dfrac{1}{\rho}\right)^{\gamma-1}$

$$\therefore \quad \eta=1-\left(\frac{1}{\rho}\right)^{\gamma-1} \qquad .. \qquad (9.3\text{ a})$$

It can be clearly seen from Eq. 9.3 that the efficiency of an engine (ideal one) is 100% only when $T_2=O$ and $T_1 \neq O$. Since nature conceals the absolute zero (on the perfect gas scale), the **complete conversion of heat into work is impossible.**

**9.4. Reversibility of the Carnot Cycle

It is important to notice that the Carnot cycle is reversible at each stage, that is, instead of abstracting heat Q_1 from a source at higher temperature T_1 and rejecting a part Q_2 to the sink at lower temperature T_2 and giving the net useful work $W=Q_1-Q_2$ to the piston and its load, it can be made to abstract heat Q_2 from the sink of heat at lower temperature T_2, a net work W is done on the system and the total energy $Q_1=Q_2+W$ is transferred to the source of heat at some higher temperature T_1. This is done by proceeding along the reverse route $DCBAD$.

An engine in which the working substance, performs a reversible cycle is called a *reversible engine.* Engines in which the cycle is irreversible, are called *irreversible engines.*

9.5. Practical Engines : Steam Engine : Petrol Engine : Diesel Engine

We will now consider the construction and working of a few practical engines, viz. **Steam engines, Petrol Engines** and **Diesel engines**.

(a) STEAM ENGINE

In 1768 James Watt of England invented this engine. The following are the essential parts of a steam engine.

1. A Boiler. This is a stout cylindrical steel vessel in which steam is generated under pressure by burning coal in a furnace. The steam generated is led into the next part of the engine, namely, a steam chest through a metal pipe. Immediately before entering into the steam chest, the supply-pipe is intercepted by a throttle valve which controls the supply of steam from the boiler to the steam-chest The boiler is provided with a safety valve which is worked by a third class-lever.

2. The steam or valve chest. It is a rectangular stout box mounted on the cylinder of the engine There are three holes bored in the wall common to the steam chest and the cylinder. The middle

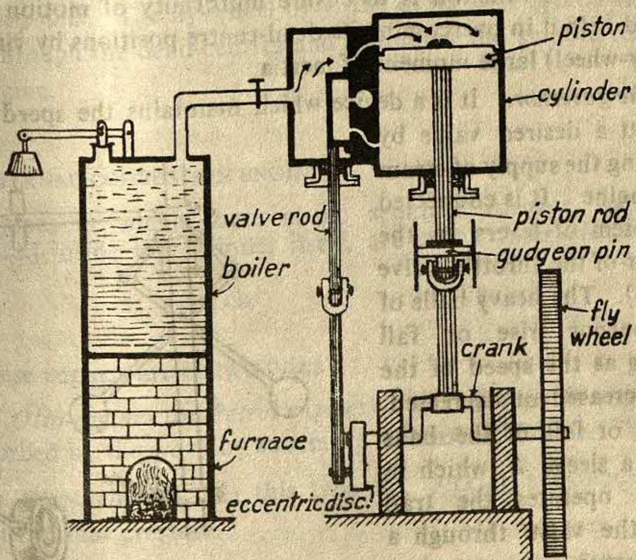


Fig. 9.5

one is connected with the exhaust pipe leading to the sink (condenser) of the engine while the other two communicate with the upper and lower parts of the cylinder.

3. Sliding Valve. It is commonly a D-shaped valve sliding to and fro over the surface containing the ports (holes) so that at a ime only one of the outer ports communicates with the cylinder and the middle one is always cut off from it. It is run by the main-shaft of the engine itself. A rod rigidly attached to the valve is joined at cross head to another rod joined to an eccentric disc mounted on the shaft of the engine. Because of this eccentric mounting the valve reciprocates (i.e., moves to-and-fro) in the steam chest.

4. The cylinder and the piston. The cylinder of the engine is a stout cylindrical vessel containing a steam-tight piston. A rod attached to the piston passes through a stuffing-box and is constrained to move in a straight line by the **gudgeon** pin travelling between two guides. To this pin is connected another rod whose other end is joined to the **crank** of the shaft. A 'crank' is simply a short off-set of the shaft off its axis of rotation. The to-and-fro motion of the piston is transformed into rotational motion of the shaft by the crank.

5. The fly-wheel. It is a large and massive wheel mounted on the main shaft. Its function is to ensure uniformity of motion of the shaft and help it in overcoming its dead-centre positions by virtue of its (of fly-wheel) large moment of inertia.

6. The governor. It is a device which maintains the speed of the engine at a desired value by controlling the supply of steam to the engine It is connected by a system of levers to the trap door of the throttle valve (Fig. 9.6). The heavy balls of the governor rise or fall according as the speed of the engine increased or decreased. This rise or fall of the balls operates a sleeve *S* which in its turns operates the trap door of the valve through a series of levers.

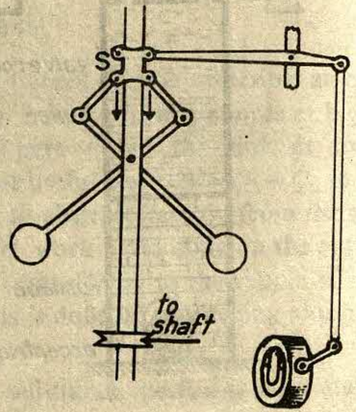


Fig. 9.6

Action. To explain the

working of the engine, let us suppose that to begin with the positions of the piston and the valve are as shown in the Fig. 9.5. Steam enters the cylinder through the upper hole and pushes down the piston with great force, the steam on the other side of the piston being thrown out through the lower and the middle holes to the exhaust pipe only to be condensed in a vessel called condenser in the condensing type of engine or to be expelled to the atmosphere in the non-condensing type. When the piston has reached the extreme end of its downward stroke, the valve takes up its other extreme position. Now steam finds its way into the cylinder through the lower hole and pushes the piston upwards, the steam above the piston being pushed to the exhaust pipe through the upper and the middle holes. When the piston reaches the upper end, one cycle is completed, and it is repeated in quick succession. The to-and-fro motion of the piston is converted into rotatory motion through the action of the crank.

Twice during each revolution of the shaft, the crank and the connecting rod come into the same straight line, when there is no turning effect on the shaft. These positions are called the *dead-centres* or *dead-points*. The fly-wheel by virtue of its large inertia carries the shaft through the dead-centres and maintains the smooth running of the engine.

(b) INTERNAL COMBUSTION ENGINES

Engines in which there is no separate furnace, but heat is generated inside the cylinder itself, are called *Internal Combustion Engines*.

These engines are of two types :

(a) *Otto-engines* (or *Petrol engines*). In this type of engines heat is absorbed by the working substance at **constant volume.**

(b) *Diesel engines*. In this type the working substance absorbs heat at **constant pressure.**

(a) *Petrol Engine*. This engine was designed by Otto in 1876.

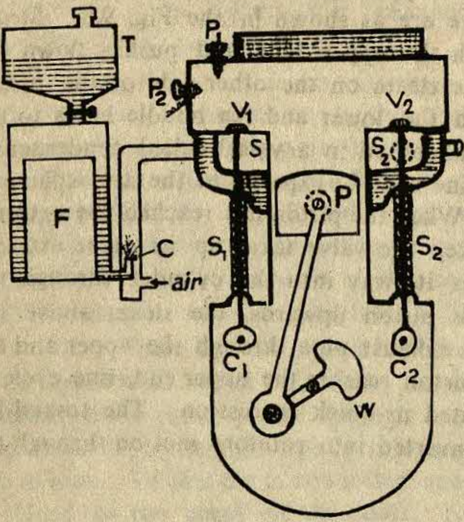


Fig. 9.10

The fuel of this engine is petrol and hence it is called *Petrol Engine*.

In the Fig. 9.10 of a petrol engine T is the petrol tank in which the fuel, namely petrol, is stored. From it petrol trickles into a chamber placed immediately below the tank. This chamber contains a float F which raises the level of petrol to produce sufficient pressure to force petrol in the form of fine spray through a narrow orifice called the *carburettor*. Here the petrol vapour and air are mixed in correct proportion which forms the explosive mixture. The carburetted mixture is led into the side chamber S through a pipe. From here the mixture enters into the combustion chamber through a valve V_1 operated by the engine itself There is another side chamber into which burnt gas is pushed through a similar valve V_2 only to be expelled to the atmosphere through the exhaust pipe. The combustion chamber contains two sparking plugs for the ignition of the explosive mixture at the right moment. The spark is led by a **magneto**, which is a magneto-electric machinery driven by the engine itself. The chamber is water-jacketed to prevent excessive heating. The valves V_1 and V_2 are operated by two rods resting at their lower ends on two eccentric mountings C_1 and C_2 on the shaft of the engine rotating at half the speed of rotation of the main shaft. These are called *cams*. The valves are held down on their seats by two powerful springs and lifted at proper moments by the action of the cams. The cylinder of the engine is in continuations of the ignition chamber and is filled with a cylinder-shaped piston P. It is connected to the crank of the shaft by a rod. There is a counter weight W which by virtue of its large mass carries the piston through its dead-centres.

Principle of Action. Thermodynamically a petrol engine may be simplified to a cylinder and a piston provided with inlet and outlet valves opening at the right moments and sparking plugs igniting the gas at the opportune moment. The working substance is **air**, the function of petrol being merely to heat the air by its combustion. The cycle in which it works is called the *Otto cycle.* In one complete cycle, there are four strokes of the piston and hence these engines are also called *four-stroke engines.*

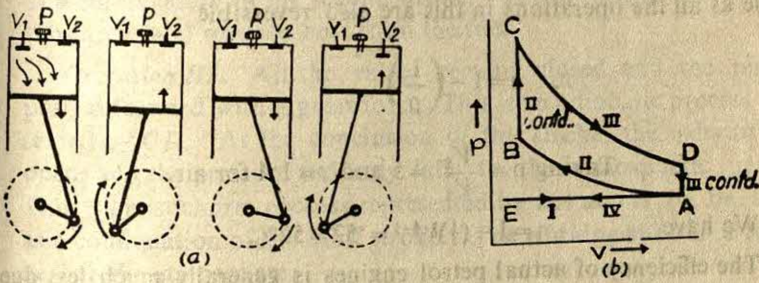


Fig. 9.11

(*i*) *First stroke* (or *charging stroke*). The inlet valve V_1 opens up and the carburetted mixture is sucked into the cylinder by the forward motion of the piston. This is an isobaric process represented by *EA* in the indicator diagram.

(*ii*) *Second stroke* (or *the compression stroke*). Both the valves remain closed and the mixture of air and petrol is compressed by the backward motion of the piston. This is an adiabatic process represented by *AB*. At the end of the compression, when the volume is reduced to 1/5 of the original volume and temperature rises to about 600°C due to compression, a series of electric sparks ignite the gas, when all of a sudden the pressure increases and the temperature rises to about 2000°C almost without change in volume. This is represented by *BC*. This is an isochoric process and is to be treated as a continuation of the second stroke. In this stroke the working substance (air) receives heat from the fuel at the last moment of the stroke.

(*iii*) *Third stroke* (or *the working stroke*). Both the valves still remain closed and the piston is pushed forward with a great force. This is an adiabatic process represented by *CD*. At the conclusion of this stroke, the exhaust valve opens and the pressure suddenly falls to the atmospheric pressure. This is an isochoric process

represented by *DA* and is to be treated as a continuation of the third stroke. The working substance releases heat to the atmosphere.

(*iv*) *Fourth stroke* (or *the exhaust stroke*). The exhaust valve remains open and the burnt gas is expelled to the atmosphere by the backward motion of the piston. This is an isobaric process represented by *AE*.

The efficiency of this cycle is the same as a reversible Carnot cycle as all the operations in this are also reversible

$$\therefore \quad \eta = 1 - \left(\frac{1}{\rho}\right)^{\gamma-1}$$

$$\text{Taking } \rho = \frac{V_2}{V_1} = 5 \text{ and } \gamma = 1{\cdot}4 \text{ for air.}$$

We have, $\eta = 1 - (\frac{1}{5})^{1\cdot4-1} = {\cdot}52 = 52\%$.

The efficiency of actual petrol engines is generally much less due to turbulence, acceleration, conduction etc. which are usually present.

(a) *Diesel Engine.* This engine was invented by Diesel. The fuel used in it is a crude oil called *Diesel oil.* Thermodynamically a Diesele ngine may also be simplified to a piston-cylinder system provided with in-let and out-let valves opening at the right moments

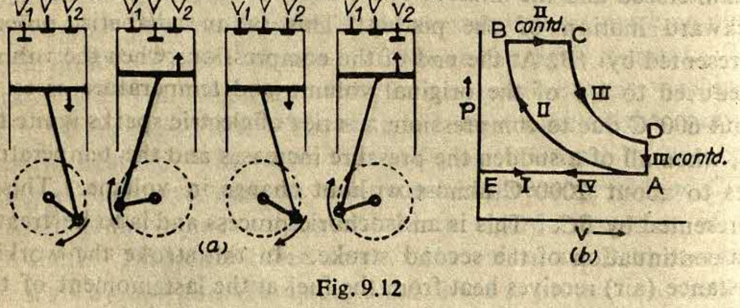


Fig. 9.12

and an oil valve to inject oil at the opportune moment. The cycle in which it works is called the *Diesel cycle.*

Operation I. The inlet valve opens up and pure air is sucked into the cylinder by the forward motion of the piston. This is an isobaric process represented by *EA* in the indicator diagram.

Operation II. All the valves remain closed an the air is compressed by the backward motion of the piston. This is an adiabatic process represented by AB. At the conclusion of the stroke the air is compressed heavily to $\frac{1}{17}$th of its volume resulting in an excessive rise in temperature (about 1000°C and pressure 36 atmosphere). At this moment the oil valve injects the liquid fuel which readily ignites at this high temperature. This takes place at constant pressure. This is represented by BC in the indicator diagram. This is to be treated as a continuation of the second stroke. In this stroke the working substance (air) receives heat from the fuel.

Operation III. All the valves remain closed and the piston is pushed forward with a great force. This is an adiabatic process represented by CD. At the conclusion of this stroke, the exhaust valve opens and the pressure suddenly falls to the atmospheric pressure. This is an isochoric process represented by DA and is to be treated as a continuation of the third stroke. The working substance releases heat to the atmosphere.

Operation IV. The exhaust valve remains open and the burnt gas is expelled to the atmosphere by the backward motion of the piston. This is an isobaric process represented by AE.

The efficiency of this cycle is approximately the same as that of the reversible Carnot engine.

$$\therefore \quad \eta \simeq 1-\left(\frac{1}{\rho}\right)^{\gamma-1}.$$

In the diesel engine $\rho \simeq 17$ and $\gamma = 1{\cdot}4$ for air.

$\therefore \quad \eta = 1-(\frac{1}{17})^{1\cdot4-1} = 1-(\frac{1}{17})^{\cdot4} = 1-{\cdot}32 = {\cdot}68 = 68\%.$

9.6. Indicated Horse Power (I.H.P.) : Brake Horse Power (B.H.P.) : Mechanical Efficiency

In an engine the real power developed is indicated by the term *Indicated Horse Power.* This depends on (i) the average pressure developed on the piston (p_m), (ii) the area of cross-section of the piston (A), (iii) the distance (L) through which the piston moves to and fro, and (iv) the number of working strokes (N) per second,

$$\therefore \quad \text{Indicated Horse power} = p_m\, L.\, A.\, N. \text{ watt} = \frac{p_m\ L.\ A.\ N.}{746} \text{ H.P.}$$

If n is the revolutions per second of the engine, then in a steam engine $N = 2n$ and in a four-stroke internal combustion engine $N = \frac{1}{2}n$.

The power is so called as it is determined from the mean pressure of the working substance on the piston, which is indicated by an instrument known as *indicator*.

All the power developed within the cylinder of an engine is not available to drive outside machineries, for part of it is wasted in the driving of the engine itself. Thus the power actually available for driving outside machineries is always less than the indicated power and is known as *Brake Horse Power*. It is so called because it is usually determined by making the engine work against brake applied on the fly-wheel.

The mechanical efficiency of the engine is defined as

$$\text{Mechanical efficiency} = \frac{\text{Brake power}}{\text{Indicated power}}.$$ It depends on the load.

In modern engines the mechanical efficiency is often greater than 80% at full load.

**9.7. Refrigerator

The converse of a heat engine is a refrigerator. A refrigerator extracts heat from a body at lower temperature T_2, net work is done on the working substance and it delivers the total energy to another body at higher temperature T_1. When the Carnot engine is worked in reverse order, it becomes a refrigerator The efficiency of a refrigerator is measured in terms of what is called the '*coefficient of performance*'. If Q_2 is the heat extracted by it at temperature T_2, W is the net work done **on** it, then its coefficient of performance is

$$e = \frac{Q_2}{W} = \frac{Q_2}{Q_1 - Q_2} = \frac{1}{\frac{Q_1}{Q_2} - 1} = \frac{1}{\frac{T_1}{T_2} - 1}$$

or

$$e = \frac{T_2}{T_1 - T_2}.$$

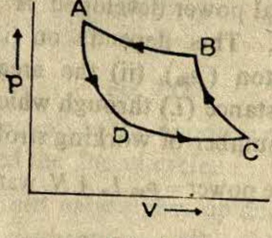


Fig. 9.13

Fig. 9.13 shows the Carnot refrigeration cycle.

**9.8. Practical Refrigerators

The practical refrigerator works on the fact that '**evaporation causes cooling**'. The liquid which is evaporated is called the *refrigerant*. Common refrigerants are NH_3, SO_2, freon (CCl_2F_2). From the thermodynamical point of view, a compression (Frigidaire) type of refrigerator consists of a pump P, a condenser C, a refrigerating chamber R and a controlling valve V.

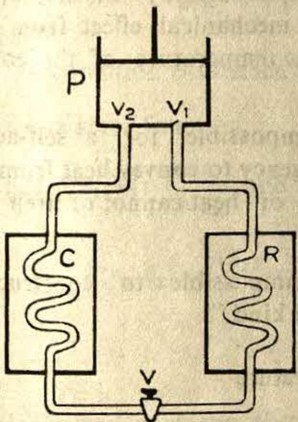


Fig. 9.14

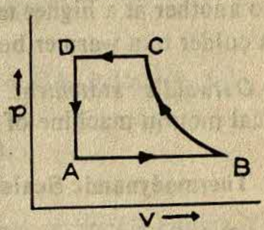


Fig. 9.15.

Practical refrigeration cycle : The piston moves outwards when V_1 opens. The refrigerant evaporates. This is an isothermal-isobaric process represented by AB. The heat of evaporation is extracted from the refrigerating chamber R. During the next inward motion the valves V_1 and V_2 remain closed and the vapour is compressed. This is an adiabatic process represented by BC. At the conclusion of the inward motion V_2 opens and the vapour suddenly condenses in the condenser, delivering heat of condensation to the surroundings. Almost at the same time the controlling valve V opens and the condensed liquid is transferred to the other side to begin a fresh cycle. This is represented by DA of the indicator diagram (Fig. 9.15).

***9.9. Second Law of Thermodynamics

'Complete conversion of **heat into work is impossible**', as learnt from the theory of Carnot's engine, forms the basis of second law of thermodynamics. It is to be accepted as a fundamental fact that a heat engine essentially rejects a certain amount of heat to a source of

heat at lower temperature. It can never be possible that an engine receives heat from a source and converts the whole of it into work without giving away **a part of the heat** received by it to a body at lower temperature. This is to be treated as something fundamental in Physics because all attempts to contradict it have failed. Different authors state this fact in different ways. So we have several standard statements of the second law of thermodynamics.

(*a*) *Kelvin's statement*. "It is impossible by means of an inanimate material agency to derive a mechanical effect from any portion of matter by cooling it below the temperature of the coldest of the surrounding objects."

(*b*) *Clausius' statement*. "It is impossible for a self-acting machine, unaided by any external agency to convey heat from one body to another at a higher temperature or heat cannot of itself pass from a colder to a warmer body."

(*c*) *Ostwald's statement*. "It is impossible to construct a perpetual motion machine of the second kind."

**9.10. Thermodynamic Scale of Temperature

The efficiency of a reversible engine is independent of the working substance and depends only on the two temperatures between which the engine works. This led Kelvin to suggest a new scale of temperature. The above fact may be mathematically stated in the form

$\eta=\dfrac{W}{Q_1}=f(\theta_1, \theta_2)$ where Q_1 is the heat taken by the engine from the source at temperature θ_1 on any arbitrary scale, W is the work delivered by it and $Q_2=Q_1-W$ is the heat rejected to the sink at the temperature θ_2 on the same scale.

$$\therefore \quad \eta=\frac{Q_1-Q_2}{Q_1}=f(\theta_1, \theta_2)$$

or
$$\frac{Q_1}{Q_2}=\frac{1}{1-f(\theta_1, \theta_2)}=F(\theta_1, \theta_2) \quad \ldots \quad \text{(i)}$$

where F denotes some other unknown function.

Let us now introduce a temperature θ_3 in between θ_1 and θ_2 and work a reversible engine in between the pairs of temperatures (θ_1, θ_3),

(θ_3, θ_2) and (θ_1, θ_2). Let Q_1, Q_2 and Q_3 be the heat taken or delivered (as the case may be) by the engine at these temperatures respectively, then

$$\frac{Q_1}{Q_3}=F(\theta_1, \theta_3); \quad \frac{Q_3}{Q_2}=F(\theta_3, \theta_2) \quad \text{and} \quad \frac{Q_1}{Q_2}=F(\theta_1, \theta_2).$$

Now
$$\frac{Q_1}{Q_2}=\frac{Q_1}{Q_3}\times\frac{Q_3}{Q_2}$$

or
$$F(\theta_1, \theta_2)=F(\theta_1, \theta_3)\times F(\theta_3, \theta_2) \quad .. \quad \text{(ii)}.$$

This condition is satisfied only when $F(\theta_1, \theta_2)$ is of the form $\frac{\psi(\theta_1)}{\psi(\theta_2)}$ where ψ is some function.

Then $F(\theta_1, \theta_3)=\frac{\psi(\theta_1)}{\psi(\theta_3)}$ and $F(\theta_3, \theta_2)=\frac{\psi(\theta_3)}{\psi(\theta_2)}$ and we have

$$F(\theta_1, \theta_2)=\frac{\psi(\theta_1)}{\psi(\theta_2)}=\frac{\psi(\theta_1)}{\psi(\theta_3)}\times\frac{\psi(\theta_3)}{\psi(\theta_2)}=F(\theta_1, \theta_3)\times F(\theta_3, \theta_2)$$

$$\therefore \quad \frac{Q_1}{Q_2}=\frac{\psi(\theta_1)}{\psi(\theta_2)} \quad .. \quad \text{(iii)}.$$

We know from the theory of Carnot reversible engine if $\theta_1>\theta_2$, then $Q_1>Q_2$ and therefore $\psi(\theta_1)>\psi(\theta_2)$, that is, $\psi(\theta)$ is a quantity which increases monotonically with θ.

Denoting the value of $\psi(\theta)$ by τ we have

$$\frac{Q_1}{Q_2}=\frac{\tau_1}{\tau_2} \quad .. \quad \text{(iv)}.$$

This relation forms the basis of defining a new scale of temperature τ, which is called the *thermodynamic* or *Kelvin Scale*. According to this relation the ratio of any two temperatures on this scale is equal to the ratio of the heats taken in or rejected by a reversible engine working between the two temperatures.

Once the basis of a scale is found, it now remains to define the absolute zero of the scale and fix up the size of the degrees on this scale. The absolute zero is that temperature which one can think of on a particular theory or principle. For example, on the perfect gas scale the absolute zero is that temperature where a perfect gas of any mass contracts to zero volume, because by the maximum stretch of our imagination we can think of zero volume of a gas, but never of a

negative volume. On the theory of the reversible engine, on which the Kelvin scale is based, we can think of an engine of hundred per cent efficiency by the maximum stretch of our imagination and never more than 100. Hence according to the thermodynamic scale, the absolute zero is that temperature of the sink, where a reversible engine working between this temperature and any other higher temperature will have hundred per cent efficiency

The zero of the scale having been determined, the size of degrees is now to be fixed. In confirmity with the common practice, the interval between the freezing point and the boiling point of water at normal pressure is divided into 100 equal parts. That is,

$$\frac{Q_{steam}}{Q_{ice}}=\frac{\tau_{ice}+100}{\tau_{ice}}$$

where τ_{ice} is the freezing point of water on the Kelvin Scale.

The thermodynamic scale is thus completely defined and fixed.

Lastly, one very pertinent question arises—how to realise this scale in practice, because the Carnot reversible engine does not exist.

If T_1 and T_2 are the temperatures of the source and sink on the perfect gas scale, then we have shown in the working of the Carnot reversible engine, that

$$\frac{Q_1}{T_1}=\frac{Q_2}{T_2}, \text{ or } \frac{Q_1}{Q_2}=\frac{T_1}{T_2} \quad \cdot\cdot \quad \text{(iv)}.$$

Equation (iv) combined (iii) yields

$$\frac{\tau_1}{\tau_2}=\frac{T_1}{T_2} \quad \cdot\cdot \quad \text{(v)}.$$

Obviously, when $T_1=0$, $\tau_1=0$ and

$$\frac{\tau_{ice}+100}{\tau_{ice}}=\frac{T_{ice}+100}{T_{ice}}, \text{ or } \tau_{ice}=T_{ice}.$$

Thus the temperatures of a body on the perfect gas scale and the thermodynamic scale are identical. Hence the thermodynamic scale is realised in practice by the perfect gas scale.

**9.11. Entropy and Second Law of Thermodynamics

The first law of thermodynamics cannot tell us in which direction any physical or chemical process involving energy changes, would take place. It simply says that in any process energy must remain conserved. The criterion for the possibility of occurrence of any

physical or chemical change is given by the second law of thermodynamics through the change of a new state variable or property of the system or systems taking part in the process. This new state variabale or property is called *entropy* introduced by Clausius.

Let us start by considering a Carnot cycle, which is also the basis of the formulation of the second law of thermodynamics. For such a cycle we have

$$\frac{Q_1}{T_1}=\frac{Q_2}{T_2}.$$

Here Q_1 is the heat taken by the working substance from the source and Q_2 is the heat rejected to the sink. If we take Q's with proper sign, Q being positive when it enters the working susbstance and negative when heat leaves it, we have to write this relation as

$$\frac{Q_1}{T_1}=\frac{-Q_2}{T_2}$$

or $$\frac{Q_1}{T_1}+\frac{Q_2}{T_2}=0, \quad \text{or} \quad \Sigma\frac{Q}{T}=0 \qquad \text{(i)}.$$

This equation states that the algebraic sum of quantities Q/T is zero for a Carnot cycle, which is a reversible cyclic process.

Let us now consider an arbitrary reversible cyclic process. Draw a set of closely spaced isotherms and then join them by short adiabatic lines cutting the cycle (Fig. 9.16). This way the arbitrary cycle is divided into a large number of tiny Carnot cycles. Now, first traverse the individual Carnot cycles and then go once round the cycle along the jagged path. See that the two traversals are exactly equivalent to each other as far heat transfer and work done are concerned, because adjacent Carnot cycles have a common isotherm and its traversal in opposite directions in the two adjacent Carnot cycles cancel each other. Thus heat transfer and work done along the jagged path are the same as the heat transfer and the work done in traversing the Carnot cycles. If ΔQ is the heat transfer at temperature T then for the Carnot cycles we have

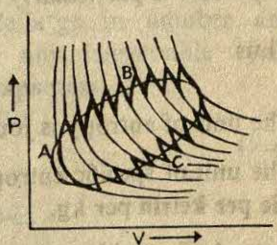


Fig. 9.16

from (i)

$$\Sigma \frac{\Delta Q}{T} = 0.$$

In the limiting case when the Carnot cycles are infinitely small, the jagged path coincides exactly with the arbitrary cyclic process. Thus for any reversible cyclic process

$$\oint_R \frac{dQ}{T} = 0$$

where $\oint$ indicated the integral (i.e., summation) for a complete traversal of a cycle and R stands for 'revèrsible cycle'.

Thus $$\oint_R \frac{dQ}{T} = 0 \qquad .. \qquad (9.4).$$

If the integral of a quantity around any closed path, reversible or irreversible, is zero, that quantity is called a state variable or property of the system. Taking a clue from the above result (Eq. 9.4), Clausius introduced a state variable (or property) of a system and named it the Entropy (S) of the system. Thus the *entropy of a system is a state variable of the system like temperature, pressure, volume and internal energy of it, whose integral around any cyclic process, reversible or irreversible is zero and along a reversible cyclic process is particularly equal to the integral of dQ/T.*

Thus $$\oint_{\text{I or R}} dS = \oint_R \frac{dQ}{T} = 0 \qquad .. \qquad (9.5).$$

The unit of entropy is joule per kelvin (JK^{-1}).

The unit of specific entropy is obviously **joule per kelvin** per kg or **calorie per kelvin** per kg.

Instead of considering a cyclic process, if we consider a change of state of the system between any two equilibrium states a and b, then a change of entropy between a and b must be independent of the processes, reversible or irreversible, of change between a and b and be equal to the integral of dQ/T from a to b along a reversible process.

That is, $$\Delta S = \int_{\text{I or R}\,a}^{b} dS = \int_{\text{R}\,a}^{b} \frac{dQ}{T}$$

or $$S_b - S_a = \int_{\mathrm{I\ or\ R}\ a}^{b} dS = \int_{\mathrm{R}\ a}^{b} \frac{dQ}{T} \quad \ldots \quad (9.6).$$

Entropy—Irreversible processes. Let us consider a simple irreversible cycle, which absorbs a heat Q_1 at T_1 and rejects a heat Q_2 at T_2. The efficiency of an engine (irreversible) performing such a cycle is given by $\frac{Q_1 - Q_2}{Q_1}$, and according to the second law of thermodynamics the efficiency of such an engine is less than that of a reversible engine working between the same two temperatures. The efficiency of a reversible engine working between T_1 and T_2 is $1 - \frac{T_2}{T_1}$.

$$\therefore \quad 1 - \frac{Q_2}{Q_1} < 1 - \frac{T_2}{T_1}$$

or $$\frac{Q_1}{T_1} - \frac{Q_2}{T_2} < 0, \quad \text{or} \quad \frac{Q_1}{T_1} + \left(\frac{-Q_2}{T_2}\right) < 0$$

or $$\Sigma \frac{Q}{T} < 0.$$

In the limiting case, when the cycles are large in number and they are infinitely small, we can write for any irreversible cyclic process, as we did for any reversible cyclic process,

$$\oint_{\mathrm{I}} \frac{dQ}{T} < 0 \quad \ldots \quad (9.7).$$

This relation combined with Eq. 9.4 gives

$$\oint \frac{dQ}{T} \leqslant 0$$

where the inequality sign holds for irreversible cyclic processes and the equality sign for reversible cyclic processes.

This is known as Clausius' inequality. Now consider three paths A, B and C joining two equilibrium states 1 and 2 of a system. Here A and B are reversible and C is irreversible. Consider the reversible cyclic process

$1 \xrightarrow{A} 2 \xrightarrow{B} 1$ i.e., (1 to 2 via A and from 2 to 1 via B)

$$_R\oint \frac{dQ}{T} = {}_A\int_1^2 \frac{dQ}{T} + {}_B\int_2^1 \frac{dQ}{T} = 0.$$

Now consider the irreversible cyclic process $1 \xrightarrow{A} 2 \xrightarrow{C} 1$

$$_I\oint \frac{dQ}{T} = {}_A\int_1^2 \frac{dQ}{T} + {}_B\int_2^1 \frac{dQ}{T} \leqslant 0.$$

Combining these two, we have

$$_C\int_2^1 \frac{dQ}{T} \leqslant {}_B\int_2^1 \frac{dQ}{T}$$

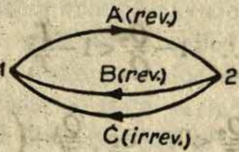


Fig. 9.17

or $$_C\int_1^2 \frac{dQ}{T} \leqslant {}_B\int_1^2 \frac{dQ}{T}$$

or $$_I\int_1^2 \frac{dQ}{T} \leqslant {}_R\int_1^2 \frac{dQ}{T}$$ ($\because$ C is an irreversible process and B is a reversible process).

According to the definition of entropy

$$\Delta S = {}_{I,R}\int_1^2 dS = {}_R\int_1^2 \frac{dQ}{T}.$$

But, $$_R\int_1^2 \frac{dQ}{T} \geqslant {}_I\int_1^2 \frac{dQ}{T}$$ as shown above.

$$\therefore \ \Delta S \geqslant {}_I\int_1^2 \frac{dQ}{T}.$$

This is also called Clausius's inequality.

The universe as a system is an isolated one. Therefore for any process taking place in the universe we have $\triangle S \geqslant 0$ because $dQ=0$ for an isolated system. Here inequality sign holds for irreversible process and equality sign holds for reversible process.

This result is of great importance. This fixes up the criterion for irreversibility of any process. The process in which $\triangle S>0$, that is, there is an increase in entropy of the universe is irreversible and the one in which $\triangle S=0$, that is, the entropy remains constant, is reversible. In this universe, all natural processes are taking place irreversibly and hence by all processes going around us the entropy of the universe is ever increasing. The natural direction of all processes is toward irreversibility. Thus the second law of thermodynamics fixes up the direction of a physical or chemical process.

"*A physical or chemical process will proceed in the direction that causes the entropy of the universe to increase.*" This is the principle of increase of entropy of the universe.

**9.12. Entropy and Availability of Heat Energy for Work

Suppose T_0 is the temperature of the **coldest** possible sink. Whenever heat will flow to this sink (coldest possible) this amount of the heat will never be available for work because to derive work out of heat from this reservoir, we need a colder reservoir to work a heat engine. But this is impossible, because the sink we have considered is the 'coldest'. How then another body be colder than that body? We will now show that heat unavailable for work is equal to the increase of the entropy of the universe.

Consider a body at temperature T. To derive the maximum work out of a certain amount of heat $\triangle Q$ of this body, we must employ the most efficient engine and that engine is obviously the Carnot reversible engine, working between T and T_0. The efficiency of a Carnot reversible engine working between T and T_0 is $\left(1-\frac{T_0}{T}\right)$. Hence the maximum available work out of $\triangle Q$, when the body is at temperature T, is $\triangle Q\left(1-\frac{T_0}{T}\right)$ Let this heat flow to a lower temperature T' by conduction, which is an irreversible process.

Now the available work out of $\triangle Q$ from a body at temperature T' is

$$\triangle Q\left(1-\frac{T_0}{T'}\right).$$

Thus the unavailable work due to conduction

$$=\Delta Q\left(1-\frac{T_0}{T}\right)-\Delta Q\left(1-\frac{T_0}{T'}\right)$$

$$=T_0\,\Delta Q\left(\frac{1}{T'}-\frac{1}{T}\right).$$

ΔS, change of entropy of the universe $=\frac{-\Delta Q}{T}+\frac{\Delta Q}{T'}$

$$=\Delta Q\left(\frac{1}{T'}-\frac{1}{T}\right).$$

$\therefore$ Unavailable work due to conduction $= T_0\Delta S$.

Thus, energy (work) unavailable $\propto \Delta S$.

Hence the principle of increase of entropy implies that the available energy in the universe is tending to zero. This is known as the *principle of degradation* or *'running downhill' of energy*.

** 9.13. Entropy and Disorder

Whenever work is dissipated into internal energy, the disorderly motion of molecules is increased. It is possible therefore to regard all natural processes from this point of view, and in all cases the result obtained is that there is a tendency on the part of nature to proceed toward a state of greater disorder.

Thus the entropy of a system may also be taken as a measure of the degree of molecular disorder existing in the system.

Examples :

1. *An ideal gas engine operates in a Carnot cycle between* 227°*C and* 127°*C. It absorbs* 60 *kilo calorie at the higher temperature. How much work per cycle is this engine capable of performing ?*

Sol. η (efficiency of the engine) $=1-\frac{127+273\cdot 15}{227+273\cdot 15}$

$$=1-\frac{400\cdot 15}{500\cdot 15}$$

$$=\cdot 2.$$

Also, $\eta=\frac{W}{Q}$; $\therefore$ $\cdot 2=\frac{W}{60 \text{ kilo cal}}$

or $W=12$ kilo cal$=12{,}000$ cal

$=50400$ joule. **Ans.**

2. *The motor in a refrigerator has a power out put of* 200 *watts. If the freezing compartment is at* 270°*K and the outside air is at* 300° *K, what is the maximum amount of heat that can be extracted from the freezing compartment in* 10 *minutes* ?

Sol. e (coefficient of performance of the refrigerator)$=\frac{Q_2}{W}$

or
$$e=\frac{Q_2}{Q_1-Q_2}=\frac{1}{\frac{T_1}{T_2}-1}=\frac{T_2}{T_1-T_2}.$$

$$=\frac{270}{300-270}=9.$$

$\therefore$ $Q_2=9\times 200=1800$ watt$=1800$ joule per second

$=\frac{1800}{4{\cdot}2}$ cal per second.

$\therefore$ Heat extracted in 10 minutes$=\frac{1800}{4{\cdot}2}\times 10\times 60$

$=25{\cdot}7\times 10^4$ calories. **Ans.**

3. 20 *gm of ice at* 0°C *is heated to* 27°C. *Calculate the change in entropy of the substance.*

(*Specific latent heat capacity*$=80$ *kilo cal* kg^{-1}.)

Sol. During the change of state, heat absorbed$=\frac{20}{1000}\times 80\times 1000$

$=1600$ calorie

$=1600\times 4{\cdot}2$ joule.

$\therefore$ Change of entropy$=\frac{1600\times 4{\cdot}2}{273}=24{\cdot}6$ J K^{-1}.

During the rise in temperature from 0°C to 27°C,

the change of entropy$=\int_{273}^{300}\frac{dQ}{T}$

$$= \int_{273}^{300} \frac{\frac{20}{1000} \times 4200 \times dT}{T} \quad (\because \; dQ = m\,c\,dT)$$

$$= 84 \ln \frac{300}{273}$$

$$= 84 \times 2{\cdot}3\,(\log 300 - \log 273)$$

$$= 84 \times 2{\cdot}3\,(2{\cdot}4771 - 2{\cdot}4362)$$

$$= 84 \times 2{\cdot}3 \times {\cdot}0409$$

$$= 7{\cdot}90.$$

$\therefore$ Total change in entropy $= 24{\cdot}6 + 7{\cdot}9$

$= 32{\cdot}5 \text{ J K}^{-1}$. **Ans.**

QUESTIONS

(A)

1. The complete conversion of heat into work is (a) always possible, (b) impossible, (c) possible only when a reversible Carnot engine is available, (d) heat is never convertible into work.

2. The complete conversion of work into heat is (a) always possible, (b) possible, (c) impossible, (d) work is never convertible into heat.

3. In completing a cycle of operation an ideal Carnot engine will take (a) finite time, (b) infinite time, (c) infinitesimally small time, (d) no time.

****4. In irreversible** processes the entropy of the universe (a) increases, (b) decreases, (c) remains stationary, (d) fluctuates.

5. In reversible processes the entropy of the universe (a) increases, (b) decreases, (c) remains stationary, (d) fluctuates.

6. The upper limit of efficiency of a steam engine working between 27°C and 100°C is about (a) 20%, (b) 40%, (c) 10%, (d) 60%.

7. The upper limit of the efficiency of a petrol engine is about (a) 10%, (b) 20%, (c) 42%, (d) 52%, (e) 70%.

8. The upper limit of the efficiency of a Diesel engine is about (a) 10%, (b) 30%, (c) 50%, (d) 70%.

9. Which is the most efficient engine? (a) a steam engine, (b) a petrol engine, (c) a Diesel engine.

10. The working substance is compressed to (a) $\frac{1}{3}$rd, (b) $\frac{1}{4}$th, (c) $\frac{1}{8}$th, (d) $\frac{1}{16}$th of the original volume in a petrol engine.

11. The working substance is compressed to (a) $\frac{1}{3}$rd, (b) $\frac{1}{10}$th, (c) $\frac{1}{17}$th, (d) $\frac{1}{30}$th of the original volume in a Diesel engine.

[**Ans.** 1. (b), 2. (b), 3. (b), 4. (a), 5. (c), 6. (a), 7. (d), 8. (d), 9. (c), 10. (c), 11. (c).]

(B)

1. Explain Rankine's cycle.

2. What are reversible and irreversible processes ?

3. Explain the thermodynamics of a refrigerator. What do you mean by its coefficient of performance ?

**4. Show that in an irreversible process $\triangle S \geqslant \int_1^2 \frac{dQ}{T}$.

**5. Show that entropy change is a measure of the non-availability of energy in the universe.

**6. Explain 'entropy' as a measure of disorder of a system.

7. Explain the action of a crank and a shaft.

(C)

**1. What are reversible and irreversible processes ? Explain with examples.

**2. Describe Carnot's reversible engine and its reversible cycle. Calculate its efficiency.

3. What is a heat engine ? Describe a steam-engine and explain the action of its different parts with a neat diagram.

4. Describe a petrol engine with a neat diagram. Explain its cycle of operation.

**5. Explain how Kelvin introduced the thermodynamic scale of temperature. How is this scale realised in practice ?

**6. What is entropy ? Show that entropy increases in all reversible processes and just remains constant in a reversible process.

(D)

1. A Carnot engine operates between a hot reservoir at 320°K and a cold reservoir at 260°K. If it absorbs 500 joules of heat at the hot reservoir, how much work does it deliver ? (b) If the same engine, working in reverse, functions as a refrigerator between the same two reservoirs, how much work must be supplied to remove 1000 joules of heat from the cold reservoir ?

(Ans. 93·75 joule; 230·8 joule)

2. How is the efficiency of a reversible heat engine related to the coefficient of performance of the reversible refrigerator obtained by running the engine backward ?

$$\left(\text{Ans.} \quad e=\frac{1-\eta}{\eta}\right)$$

**3. A brass rod is in contact with a heat reservoir at 127°C at one end and a heat reservoir at 27°C at the other end. Compute the total change in the entropy arising from the process of conduction of 1200 calories of heat through the rod. Does the entropy of the rod change in the process ?

(Ans. 1 cal K^{-1}; No)

**4. Show that when a substance of mass m having a constant specific heat capacity c is heated from T_1 to T_2 the entropy change is

$$S_2-S_1=mc \ln \frac{T_2}{T_1}.$$

(E)

1. The working substance of a petrol engine is.................. (air, petrol vapour)
2. The working substance of a Diesel engine is.................. (air, diesel vapour)
3. The source of energy of a petrol engine is the energy of.................. (air, petrol)

4 When a tub of water is poured over a burning log of wood, fire is extinguished; but when water is poured over a petrol soaked wood or cloth, it is all the more aggravated. Why ?

5. The petrol engine receives energy from the fuel (petrol) at constant......... (pressure, volume)

6. The Diesel engine receives energy from the fuel (diesel) at constant......... (pressure, volume)

7. The thermal efficiency of an engine is..................(less, greater) than its mechanical efficiency.

8. The indicated power of an engine is..................(less or greater) than its brake power.

9. The indicated power is defined as..................

10. The brake power of an engine is..................(less, greater) than the indicated power.

[Ans. 1. air, 2. air, 3. petrol, 4. After petrol burns, temperature rises to about 2000°C (see working of petrol engine) and so water is at once vaporised Moreover, due to spreading of petrol over water, greater area catches fire. 5. volume, 6. pressure, 7. less, 8. greater; 9. power developed in the cylinder, 10. less.]

CHAPTER 10

KINETIC THEORY OF GASES

10.1. An Ideal Gas (Perfect Gas)—a Macroscopic Description

To describe the behaviour of any system, we select some gross properties of the system which can be measured by simple laboratory operations and we classify them as 'macroscopic' (large scale). For a gas, quantities such as pressure, temperature, volume, internal energy, entropy etc. are 'macroscopic'. Macroscopically an ideal gas (perfect gas) is that which thoroughly obeys Boyle's law and Charles' law. That is

(a) $p \propto \frac{1}{V}$ when temperature (T) constant .. Boyle's law.

(b) $p \propto T$ when V constant .. Charles' law.

Combining the two we have

$$p \propto \frac{T}{V} \text{ when } T \text{ and } V \text{ both vary}$$

or
$$pV = RT \qquad \ldots (10.1)$$

where R is a universal constant for a mole of a gas and is called the universal gas constant. Its value is 8·314 joule $mol^{-1}K^{-1}$ or $8{\cdot}3 \times 10^3$ JK^{-1} $kgmol^{-1}$. Another striking characteristic of an ideal gas is that its internal energy depends only on temperature.

That is, $U = f(T)$. ... This is called Joule's law.

10.2. The Kinetic Theory of Gases—a Microscopic Description of an Ideal Gas

The kinetic theory of gases presents an explanation of the macroscopic behaviour of an ideal gas by going into minute (microscopic) details of the atoms and molecules that make up the gas. The theory was developed by a number of physicists—among whom Robert Boyle, Daniel Bernoulli, James Joule, Rudolph Clausius, Clerk Maxwell are prominent. The theory is based on two basic assumptions :

(a) A gas consists of particles called molecules.

(b) The molecules are in motion and what we understand by heat consists in the motion of the molecules.

The following are details of the molecules and their kinetic state :

1. The molecules are in random motion and obey Newton's laws of motion. The molecules move in all directions with all possible velocities from zero to infinity and collide constantly with each other and also on the walls of the container. Though the molecules are constantly having their velocities changed in magnitude and direction due to mutual collisions, yet at a particular temperature, their over-all distribution in every element of volume remains unaffected, because of their large number and incessant collisions.

2. The volume of the molecules is a negligibly small fraction of the volume occupied by the gas, i.e., the molecules are mere point spheres.

3. Collisions are perfectly elastic and are of negligible duration. Collisions of the molecules with each other and with the wall of the container conserve momentum and kinetic energy. The time of collision is negligible in comparison to the time spent in traversing the paths between two collisions called *free paths*.

4. No appreciable forces of attraction or repulsion act on the molecules, except during a collision i.e., all energy of the gas is kinetic and nothing potential.

10.3. Calculation of the Pressure on the Kinetic Theory

Let us consider a gas in a cubical vessel of length l on each edge.

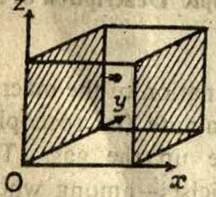


Fig. 10.1

Consider a molecule which has a velocity c and component velocities u, v and w in the direction of the edges of the cube meeting at a corner.

Then $\quad c^2 = u^2 + v^2 + w^2. \quad \ldots \text{(i)}$

If this molecule collides with the wall perpendicular to the x-axis, it will rebound with u reversed and v and w will remain unchanged. This is so because the collisions are perfectly elastic according to the third assumption of the theory. Thus the momentum of the molecule along the x-axis before collision = mu

and its momentum along the x-axis after collision $= -mu$.

$\therefore$ Change in momentum per collision $= mu - (-mu) = 2mu$.

Suppose this particle reaches the opposite wall without striking any other particle on the way. The time to traverse the cube will be l/u. So the molecule will move up and down, u/l times in one second in between the two walls perpendicular to the x-axis. The molecule will make as many collisions as it will move up and down between the walls. Hence the molecule will make u/l collisions on the two walls—one half of these collisions will be on one wall and the other half on the cther opposite wall.

$\therefore$ The number of collisions of the molecule on each wall $= \frac{1}{2}\frac{u}{l}$.

Hence the rate of change of momentum of the molecule on each wall $= 2mu \times \frac{u}{2l} = \frac{mu^2}{l}$.

According to Newton's second law this is the force exerted by the wall on the molecule and the same force is exerted by the molecule on the wall in the opposite direction according to the third law. Thus the force exerted by the molecule on the wall $= \frac{mu^2}{l}$.

To obtain the total force on the wall, we must sum up $\frac{mu^2}{l}$ for all the molecules.

$\therefore$ $F_x = \Sigma \frac{mu^2}{l}$ where the summation Σ (sigma) extends over all the molecules. Then to find the pressure, we divide this force by the area of the wall, namely, l^2.

$$\therefore \quad p_x = \frac{F_x}{l^2} = \frac{1}{l^2}\Sigma\frac{mu^2}{l} = \frac{m}{l^3}\Sigma u^2 \qquad \text{(ii)}$$

$$= \frac{m}{V}\Sigma u^2 \qquad (\because \; V = l^3)$$

or $p_x = \frac{mN}{V} \cdot \frac{\Sigma u^2}{N}$ where N is the number of molecules in the cube

or $$p_x=\frac{M}{V}\overline{u^2}$$

where $mN=M$, mass of the gas in the vessel and $\overline{u^2}$ = mean square speed along the x-axis.

Similarly $$p_y=\frac{M}{V}\overline{v^2}$$

$$p_z=\frac{M}{V}\overline{w^2}.$$

$\because$ $$c^2=u^2+v^2+w^2,$$

$\therefore$ $$\Sigma c^2=\Sigma u^2+\Sigma v^2+\Sigma w^2$$

or $$\frac{\Sigma c^2}{N}=\frac{\Sigma u^2}{N}+\frac{\Sigma v^2}{N}+\frac{\Sigma w^2}{N}$$

or $$\overline{c^2}=\overline{u^2}+\overline{v^2}+\overline{w^2}. \quad \text{.. (iii)}$$

As the velocities of molecules are symmetrically distributed, we have $$\overline{u^2}=\overline{v^2}=\overline{w^2}.$$

This combined with (iii) gives, $\overline{u^2}=\overline{v^2}=\overline{w^2}=\frac{1}{3}\ \overline{c^2}$.

$\therefore$ $$p_x=p_y=p_z=\frac{M}{V}\ \frac{1}{3}\ \overline{c^2}.$$

Thus a gas exerts equal pressure in all the directions which is a macroscopic fact. Let p be the pressure in any direction. Then

$\therefore$ $$p=\frac{1}{3}\ \frac{M}{V}\ \overline{c^2}. \quad \text{.. (10.2)}$$

Since $\frac{M}{V}=\rho$, density of gas and also $\rho=mn$ where n is the number of molecules per unit volume,

$$p=\tfrac{1}{3}\rho\ \overline{c^2} \quad \text{.. (10.2a)}$$

or $$p=\tfrac{1}{3}\,mn\ \overline{c^2}. \quad \text{.. (10.2b)}$$

The square root of $\overline{c^2}$ (mean square speed) is called the root-mean-square speed. Thus,

$$c_{rms}=\sqrt{\overline{c^2}}=\sqrt{\frac{3p}{\rho}}. \quad \text{.. (10.3)}$$

This equation relates a macroscopic quantity (the pressure) to an average value of a microscopic quantity (that is, c_{rms}).

10.4. Kinetic Interpretation of Temperature

According to the kinetic theory all the energy of an ideal gas is kinetic and nothing is potential, and the energy that we understand as heat consists in the motion of the molecules of the gas. So the temperature and the kinetic energy of an ideal gas must be proportional to each other. Or, in other words, *the kinetic energy manifests itself as the temperature of the gas.* This is the kinetic interpretation of temperature. Consider a mole of a perfect gas.

Then, the kinetic energy of a mole $=\frac{1}{2}mc_1^2+\frac{1}{2}mc_2^2+\ldots\ldots+\frac{1}{2}mc_N^2$ where N is the number of molecules in a mole of a gas. This is called *Avogadro's number.*

or the kinetic energy of a mole $=\frac{1}{2}m\,\Sigma c^2$

$$=\frac{1}{2}mN\,\frac{\Sigma c^2}{N}$$

$$=\frac{1}{2}M\overline{c^2} \qquad (\because\; mN=M).$$

If T is the temperature of the gas, we can write

$$T\propto\frac{1}{2}M\overline{c^2} \text{ or } T\propto M\overline{c^2}. \quad \text{.. (10.5)}$$

Obviously, the absolute zero on the kinetic theory is that temperature where $\overline{c^2}=0$

or $\qquad c_1^2+c_2^2+c_3^2+\ldots.+c_N^2=0.$

This is possible only when $c_1=c_2=c_3=\ldots.=0.$

Thus the absolute zero on the kinetie theory is that temperature where all the molecules are **motionless.** It is to be noted that the thermodynamic interpretation of the absolute zero does not require the molecule to be motionless.

Remember that the temperature of a gas is entirely due to only random motion of its molecules and the motion of its centre of mass has no bearing on it. The temperature of a gas in a container does

not increase when we put the container on a moving train. However, if it is assumed that the mass motion is converted into random motion of the molecules when the train suddenly stops there may be rise in temperature.

10.5. Explanation of the Macroscopic Behaviour of an Ideal Gas (i.e., Deduction of the Gas Laws)

We have by Eq. 10.2

$$pV = \tfrac{1}{3}M\overline{c^2}$$

and by Eq. 10.5

$$M\overline{c^2} \propto T.$$

$$\therefore \quad pV \propto T$$

or

$$pV = RT \qquad .. \quad (10.6)$$

where R is a universal constant. This is the famous equation of a perfect gas.

Comparing Eq. 10.2 and 10.6, we have

$$\tfrac{1}{3}M\overline{c^2} = RT$$

or

$$c_{rms} = \sqrt{\frac{3RT}{M}}. \qquad .. \quad (10.7)$$

This equation relates another macroscopic quantity (the temperature) to an average of microscopic quantity (that is, to $\overline{c^2}$ or c^2_{rms}).

(*a*) *Boyle's law.* When the temperature remains constant $p \propto \frac{1}{V}$ from Eq. 10.6. This is Boyle's law.

(*b*) *Charles' law.* When the pressure remains constant, by Eq. 10.6, $V \propto T$. This is Charles' law.

(*c*) *Pressure law* (also Charles' law for pressure). When the volume remains constant, $p \propto T$. This is the pressure law.

(*d*) *Avogadro s law.* The law states that under identical conditions of temperature and pressure, equal volumes of all gases contain an equal number of molecules.

Let us consider two gases occupying the same volume at the same temperature and pressure. Since the pressure is the same, we have from Eq. 10.2b,

$$p = \tfrac{1}{3}m_1 n_1 \overline{c_1^2} = \tfrac{1}{3}m_2 n_2 \overline{c_2^2} \qquad .. \quad \text{(i)}$$

where the subscripts 1 and 2 refer to the first and the second gas respectively.

Further, if the two gases having the same temperature are mixed, there should be no transfer of heat (that is, kinetic energy) as the two are at the same temperature. However, on mixing them, the two types of molecules will collide against one another and there will be a mutual sharing of energy. Clausius and Maxwell showed purely from dynamical considerations that the condition for over-all no transfer of energy from one type to the other is that the mean translational energy of molecules of the first gas is equal to that of the second. Hence if the two gases are at the same temperature

$$\frac{\Sigma\frac{1}{2}m_1c_1^2}{n_1}=\frac{\Sigma\frac{1}{2}m_2c_2^2}{n_2}$$

or $$m_1\overline{c_1^2}=m_2\overline{c_2^2}, \qquad \text{.. (ii)}$$

Combining (i) and (ii), we get

$$n_1=n_2.$$

(*e*) *Graham's law of diffusion.* Graham's law of diffusion states that if two gases at the same temperature and pressure be allowed to diffuse, the rate of diffusion is inversely proportional to their densities.

The rate of diffusion of a gas is evidently proportional to the mean speed of the molecules of the gas. Then

$$\frac{V_1\text{ (rate of diffusion of the first gas)}}{V_2\text{ (rate of diffusion of the second gas)}}=\frac{\bar{c}_1}{\bar{c}_2}. \qquad \text{.. (i).}$$

But the mean speed is directly proportional to the rms speed c_{rms} or the square of the mean speed is proportional to the mean square speed. That is

$$\frac{\bar{c}_1^{\,2}}{\bar{c}_2^{\,2}}=\frac{\overline{c_1^2}}{\overline{c_2^2}} \qquad \text{.. (ii)}$$

or $$\frac{\bar{c}_1}{\bar{c}_2}=\sqrt{\frac{\overline{c_1^2}}{\overline{c_2^2}}}. \qquad \text{.. (iia)}$$

As the pressure is the same

$$p=\tfrac{1}{3}m_1n_1\overline{c_1^2}=\tfrac{1}{3}m_2n_2\overline{c_2^2}$$

or $$\rho_1\overline{c_1^2}=\rho_2\overline{c_2^2} \quad \text{or} \quad \sqrt{\frac{\overline{c_1^2}}{\overline{c_2^2}}}=\sqrt{\frac{\rho_2}{\rho_1}}. \qquad \text{.. (iii)}$$

Combining (ii) and (iii),

$$\frac{\bar{c}_1}{\bar{c}_2}=\sqrt{\frac{\rho_2}{\rho_1}}. \qquad \text{.. (iv)}$$

Combining (iv) and (i) we have

$$\frac{V_1}{V_2}=\sqrt{\frac{\rho_2}{\rho_1}}.$$

(*f*) *Dalton's law of partial pressures.* The law states that the pressure exerted by a mixture of several gases is equal to the sum of the pressure exerted by the constituent gases occupying the same volume as that of the mixture. These are called partial pressures.

Let K be the kinetic energy per unit volume of an ideal gas.

Then
$$K=\tfrac{1}{2}mc_1^2+\tfrac{1}{2}mc_2^2+\tfrac{1}{2}mc_3^2+\ldots+\tfrac{1}{2}mc_n^2$$
$$=\tfrac{1}{2}m\Sigma c^2$$
$$=\tfrac{1}{2}mn\frac{\Sigma c^2}{n}=\tfrac{1}{2}\rho\overline{c^2}\quad (\because\ mn=\rho,\text{ density of the gas}$$

and n is the number of molecules per unit volume).

But
$$p=\tfrac{1}{3}\rho\overline{c^2}$$
$$K=\tfrac{1}{2}.\,3p\quad\text{or}\quad p=\tfrac{2}{3}K \qquad \ldots (10.8)$$

If $m_1,\ m_2,\ m_3\ldots$ be the masses of different gas molecules and $n_1,\ n_2,\ n_3\ldots$ their respective numbers per unit volume, then the total energy per unit volume of the mixture is given by,

The kinetic energy per unit volume of the mixture

$$=\tfrac{1}{2}m_1n_1\overline{c_1^2}+\tfrac{1}{2}m_2n_2\overline{c_2^2}+\tfrac{1}{2}m_3n_3\overline{c_3^2}+\ldots$$

By Eq. 10.8. the pressure of the mixture is

$$p=\tfrac{2}{3}\left(\tfrac{1}{2}m_1n_1\overline{c_1^2}+\tfrac{1}{2}m_2n_2\overline{c_2^2}+\tfrac{1}{2}m_3n_3\overline{c_3^2}+\ldots\right.$$
$$=\tfrac{1}{3}m_1n_1\overline{c_1^2}+\tfrac{1}{3}m_2n_2\overline{c_2^2}+\tfrac{1}{3}m_3n_3\overline{c_3^2}+\ldots$$

But the partial pressures are $p_1=\tfrac{1}{3}m_1n_1\overline{c_1^2};\ p_2=\tfrac{1}{3}m_2n_2\overline{c_2^2}\ldots$

$$\therefore\quad p=p_1+p_2+p_3+\ldots \qquad \ldots (10.9)$$

10.6. Physical Evidences of Random Motion of Molecules

Brownian Motion. There are many indirect evidences of the existence of chaotic motion of molecules. Diffusion, effusion, evaporation etc. are due to molecular agitation.

A gas tends to expand. It is endowed with this property of expansibility, because of the tendency of its molecules to fly away. But the most striking evidence of random motion of molecules is provided by the **Brownian motion.** This phenomenon was discovered by the English botanist Robert Brown in 1827. He found that pollen (the powdery substance of a flower) suspended in water, shows

a continuous random motion when viewed under a microscope. At first these motions were considered a form of life, it was soon found that small inorganic (hence lifeless) particles also behave similarly. This phenomenon can be observed if a colloidal solution be examined under a high power microscope. The suspended particles appear like bright stars, moving to and fro in an entirely haphazard fashion. The motion is perpetual and spontaneous. This motion is due to the molecular agitation of the liquid molecules. The suspended particles are in fact in a sea of liquid molecules. Each particle is pushed by liquid molecules from all sides, which give rise to a resultant unbalanced force. For bigger particles, the random impacts tend to balance out and so they do not move out. The suspended particles (called Brownian particles also) behave exactly in the same way as the molecules of the host liquid. This fact was established experimentally by Perrin in 1909. Thus the suspended particles in a colloidal solution can be utilised to determine Avogadro's number, as the suspended particles are visible and their motion, if required, can be photographed. The complete theory of the Brownian movement was developed by Einstein. Perrin determined Avogadro's number by observing the motion of a particle over a long time. This is the greatest importance of the Brownian movement.

10.7. Real Gases : Vander Waals' Equation of State

So far we have discussed the behaviour of an ideal gas which is microscopically described by the equation of state

$$pV = RT \text{ for a mole}$$

or

$$pV = mRT \text{ for } m \text{ mole of a gas.} \qquad \ldots (10.10)$$

For an ideal gas its internal energy depends only on temperature. That is

$$U = f(T) \qquad \ldots \text{ Joule's law.}$$

Gases in real existence hardly obey these relations. Real gases obey these relations and that, too approximately only **at low densities.** The kinetic theory provides a microscopic description of the behaviour of an ideal gas, where two drastically oversimplified assumptions, namely, **molecules are mere point masses and that molecules do not attract or repel each other** are far from the actual state of affair. J. D. Vander Waals deduced a modified equation of state which takes these factors into account. He obtained

$$\left(p + \frac{a}{V^2}\right)(V - b) = RT \qquad \ldots (10.11)$$

where a and b are two constants called Vander Waals' constants. of the gas. All real gases obey this equation of state, though not exactly. For real gases the internal energy depends on the volume as well as on the temperature.

10.8. Degree of Freedom : Equipartition of Energy

The kinetic theory of gases depicts a molecule as a hard elastic sphere and neglects altogether its internal structure. In the matter of calculation of specific heat capacity, we are concerned with all the possible ways of absorbing energy by a molecule. According to the kinetic theory, a molecule may absorb energy only in the form of purely translational kinetic energy. But when the molecule has an internal structure, such as when it is diatomic, triatomic etc. it may store energy in the form of vibrational, rotational as well as translational kinetic energy. *Each independent way in which a molecule may absorb energy is called a degree of freedom. At a particular temperature the energy of a molecule is equally shared among the different ways in which the molecule can absorb energy. That is, in other words, the energy of a molecule is equally divided among its degrees of freedom and per degree of freedom it is equal to* $\frac{1}{2}$ kT *where* k *is the Boltzmann constant and* T *is the absolute temperature of the gas.* This theorem, stated here without proof, is called the equipartition of energy.

For monatomic gases, the molecules have only translational motion (no internal structure in the kinetic theory). The kinetic energy of translation is the sum of three terms i.e.,

$K=\frac{1}{2}mu_x^2+\frac{1}{2}mu_y^2+\frac{1}{2}mu_z^2$. So it has three degrees of freedom.

$\therefore$ Energy of each molecule $=3.\ \frac{1}{2}\ kT=3/2\ kT$.

$\therefore$ The energy of a mole of the gas $=N.\ 3/2\ kT$.

where $N=$ Avogadro's number

$$=3/2\ RT \quad (\because\ R=Nk).$$

Thus
$$E=3/2\ RT.$$

Now,
$$C_v=\frac{\partial E}{\partial T} \text{ (by definition)}$$
$$=3/2\ R.$$

We have $C_p-C_v=R$.

$$\therefore \qquad C_p=3/2\ R+R=5/2\ R.$$

Then ratio of the specific heat capacities is given by

$$\frac{C_p}{C_v}=\frac{5/2\ R}{3/2\ R}=1{\cdot}67.$$

For a diatomic gas each molecule may be considered to have a dumbell shape (two rigid spheres joined by a thin rigid rod). The kinetic energy of such a molecule is the sum of six terms. That is

$$K=(\tfrac{1}{2}mu_x^2+\tfrac{1}{2}mu_y^2+\tfrac{1}{2}mu_z^2)+(\tfrac{1}{2}I_xw_x^2+\tfrac{1}{2}I_yw_y^2+\tfrac{1}{2}I_zw_z^2).$$

However, the rotational inertia about its axis (i.e., I_x) is negligible. Hence in fact the energy is the sum of five terms. Therefore, a diatomic molecule has five degrees of freedom.

$\therefore$ Energy of each diatomic molecule $=5.\frac{1}{2}\ kT$.

$\therefore$ E, energy of a mole of diatomic gas $=5/2\ NkT=5/2\ RT$.

$$\therefore \qquad C_v=\frac{\partial E}{\partial T}=5/2\ R \quad \text{and} \quad C_p=C_v+R=7/2\ R.$$

Then ratio of specific heat capacities of diatomic gas is

$$=\frac{7/2\ R}{5/2\ R}=1{\cdot}4.$$

For a triatomic molecule, the energy term will contain six squared terms and hence it has six degrees of freedom.

$\therefore$ Energy of triatomic molecule $=6.\ \frac{1}{2}\ kT=3\ kT$.

E, Energy of a mole of polyatomic gas $=3\ NkT=3\ RT$.

$$\therefore \qquad C_v=\frac{\partial E}{\partial T}=3\ R$$

and $$C_p=C_v+R=4\ R.$$

The ratio of specific heat capacities of a polyatomic gas

$$=\frac{4R}{3R}=1{\cdot}33.$$

Examples :

1. *The mass of the H_2 molecule is $3{\cdot}32\times10^{-27}$ kg. If 10^{23} hydrogen molecules per second strike 2 cm^2 of wall at an angle of 45° with the normal when moving with a speed of 10^3 ms^{-1}, what pressure do they exert on the wall ?*

Sol. Resolved component of velocity perpendicular to the wall

$$=10^3 \cos 45° = \frac{10^3}{\sqrt{2}} = 707{\cdot}2 \text{ ms}^{-1}.$$

Force = change in momentum per sec

$$=[3{\cdot}32\times10^{-27}\times707{\cdot}2-(-3{\cdot}32\times10^{-27}\times707{\cdot}2)]\times10^{23}$$
$$=2\times3{\cdot}32\times10^{-27}\times707{\cdot}2\times10^{23}.$$

$$\therefore \quad \text{Pressure} = \frac{\text{force}}{\text{area}} = \frac{2\times3{\cdot}32\times10^{-4}\times707{\cdot}2}{2\times10^{-4}}$$

$$=2{\cdot}35\times10^3 \text{ Nm}^{-2}. \textbf{ Ans.}$$

2. *Show that the pressure of an ideal gas is 2/3 its kinetic energy per unit volume.*

Sol. u_E (kinetic energy per unit volume)

$$=\tfrac{1}{2}mc_1^2+\tfrac{1}{2}mc_2^2+\tfrac{1}{2}mc_3^2+\ldots+\tfrac{1}{2}mc_n^2$$

$$=\tfrac{1}{2}m\Sigma c^2=\tfrac{1}{2}mn\frac{\Sigma c^2}{n}=\tfrac{1}{2}\rho\overline{c^2}$$

Now $\quad p=1/3\ \rho\overline{c^2}.$

$\therefore \quad p=1/3.\ 2u_E=2/3\ u_E.$ **Proved.**

3. *Show from the kinetic theory that the energy per degree of freedom is $\frac{1}{2}kT$ where k is Boltzmann's constant and T is the temperature of the gas.*

Sol. According to the kinetic theory, a molecule of an ideal gas has only three degrees of freedom because according to this theory a molecule (irrespective of its internal structure) is a solid elastic sphere. We have for an ideal gas $pV=RT$.

If N is the Avogadro's number, then

$R=Nk$ where k is Boltzmann's constant (the gas constant per molecule).

$$\therefore \quad p=\frac{NkT}{V}=nkT \qquad \text{.. Clayperon's eq.}$$

where $n=\frac{N}{V}$, number of molecules per unit volume.

u_E (energy per unit volume) $=3/2\ p=3/2\ nkT.$

The number of degrees of freedom of n molecules is $3\ n$.

$\therefore$ energy per degree of freedom $= \dfrac{3/2\ nkT}{3n} = \frac{1}{2} kT$. **Proved.**

4. *Compute the number of molecules per unit volume at a pressure* 10^{-3} *atm and a temperature of* 200 *K*.

($1\ atm = 1{\cdot}013 \times 10^5\ Nm^{-2}$;
Boltzmann constant $= 1{\cdot}38 \times 10^{-23}$ JK^{-1} mol^{-1})

Sol. We have $p = nkT$.

$$\therefore \quad 1{\cdot}013 \times 10^5 \times 10^{-3} = n \times 1{\cdot}38 \times 10^{-23} \times 200;$$

$$\therefore \quad n = \frac{1{\cdot}013 \times 10^2}{1{\cdot}38 \times 2 \times 10^{-21}} = 3{\cdot}67 \times 10^{22}. \quad \textbf{Ans.}$$

5. *Compute the root-mean-sqaure speed of an argon atom at* 20°C. *At what temperature will the root-mean-square speed be half that value ? Twice that value ?*

($R = 8{\cdot}3$ *joule per mole per kelvin; molecular weight of argon* $= 40$)

Sol. We have $p = 1/3\ \dfrac{M}{V}\ \overline{c^2}$ or $pV = 1/3\ M\overline{c^2}$.

Also $$pV = RT.$$

$$\therefore \quad RT = 1/3\ M\overline{c^2}$$

or $$\overline{c^2} = \frac{3\ RT}{M} \quad \text{or} \quad c_{rms} = \sqrt{\frac{3\ RT}{M}} = \sqrt{\frac{3 \times 8{\cdot}3(273+20)}{40 \times 10^{-3}}}$$

$$= 1{\cdot}33 \times 10^3\ ms^{-1}.$$

Since $c_{rms} \propto \sqrt{T}$,

$$\therefore \quad \frac{c_{rms} \text{ at } t_1\text{°C}}{c_{rms} \text{ at } 20\text{°C}} = \sqrt{\frac{273+t_1}{273+20}} \quad \text{or} \quad \frac{1}{2} = \sqrt{\frac{273+t_1}{273+20}}$$

or $$4\,(273+t_1) = 293 \quad \text{or} \quad t = -199{\cdot}75°. \ \textbf{Ans.}$$

Again $$2 = \sqrt{\frac{273+t_1}{273+20}} \quad \text{or} \quad 4 \times 293 = 273 + t_1$$

or $$t_1 = 899\text{°C}. \ \textbf{Ans.}$$

QUESTIONS

(A)

1. According to the kinetic theory of gases (a) the momentum of molecules is conserved and not the kinetic energy, (b) kinetic energy is conserved and not their

momentum, (c) both are conserved, (d) both are not conserved.

2. According to the kinetic theory the energy of a gas molecule is (a) entirely kinetic, (b) entirely potential, (c) partly kinetic and partly potential, (d) neither kinetic nor potential.

3. The internal energy of an ideal gas depends on (a) temperature and volume, (b) volume only, (c) temperature only, (d) temperature, pressure and volume.

4. If U_E is the kinetic energy per unit volume of an ideal gas and p is the pressure of the gas then (a) $U_E = 2/3\, p$, (b) $U_E = 1/3\, p$, (c) $p = 1/3\, U_E$, (d) $p = 2/3\, U_E$.

5. The internal energy of a real gas depends on (a) temperature and volume, (b) volume, (c) temperature only, (d) neither on temperature nor on volume.

6. A real gas is close to an ideal gas at (a) high temperature and high pressure, (b) high temperature and low pressure, (c) low pressure and low temperature, (d) low temperature and high pressure.

7. The internal energy of an ideal gas depends only on temperature. This is called (a) Boyle's law, (b) Charles' law, (c) Avogadro's hypothesis, (d) Joule's law.

Ans. 1. c, 2. a, 3. c, 4. d, 5. a, 6. b, 7. d.

(B)

1. What is Brownian movement ? What is its importance ?

2. Explain degrees of freedom and principle of equipartition of energy.

**3. Show from principle of equipartition of energy that the ratio of specific heat capacities of gases is constant.

4. Explain Vander Waals' equation of state (deduction not required).

5. Outline the essential features of the kinetic theory of gases.

(C)

1. What is an ideal gas macroscopically and microscopically ?
Outline the essential features of the kinetic theory of gases. Derive an expression for the pressure of a gas on the basis of the kinetic theory.

2. Deduce Avogadro's hypothesis and Dalton's law of partial pressure for a perfect gas from the principle of the kinetic theory of gases. What interpretation of temperature is given according to this theory ?

3. Deduce an expression for the pressure exerted by a perfect gas according to the kinetic theory and explain how Boyle's law and Graham's law of diffusion follow from this expression.

4. Show that the mean kinetic energy of a molecule of a perfect gas is $3/2\, kT$, where k is the Boltzmann's constant and T is the absolute temperature of the gas.

5. Deduce the following on the basis of the kinetic theory : (a) Boyle's law, (b) Charles' law.

(D)

1. Calculate the value of the kinetic energy of the molecules of an ideal gas at 0°C and at 100°C. What is the kinetic energy per mole of an ideal gas at these temperatures ? $k = 1{\cdot}38 \times 10^{-23}$; $N = 6{\cdot}023 \times 10^{23}\ \text{mol}^{-1}$.

(Hint : Mean kinetic energy per molecule $= 3/2\ kT$)

(Ans. 565×10^{-23} J; 772×10^{-23} J; 3390 J; 4630 J)

2. Calculate the temperature at which the root-mean-square speed of hydrogen is equal to the speed of escape from the surface of the earth. Velocity of escape $= 1{\cdot}12 \times 10^4\ \text{ms}^{-1}$.

$$\left(\text{Hint : } c_{rms} = \sqrt{\frac{3RT}{M}}\right)$$

(Ans. $1{\cdot}01 \times 10^4$K)

3. One gram mole of an ideal gas occupies 22·4 litres at S. T. P. Compute the value of R. (standard pressure $= 1{\cdot}013 \times 10^5\text{Nm}^{-2}$)

(Ans. $8{\cdot}3\ \text{JK}^{-1}\ \text{mol}^{-1}$)

4. Calculate the root-mean-square speed of helium molecules at 40°C, given that rms speed of oxygen molecules at 0°C is 460 ms^{-1}. The molecular weight of oxygen is 32 and that of helium is 4.

(Ans. 1400 ms^{-1})

5. Show that $\frac{v_1}{v_2} = \sqrt{\frac{m_2}{m_1}}$ at constant temperature where v_1 is the speed of sound in the gas of molecular weight m_1 and v_2 is the speed of sound in the gas of molecular weight m_2.

6. One mole of oxygen is heated at constant pressure starting at 0°C. How much heat energy must be added to the gas to double its volume ?
($R = 8{\cdot}3\ \text{Jmol}^{-1}\ \text{K}^{-1}$ and treat oxygen as an ideal gas.)

(Ans. 1888 cal)

(E)

1. Macroscopically an ideal gas is that one which obeys.....

2. Microscopically an ideal gas is that one whose molecules do not......each other.

3. The energy density of an ideal gas is 3/2 times the pressure of the gas. (True or false ?)

4. The internal energy of an ideal gas depends only on......(temperature, volume).

5. According to the kinetic theory of gases, the molecules of a gas may have kinetic energy as well as potential energy. (True or false ?)

6. The volume of a real gas is doubled. Does it approach the ideal state or deviates away from it ?

7. Which one of the following describes the state of a real gas ?

Vander Waals' equation, Joule's law, Charles' law.

8. According to the kinetic theory, has a molecule of a perfect gas any internal structure ?

9. Explain why there is fall of temperature when a gas under pressure issues through an orifice.

10. Does the temperature of a gas increase when the container of the gas is moved with a constant velocity ? Does the temperature increase if the container is suddenly stopped ?

Ans. 1. Boyle's law and Charles' law. 2. Attract or repel. 3. true. 4. temperature, 5. false. 6. approaches the ideal state. 7. Vander Waals' equation. 8. No. Molecules are solid spheres of negligible size.

9. When a gas issues through an orifice, the molecules are streamlined. Here random motion of the molecules is converted into mass motion. The random motion being less, the temperature of the gas is reduced.

10. No, because the motion of mass has no bearing with the temperature of the gas. When the container is suddenly stopped, if the mass motion is converted into random motion (this is most likely due to sudden stoppage of the container) the temperature of the gas may rise.

□□□

CHAPTER 1

ELEMENTS OF WAVES : SOUND AS WAVE : SUPERPOSITION OF WAVES : INTERFERENCE : BEATS : STATIONARY WAVES : REFLECTION AND REFRACTION OF SOUND : ECHO

1.1. Wave-Motion

The concept of 'wave-motion' is very important in Physics. Almost in every branch of Physics we have waves—sound waves, light waves, radio waves, matter waves (De Broglie waves) and electromagnetic waves. Here we will deal with elastic waves or mechanical waves of which sound waves in air are one example. According to the wave theory of light, light is also an elastic wave-motion in a hypothetical medium called ether.

An 'elastic wave' originates in the displacement of some portion of an elastic medium from its normal position. Because of the elastic properties of the medium, the disturbance (i.e., displacement) produced does not remain confined there but is transmitted from layer to layer. This way the 'disturbance' (wave) advances in the medium, but the medium itself does not move. Note that along with the disturbance (wave), it is the **energy** which is progressing in the medium. Thus a wave is a means of transmission of energy and we may put a 'wave' as :

A 'wave' is the process of transmission of a 'disturbance' created somewhere in an elastic medium in all directions around it through successive vibrations of the particles of the medium about their respective mean positions.

It is quite obvious that there are essentially three requirements in a wave : (i) A vibrating body to cause a disturbance called the '*source*' *of the wave*, (ii) an elastic medium called the *propagating medium* and (iii) particles of the medium participating in the process of onward transmission of the '*disturbance*' by executing successively

similar vibrations as the source about their respective mean position.

TERMS OF 'WAVE'

(i) *Amplitude of a wave.* The amplitude (maximum displacement from mean position) of vibration of the particles of the medium is called the *amplitude of the wave.* Since the amplitude of wave is defined as the amplitude of vibration of the particles of the medium and **not** the amplitude of vibration of the source, it follows that the amplitude of the wave near the source is greater than that at far off places. A source of given amplitude will set up waves of different amplitudes in different media. This is denoted by 'a'. Unit, metre (m).

(ii) *Frequency of a wave.* The number of oscillations executed by the particles of the medium in one second is called *the frequency of the wave.* It is to be noted that though the amplitude of the source is **not** the amplitude of the wave, the frequency of the source is definitely the frequency of the wave, because particles of the medium are 'forced' to vibrate by the source. A source of frequency n will set up a wave of the same frequency in any medium. It is denoted by n or ν. Unit, hertz (Hz) or s^{-1}.

(iii) *Phase of a wave at a given point of the medium at a given time.* The 'state of motion' of the particle of the medium at a given point and time is called *the phase of the wave* at that given point and time. The 'state of motion' of a particle at a time means 'where it is' and 'what is its direction of motion' at that instant. The phase of a wave is pictorially represented by a dot with an arrow over it, the head of the arrow showing the direction of motion. For example, the phase of a particle which is passing through its mean position to the right will be shown as , the phase of the particle at its extreme right end will be shown as , the phase of the particle at its extreme left end will be shown as , the phase of $+a/2$ and leftward direction of motion will be shown as  and so on.

(iv) *Wavelength of a wave.* The wave-length of a wave is the minimum distance between two particles of the medium which vibrate in the same phase. It is denoted by λ. Unit, metre (m).

(v) *Time period of a wave.* The time-period of a wave is the interval of time in which particles of the medium complete one oscillation. The time period of a wave is equal to the time period of its source. It is denoted by T. The simple relation of T with n (frequency) is

$$n=\frac{1}{T}.$$

1.2. Graphical Representation of wave with Complete Specification of Phases of Particles

Let us consider a wave of amplitude a and time period T. Particles of the medium will, therefore, vibrate in such a wave with amplitude a and time period T. One oscillation is $4a$ and hence each quarter of an oscillation, that is, a, will be covered by particles in $\frac{T}{4}$.

The first row of Fig. 1.1 represents an undisturbed elastic medium. We have shown the medium by a number of equidistant particles in a line connected by some kind of elastic force. We have done this for clarity of the diagram. Let the first particle be made to start its vibration upwards at the instant $t=0$ by the source in a direction

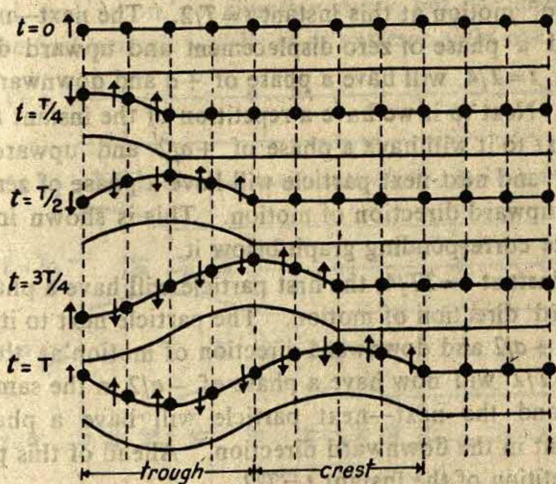


Fig. 1.1

perpendicular to the line of alignment of the particles. The first particle being made to start its vibration will trigger the vibration

in the second particle, the second will trigger the vibration in the third and so on, not simultaneously but one after another. At $t=\frac{T}{4}$ the first particle will be at its extreme upper end (displaced by a) and its phase will be as shown in the second row. The next particle will have a phase of less displacement and upward direction motion, the next—next particle will have a phase of still less displacement and upward direction of motion and so on. This is shown in the same row (second row). For clarity we have shown the very next particle to have a phase of $+a/2$ and upward direction of motion and the next—next particle to have a phase of zero displacement and upward direction of motion. Let us draw a graph by taking the mean position of the particles along the x-axis and their displacements along the y-axis. The plot of displacements of the particles at the instant $t=T/4$ is shown in the figure below the second row. This is the graphical representation of wave at $t=\frac{T}{4}$.

At the instant $t=T/2$, the first particle will have a phase of zero displacement and downward direction of motion. The next particle which had a phase by $+a/2$ and upward direction of motion at the previous instant $t=T/4$ will have a phase by $+a/2$ and downward direction of motion at this instant $t=T/2$. The next—next particle which had a phase of zero displacement and upward direction of motion at $t=T/4$ will have a phase of $+a$ and downward direction of motion. Next to it we have a repetition of the instant $t=T/4$, i e., particle next to it will have a phase of $+a/2$ and upward direction of motion and next-next particle will have a phase of zero displacement and upward direction of motion. This is shown in the third row and the corresponding graph below it.

At the instant $t=3T/4$, the first particle will have a phase by $-a$ and upward direction of motion. The particle next to it which had a phase of $+a/2$ and downward direction of motion at the previous instant $t=T/2$ will now have a phase of $-a/2$ in the same direction of motion and the next—next particle will have a phase of zero displacement in the downward direction. Ahead of this particle we have a repetition of the instant $t=T/2$.

At the instant $t=T$, the first particle will be at its mean position and direction of motion upwards, the next particle will have a phase of $-a/2$ and upward direction and the next—next particle will have

a phase of $-a$ upward direction. Ahead of this, we have a repetition of instant $t=3T/4$. Continuing the above representation, we can find the graphical representation at any instant. For example, to find the graphical representation at the instant $t=15T/4=3T+3T/4$, take the graphical representation of $t=3T/4$ and ahead of this add three full waves.

The wave discussed above is called a transverse wave, because here particles of the medium vibrate at right angles to the direction of propagation of the wave. It is to be noted that the graphical representation of the transverse wave conforms to the shape of the medium obtained by joining the actual positions of the particles of the medium. The spherical bowl-shaped portion of the graphical representation is called the *trough* of the wave and the inverted spherical bowl-shaped portion is called the *crest* of the wave.

Let us now consider the wave in which the particles of the medium vibrate along the direction of wave propagation. This type of wave is called a longitudinal wave. The first row of Fig. 1.2 represents the undisturbed medium. Let the first particle be made to start its vibration at $t=0$ by the source along the line of alignment of the particles towards the right. The first particle being made to start its vibration will trigger a vibration in the second particle, the second in the third and so on, not simultaneously, but one after another. At $t=T/4$ the first particle will be at its extreme right end (displace-

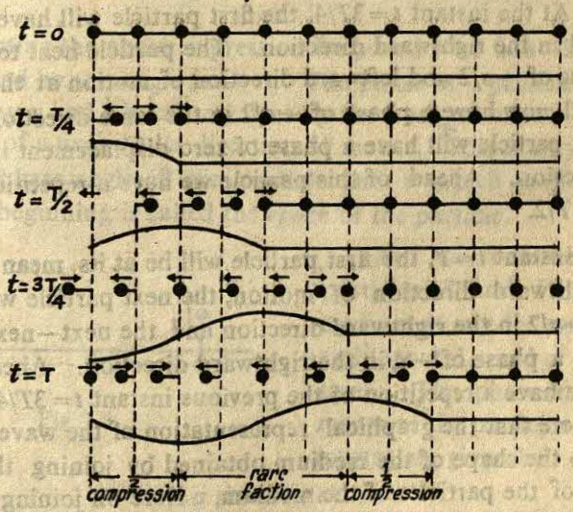


Fig. 1.2

ment a) and its phase will be as shown in the second row. The next particle will have a phase of less displacement and rightward motion, the next—next particle will have a phase of 'still less displacement and rightward motion' and so on. This is shown in the same row (second row). For clarity we have shown the very next particle to have a phase of $+a/2$ and rightward direction of motion and the next—next particle to have a phase of zero displacement and rightward motion. Let us draw a graph by taking the mean positions of the particles along the x-axis and their dis-placements along the y-axis. The plot of the displacements of the particles at the instant $t=T/4$ is shown in the figure below the second row. This is the graphical representation at $t=T/4$.

At the instant $t=T/2$ the first particle will have a phase of zero displacement and leftward direction of motion. The next particle which had a phase of $+a/2$ and rightward direction of motion at the previous instant $t=T/4$ will have a phase of $+a/2$ and leftward direction of motion at the instant $t=T/2$. The next-next particle which had a phase of zero displacement and rightward direction of motion at the previous instant $t=T/4$ will have phase of $+a$ and leftward direction of motion. Ahead of this particle we have repetition of the instant $t=T/4$, i.e., the particle next to it will have phase of $+a/2$ in the rightward direction and the next—next particle will have a phase of zero displacement in the rightward direction. This is shown in the third row and corresponding graphical representation below it. At the instant $t=3T/4$, the first particle will have a phase of $-a$ and in the rightward direction. The particle next to it which had a phase of $+a/2$ and leftward direction of motion at the instant $t=T/2$ will now have a phase of $-a/2$ in the same direction and the next—next particle will have a phase of zero displacement in the leftward direction. Ahead of this particle we have a repetition of the instant $t=T/2$.

At the instant $t=T$, the first particle will be at its mean position in the rightward direction of motion, the next particle will have a phase of $-a/2$ in the rightward direction and the next—next particle will have a phase of $-a$ in the rightward direction. Ahead of this particle we have a repetition of the previous instant $t=3T/4$.

Note here that the graphical representation of the wave does not conform to the shape of the medium obtained by joining the actual positions of the particles of the medium. Here on joining the positions of the particles one will always get a straight line. Here we

have one thing instead and that is a variation of density of the medium. In the region where the slope of the graphical representation is positive, there is a gradation of degree of rarefaction and in the adjoining portion where the slope is negative, there is compression. Thus in a longitudinal wave, we have the sequence of alternate compression and rarefaction.

1.3. Analytical Representation of Wave (Equation of Wave)

Let us consider a wave of amplitude a and frequency ν. This means particles of the medium are vibrating with an amplitude 'a' and a frequency ν because the amplitude and the frequency of a wave are respectively defined by the amplitude and frequency of the particles of the medium. Assuming that the particles of the medium vibrate simple harmonically (simple harmonic wave) we can write for the displacement of any particle.

$y = a \sin(2\pi\nu t + \alpha)$ where α is a constant varing from particle to particle.

or $y = a \sin(\omega t + \alpha)$ where $\omega = 2\pi\nu$, called the *cyclic frequency of the wave.*

The velocity of the particle is given by,

$$v = a\omega \cos(\omega t + \alpha) \qquad \left(\because v = \frac{dy}{dt}\right)$$

By inspecting the above expression for y and v we find that both are determined by the angle $(\omega t + \alpha)$. The phase of a particle is fully determined by y and v because y fixes up the position of the particle and the sign of v fixes up its direction of motion. This is why $(\omega t + \alpha)$ is called the phase angle of the particle and α which is the phase angle in the beginning is called the *epoch* of the particle.

Fig 1.3

Let us consider two particles P and P' of the medium at any distance apart and 'epochs' α and α' respectively. The phase displacements of the particles at any instant are given by

$$y = a \sin(\omega t + \alpha)$$

and
$$y' = a \sin(\omega t + \alpha')$$

If the two particles are λ distance apart, then by definition of wavelength we have

$$y = y' \text{ and } v = v'$$

or $$\sin(\omega t + \alpha) = \sin(\omega t + \alpha')$$

either $\alpha = \alpha'$

or $$\alpha = \alpha' \pm 2\pi,\ \alpha' \pm 4\pi,\ \alpha' \pm 6\pi \text{ and so on.}$$

The solution $\alpha = \alpha'$ is not acceptable, because the 'epochs' of the two particles can never be identical as the particles of the medium start their vibration one after another. Thus two particles in the same phase will have their phase angle difference as $\pm 2\pi$ or $\pm 4\pi$ or $\pm 6\pi$ ...... and in the minimum it is $\pm 2\pi$.

Thus path difference $\lambda \equiv$ phase angle difference by 2π

or path difference $x \equiv$ " " " " $\frac{2\pi}{\lambda}x$.

So if the phase angle of any particle selected at random is $(\omega t + \alpha)$, the phase angle of the particle at a distance x from it is $\left(\omega t + \alpha - \frac{2\pi}{\lambda}x\right)$. We have subtracted $\frac{2\pi}{\lambda}x$ to account for the fact that the phase generally decreases with distance.

Thus the equation of displacement (wave) of any particle at a distance x from any **arbitrary origin** is

$$y = a \sin\left(\omega t + \alpha - \frac{2\pi}{\lambda}x\right)$$

or $$y = a \sin\left(\omega t - \frac{2\pi}{\lambda}x + \alpha\right) \qquad \text{.. (1.1).}$$

Further if we take time from the instant when the particle at the origin passes through its mean position, then

$$y = a \sin\left(\omega t - \frac{2\pi}{\lambda}x\right) \qquad \text{.. (1.2).}$$

Equation 1.1 is the general equation of a wave and Eq. 1.2 is a particular case of it, when time is reckoned from the instant when particle at the origin passes through the mean position. When we have to write the equation of only one wave, we may always represent it by Eq. 1.2; but if we have to write the equation of two waves, one of them may be written by Eq. 1.2, but the other must be written by Eq 1.1, because the starting of the two waves need not be 'simultaneous'.

1.4. To Show that $y=a\sin\left(\omega t-\frac{2\pi}{\lambda}x+\alpha\right)$ Represents a Progressive Wave of Speed c Given by $c=\nu\lambda$

According to this equation, y is the disturbance (i.e., wave) at time t at a distance x from any arbitrary origin. Consider a point at a distance $\left(x+\frac{\lambda\omega}{2\pi}\right)$ from the origin. The disturbance at this point at the same instant is

$$y'=a\sin\left\{\omega t-\frac{2\pi}{\lambda}\left(x+\frac{\lambda\omega}{2\pi}\right)+\alpha\right\}$$

$$=a\sin\left(\omega t-\frac{2\pi}{\lambda}x-\omega+\alpha\right)\neq y$$

Thus a different disturbance exists at this point. Let us now see what disturbances (waves) exist at these two points after 1 second. At the first point the distrubance is

$$y''=a\sin\left\{\omega(t+1)-\frac{2\pi}{\lambda}x+\alpha\right\}=a\sin\left(\omega t+\omega-\frac{2\pi}{\lambda}x+\alpha\right)\neq y$$

Thus a different disturbance (wave) is found to appear at the first point after 1 second. The distrubance at the second point at time $t+1$ is

$$y'''=a\sin\left\{\omega(t+1)-\frac{2\pi}{\lambda}\left(x+\frac{\lambda\omega}{2\pi}\right)+\alpha\right\}$$

$$=a\sin\left(\omega t+\omega-\frac{2\pi}{\lambda}x-\omega+\alpha\right)$$

$$=a\sin\left(\omega t-\frac{2\pi}{\lambda}x+\alpha\right)=y$$

Thus according to this equation the distrubance (wave) is not fixed at any point. It is constantly advancing in the medium. The disturbance (wave), which exists at a distance x from the origin at any instant, that very disturbance occurs at a distance $\left(x+\frac{\lambda\omega}{2\pi}\right)$

from the origin after 1 second. Hence the equation represents a progressive wave of velocity c given by

$$c=\frac{\lambda\omega}{2\pi}=\frac{\lambda.2\pi\nu}{2\pi} \qquad (\because \quad \omega=2\pi\nu)$$

$$\therefore \qquad c=\nu\lambda \qquad \ldots (1.3)$$

This velocity is called the *phase velocity* of the wave as it is the velocity with which the phase of the wave advances in the medium. The *wave velocity* is the velocity with which energy is transported. In a pure harmonic wave, the phase velocity and the wave velocity are the same. In a group of waves, the phase velocity and the wave velocity are two separate entities.

Introducing c in place of ω (i.e. $2\pi\nu$) we can write the equation of the wave as

$$y=a\sin\frac{2\pi}{\lambda}(ct-x+\alpha) \qquad \ldots (1.4)$$

Velocity of a particle in a progressive wave : The velocity of a particle is the rate of change of its displacement. Differentiating y with respect to t we have, when

$$y=a\sin\left(\omega t-\frac{2\pi}{\lambda}x\right)$$

$$\frac{dy}{dt}=a\,\omega\cos\left(\omega t-\frac{2\pi}{\lambda}x\right)$$

Differentiating y with respect to x

$$\frac{dy}{dx}=a\cos\left(\omega t-\frac{2\pi}{\lambda}x\right)\left(-\frac{2\pi}{\lambda}\right)$$

$$=-\frac{2\pi}{\lambda\omega}\frac{dy}{dt}$$

or $$\frac{dy}{dt}=-\frac{\lambda\omega}{2\pi}\frac{dy}{dx}=-\frac{\lambda.2\pi\nu}{2\pi}.\frac{dy}{dx}$$

or $$\frac{dy}{dt}=-c\frac{dy}{dx}. \qquad (\because \quad c=\nu\lambda)$$

Thus, the velocity of the particle $=-c\times$ slope of the displacement curve ($y-x$ plot).

Acceleration of a particle in the progressive wave : The acceleration of a particle is the rate of change of its velocity. Differentiating $\frac{dy}{dt}$ with respect to t we have

$$\frac{d^2y}{dt^2} = -a\omega^2 \sin\left(\omega t - \frac{2\pi}{\lambda}x\right)$$

Differentiating $\frac{dy}{dx}$ with respect to x we have

$$\frac{d^2y}{dx^2} = -a\sin\left(\omega t - \frac{2\pi}{\lambda}x\right)\left(-\frac{2\pi}{\lambda}\right)\left(-\frac{2\pi}{\lambda}\right) = -\frac{4\pi^2}{\lambda^2}\, a\sin\left(\omega t - \frac{2\pi}{\lambda}x\right)$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{4\pi^2}{\lambda^2\omega^2}\,\frac{d^2y}{dt^2}.$$

$$\text{or} \quad \frac{d^2y}{dt^2} = \frac{\lambda^2 4\pi^2\nu^2}{4\pi^2}\cdot\frac{d^2y}{dx^2} \quad \text{or} \quad \frac{d^2y}{dt^2} = c^2\frac{d^2y}{dx^2} \qquad (\because \; c = \nu\lambda)$$

or the acceleration of the particle $= c^2 \times$ the curvature of the displacement curve.

Ray of a wave : The direction in which energy flow takes place is called the *ray of wave.*

Wave front : A *wave-front* is the locus of points where the particles of the medium are in the same phase of vibration.

If the source is a point, the wave-fronts are concentric spheres called spherical wave wave-fronts. At a great distance a spherical wave-front will have a large radius of curvature and a small portion of it may be regarded as plane and is called a plane wave-front.

1.5. Intensity of a Wave

In a wave-motion there is a flow of energy. The rate at which the flow of energy takes place through a unit area of a plane, perpendicular to the direction of wave propagation, is called the intensity of the wave.

Consider a thin layer of the medium at a distance x and thickness dx. Since the thickness of the layer is small all the particles are of the same kinetic and potential energy. We consider the layer to have unit area of cross-section.

The kinetic energy of a particle having instantaneous displacement $y = \frac{1}{2} m \left(\frac{dy}{dt}\right)^2$ where m = mass of the particle. In a wave-motion, $y = a \sin\left(\omega t - \frac{2\pi}{\lambda}x + \alpha\right)$

$$\therefore \quad \frac{dy}{dt} = a\omega \cos\left(\omega t - \frac{2\pi}{\lambda}x + \alpha\right)$$

$\therefore$ the kinetic energy contained in the layer

$$= \tfrac{1}{2}(\rho dx)\, a^2\omega^2 \cos^2\left(\omega t - \frac{2\pi}{\lambda}x + \alpha\right)$$

where ρ = density of the medium.

The potential energy of the same particle = work done in producing the displacement y.

The instantaneous force on the particle $= -m\frac{d^2y}{dt^2}$, the minus sign is put to account for the fact that the force is opposite to the displacement. The elementary work done $= -m\frac{d^2y}{dt^2}.dy$ (work done = force $\times$ displacement) and the total work done

$$= \int_0^y -m\frac{d^2y}{dt^2}\,dy = \int_0^y -m(-\omega^2 y)\,dy = \tfrac{1}{2}m\omega^2y^2$$

$\therefore$ the potential energy of the particle

$$= \tfrac{1}{2}ma^2\omega^2 \sin^2\left(\omega t - \frac{2\pi}{\lambda}x + \alpha\right)$$

$\therefore$ the potential energy contained in the layer at time t

$$= \tfrac{1}{2}(\rho dx)\, a^2\omega^2 \sin^2\left(\omega t - \frac{2\pi}{\lambda}x + \alpha\right)$$

$\therefore$ the total energy contained in the layer at time t

$$= \tfrac{1}{2}\rho dx a^2\omega^2 \cos^2\left(\omega t - \frac{2\pi}{\lambda}x + \alpha\right) + \tfrac{1}{2}\rho dx\, a^2\omega^2 \sin^2\left(\omega t - \frac{2\pi}{\lambda}x + \alpha\right)$$

$$= \tfrac{1}{2}\rho a^2\omega^2 dx.$$

Thus the total energy contained in a layer is independent of time. This means the total energy in the medium is the same in all the layers. The volume of the layer considered is dx.

The energy density of the medium $= \frac{1}{2}\rho a^2\omega^2$.

To calculate the rate of flow of energy through unit area of a plane perpendicular to the wave propagation, consider a cylinder of length c (velocity of wave) and unit area of cross-section.

The energy contained in this cylinder will pass out through its end section in one second.

$\therefore$ The rate of flow of energy in the medium $= \frac{1}{2}\rho a^2\omega^2 c$

This quantity is the 'energy current' or 'intensity of the wave'. Therefore, the intensity of a wave $= \frac{1}{2}\rho a^2\omega^2 c$

Since $\omega = 2\pi v$

I (intensity) $= 2\pi^2\rho a^2 v^2 c$ watt per square metre (Wm^{-2}) .. (1.5).

Thus, the intensity of a wave is directly proportional to the square of the amplitude of the wave.

1.6. Sound as a Wave

The word 'sound' is generally used in two senses : (i) to denote the external disturbance (stimulus) that gives rise to the sensation of hearing, and (ii) to denote the 'sensation' perceived.

Sound is produced by the **rapid vibration** of material bodies. The sensation of sound is produced by a vibrating body, provided its frequency of vibration lies within certain limits called the *limits of audibility*. The lower limit of audible frequencies is about 16 vibrations per second and the upper limit lies round 20,000 vibration per second. The source of sound having frequency within this range of frequencies is called a 'sonic' source and the disturbances produced by it are called 'sonic' waves. The disturbances of frequencies less than the lower limit of audible frequencies are called '*infra-sonics*' and those above the upper limit of audible frequencies are called *super-sonic* or *ultra-sonic* waves (Chapter 8).

Sound is propagated in the form of a 'wave'. There are many evidences to believe so. The famous electric bell and jar experiment of Hawkbee shows that sound needs a material medium for its propagation. Sound is reflected, refracted and diffracted. Two sound waves may interfere and produce permanent '**silence**'. Thus all the characteristic properties of a wave are exhibited by sound and so it

is believed that sound is a wave-motion, and that too a longitudinal one. The reason for believing sound to be a 'longitudinal wave' is that 'sound' freely propagates through all media—solid, liquid and gas, and it is only a longitudinal wave which can propagate through solids, liquids and gases. Transverse waves cannot propagate through gaseous medium.

To make this point clear, suppose a wave travels along AB and L_1, L_2, L_3 are the different layers of the medium (gaseous). If the wave is transverse, L_1 must move up in course of its vibratory motion, perpendicular to the direction of wave propagation and exert a force on the next layer L_2 to drag it upwards Such a force can only arise out of the shear elasticity (or rigidity) of the medium. But a gaseous medium is not possessed of any rigidity. The only force that L_1 may exert on L_2 is the 'viscous force' due to the relative motion between the layers. A 'viscous force' is frictional in nature and it always opposes motion. So there is a heavy loss of energy in overcoming the viscous forces Thus although a transverse wave in air is possible in this way, it cannot go very far. Since with sound waves there is no evidence of subsidence within a short distance, they cannot be transverse.

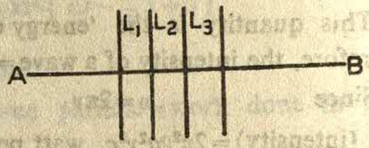


Fig. 1.4 ·

Another conclusive evidence on the longitudinal character of sound waves is the phenomenon of **polarisation.** This phenomenon can be shown by transverse waves alone. Sound waves fail to show 'polarisation' and hence they are conclusively longitudinal.

1.7. Mechanism of Propagation of Sound

Sound is a longitudinal wave. Let us now see the mechanism of propagation of sound through a gaseous medium. Any source of sound is a vibrating body moving forward and backward in quick succession. Let us consider one prong of a tuning fork as a source of sound. As the prong moves from left to right its motion is continuous no doubt, but not uniform. It starts its journey with zero velocity, passes through the mean position with maximum velocity

and finally stops momentarily at the right end to start its backward journey with zero velocity. Let us divide the entire journey from left to right in several steps to have a clear insight into the whole state of affairs.

As the prong takes the first step of its motion from left to right, it compresses the layer of air in front of it. This compressed layer in its attempt to return to the normal state compresses the next layer, which in its turn compresses the next—next layer and so on. This way the disturbance created by the source in the form of a pulse of compression is passed on from layer to layer with speed depending on the elasticity and density of the medium. The pulse of compression of the second step is also passed on from layer to layer with the same speed behind the first one. This is continued till the entire forward journey is not completed. As the degree of compression depends on the instantaneous velocity of the prong, the compression in the first and last steps will be minimum and maximum in the middle step. Hence in the forward journey from left to right the prong sends out a packet of compressions of varying degree. On reaching the extreme right position the prong swings back and starts its journey leftwards. As the prong takes first step of its backward journey, it leaves a partial vacuum causing a rarefaction in the layer of air in front of it. This rarefied layer in its attempt to return to normal condition causes a rarefaction in the next layer, which in its turn causes rarefaction in the next—next layer and so on. In this way a disturbance, created by the source in the form of a pulse of rarefaction, is handed over from layer to layer with the same speed. The pulse of rarefaction of the second step is also passed on from layer to layer with the same speed behind the first one. This is continued till the entire

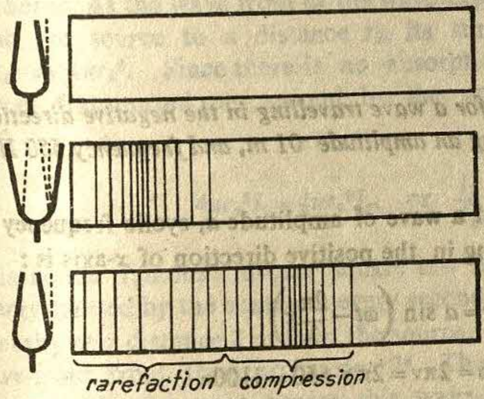


Fig. 1.5

backward journey is completed. As the degree of rarefaction depends on the instantaneous velocity of the prong, it is minimum in the first and last steps and maximum in the middle step. Hence in the backward journey the prong sends out a packet of rarefactions of varying degree. This packet of rarefactions follows the previous packet of compressions. After this, as the prong moves forward and backward, it constantly sends out packets of compressions and rarefactions. This train of packets of compressions and rarefactions, on reaching the listener's ears produce variations of pressure on his ear-membranes. These pressure variations set up impulses on the auditory nerves which carry the message to the hearing centre of the brain.

Examples

1. *Write the equation for a wave travelling in the negative direction along the x-axis and having an amplitude ·01 m, and frequency 550 Hz. and a speed 330 ms*$^{-1}$.

Sol. The equation of a wave of amplitude a, cyclic frequency ω and wavelength λ travelling in the positive direction of x-axis is :

$$y = a \sin\left(\omega t - \frac{2\pi}{\lambda}x\right)$$

Here $a = {\cdot}01$ m, $\quad \omega = 2\pi\nu = 2\pi \times 550 = 1100\pi$

and $\quad \lambda = \dfrac{c}{\nu} = \dfrac{330}{550} = \dfrac{3}{5}\, m$

$\therefore \quad y = {\cdot}01 \sin\left(1100\pi t + \dfrac{10\pi}{3}x\right)$ is the required equation.

Ans.

2. *A wave of frequency 500 Hz has a phase velocity of 350 ms*$^{-1}$. *(a) How far apart are the two points* $\pi/2$ *radian out of phase ? (b) What is the phase difference between two displacements at a certain point at times millisecond apart ?*

Sol. (a) We have $\nu\lambda = c \quad$ or $\quad \lambda = \dfrac{c}{\nu} = \dfrac{350}{500} = {\cdot}7$ m

Thus, two points ·7m apart $\equiv 2\pi$ radian out of phase

or $\quad 2\pi$ radian out of phase $\equiv$ two points ·7m apart

$\pi/2$ „ „ „ „ $\equiv$ „ „ $\dfrac{{\cdot}7}{2\pi} \times \dfrac{\pi}{2} = {\cdot}175$m. **Ans.**

(b) Again T (period of the wave) $= \frac{1}{500} = 2\times10^{-3}$s,

Thus time difference $2\times10^{-3}\text{s} \equiv 2\pi$ radian phase difference between displacements;

$\therefore$ time difference $10^{-3}\text{s} \equiv \frac{2\pi}{2\times10^{-3}} \times 10^{-3} = \pi$,, ,, ,, **Ans.**

3. *Spherical waves are emitted from a ·5 watt source in an isotropic non-absorbing medium. What is the wave intensity 2m from the source ?*

Sol : As the wave front of the wave expands from a distance r_1 from the source to a distance r_2, its surface area increases from $4\pi r_1^2$ to $4\pi r_2^2$. Since there is no absorption of energy the total energy transported per second by the wave remains constant, so that

$$4\pi r_1^2 I_1 = 4\pi r_2^2 I_2, \quad \text{or} \quad \frac{I_1}{I_2} = \frac{r_2^2}{r_1^2}.$$

This is the famous 'inverse square law' in sound. Here the total energy emitted by the source in every second is ·5 watt. Let I be the intensity at a distance 2 m from the source. The surface area of the wave-front from the source is $4\pi\times2^2$. The total energy transported per second by the wave across this wave-front is $4\pi\times2^2.I$. Since there is no absorption by the medium, $4\pi\times2^2 I = {\cdot}5$

or $$I = \frac{\cdot 5}{16\pi} = {\cdot}0099 \text{ Wm}^{-2}$$

or $$I = 9{\cdot}9\times10^{-3} \text{ Wm}^{-2}.$$ **Ans.**

1.8. Principle of Superposition of Waves

From a physical stand point it is a necessity that, when two or more than two waves meet in a medium, a resultant wave be formed. For, the same element of the medium cannot have two or more than two displacements at the same time. This basic necessity is known as the principle of superposition of waves. The principle states that—*The resultant displacement due to a number of waves at any point in a medium is the vector sum of the displacements produced by the component waves.*

If $\vec{y}_1, \vec{y}_2, \vec{y}_3$ ....are the instantaneous displacements produced by the component waves at a given place, then the resultant displacement (wave) at the same point is given by

$$\boxed{\vec{y} = \vec{y}_1 + \vec{y}_2 + \vec{y}_3 + \ldots} \quad \ldots (1.6)$$

This formula is mathematically the principle of superposition of waves. In sound interference, beats and stationary waves are the direct outcome of this principle.

1.9. Interference and Theory of Interference

Interference, in general, is the phenomenon of sustained cancellation or reinforcement of two waves, when they meet under certain specified conditions. When the effect of one wave is constantly neutralised by the other, the two waves are said to interfere destructively (destructive interference) and when their effects are reinforced they are said to interfere constructively (constructive interference). Interference is a basic characteristic of a wave. Sound is a wave. In sound when two sound wave trains meet at a point of a medium under conditions stated below, they either interfere destructively producing permanent **'Silence'** or interfere constructively producing permanent **'anti-silence'**. However, it should not be mistaken that sound energy is perhaps destroyed by interference. There is no loss of energy due to interference. Sound energy is transferred from the regions of 'silence' to the regions of anti-silence.

Conditions for interference in sound : (i) The two waves must be coherent. Two waves are said to be coherent if their phase difference is independent of time.

(ii) The two waves must be of the same amplitude* and frequency.

(iii) They must travel along the same line or different lines at small inclinations.

(iv) For 'silence' the path difference of the wave must be odd integral multiples of half waves,

i.e., path difference $= (2s+1)\ \lambda/2$ where $s = 0, 1, 2, 3 \ldots\ldots$

and for **'anti-silence'** the path difference of the waves must be any integral multiples of the full waves,

i.e., path difference $= s\lambda$ for 'anti-silence'.

* or nearly equal amplitude.

Theory of interference : Suppose two sound waves from sources S_1 and S_2 proceed almost along the same line and meet at a point P of the medium. Let y_1 and y_2 be the displacements of the particle at time t due to the two waves. Then

$$y_1 = a \sin\left(\omega t - \frac{2\pi}{\lambda} x_1\right)$$

and
$$y_2 = a \sin\left(\omega t - \frac{2\pi}{\lambda} x_2 + \delta\right)$$

where a is the amplitude of either wave, x_1 and x_2 are the distances of the point from the sources and δ is the phase difference of the two waves on account of the phase difference between the sources.

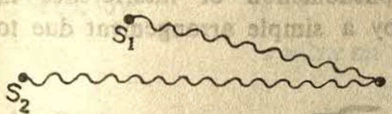


Fig. 1.6

By the principle of superposition, the resultant displacement at P is given by

$y = y_1 + y_2$ ($\because$ the two waves travel almost along the same line, y_1 and y_2 are also along the same line)

or
$$y = a\sin\left(\omega t - \frac{2\pi}{\lambda} x_1\right) + a\sin\left(\omega t - \frac{2\pi}{\lambda} x_2 + \delta\right)$$

$$= 2a\sin\{\omega t - \pi/\lambda(x_1 + x_2) + \delta/2\}\cos\left\{\frac{\pi}{\lambda}(x_2 - x_1) - \delta/2\right\}.$$

This is a simple harmonic term of amplitude $2a\cos\left\{\frac{\pi}{\lambda}(x_2 - x_1) - \delta/2\right\}$.

The intensity is proportional to the square of the amplitude.

$\therefore$ The intensity of sound at $P = k\cos^2\{\pi/\lambda(x_2 - x_1) - \delta/2\}$.

where k is a constant.

Clearly intensity at P will not be constant unless δ is a constant i.e. is independent of time. Hence the first essential condition that the intensity at any point may remain permanently at maximum or minimum value is that the two waves be 'coherent'.

Further, the condition for complete 'Silence' is that δ must be zero. When $\delta = 0$, the intensity of the wave at $P = k\cos^2 \pi/\lambda(x_2 - x_1)$. The intensity at P is zero (complete silence) when

$$\cos \pi/\lambda(x_2 - x_1) = 0$$

or $\quad \pi/\lambda(x_2 - x_1) = (2s+1)\pi/2$ where $s = 0, 1, 2, 3$

or $\quad (x_2 - x_1) = (2s+1)\lambda/2$

i.e. $\quad$ the path difference $= (2s+1)\lambda/2$

and the intensity at P is maximum (anti-silence) when

$$\cos \pi/\lambda(x_2-x_1) = \pm 1$$

or
$$\pi/\lambda(x_2-x_1) = s\pi$$

or
$$(x_2-x_1) = s\lambda$$

or
$$\text{the path difference} = s\lambda.$$

$$\left.\begin{array}{l}\text{Thus path difference} = (2s+1)\lambda/2 \ldots \text{condition for silence} \\ \quad\text{path difference} = s\lambda \ldots \text{condition for anti-silence}\end{array}\right\} \quad \ldots (1.7.)$$

EXPERIMENT TO DEMONSTRATE INTERFERENCE IN SOUND :

Quincke's experiment : The phenomenon of interference in sound is demonstrated directly by a simple arrangement due to Quincke. The arrangement consists of a 'trombone' tube i.e. a tube consisting of two U-tubes, one sliding over the other. The fixed tube T has two short side tubes in its two arms. A tuning fork of very high frequency is held near one of them and the other is connected by a rubber tube to a funnel held close to the ear. If the two parts of the trombone are of exactly the same length, then the waves sent down the short side tube by the source are divided into two waves travelling along the two parts of the trombone and they recombine at the end of the other short tube after traversing equal paths. A fairly loud sound is expected and is heard also. Now as the sliding tube is drawn out slowly, the loudness of the sound starts diminishing and is reduced to complete 'silence'. This happens when the path of the wave travelling along the sliding tube is increased by $\lambda/2$. This way as the tube is drawn out slowly, there are alternate 'silence' and 'anti-silence' in the funnel, according to the the theory of interference. The apparatus can be used to determine the velocity of sound in air or CO_2, if the frequency of the source is known.

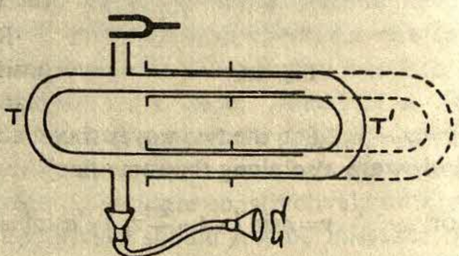


Fig. 1.7

The distance by which the tube is drawn out from a 'silence' to the next 'silence' is $\lambda/2$. Suppose x_1 is the length of the path of the

wave through the fixed tube and x_2 that of the other, then for a 'silence' we have

$$x_2 - x_1 = (2n+1)\lambda/2 \text{ where } n = 0, 1, 2, 3\ldots\ldots$$

Suppose x is the distance by which the sliding tube has to be drawn out for the next 'silence'. Then

$$(2x + x_2) - x_1 = (2n+3)\lambda/2$$

or $$2x = \lambda.$$

We know, $$v\lambda = c$$

$$c = 2vx \text{ ms}^{-1}.$$

1.10. Beats

When two sound waves of almost the same amplitude, but slightly different in frequencies (not more than 16 Hz) travel along the same line, the loudness of the resultant sound wave formed by their superposition fluctuates periodically, alternately rising to a peak value with waxing noise and then fading out with waning noise. This phenomenon of 'waxing and waning' in the loudness of resultant wave is known as 'Beats'.

Beats are formed due to the superposition of waves of the description given above. Interferences and beats are due to the superposition of waves and both are concerned with variations of the intensity of the wave. Interference is the phenomenon of sustained destruction or reinforcement of two identical waves and there is a spatial distribution of 'silence' (destructive interference) and 'anti-silence' (constructive interference) in this phenomenon, but in beats 'silence' and 'anti-silence' occur preiodically at the same place. So we may call it *'interference in time'*. In producing interference the two interfering waves must be essentially 'coherent', but in producing 'beats' any two sources will do.

Due to the slight difference in frequencies of the waves, their

phase difference will change with time. When they come in anti-phase, they produce almost silence and when in phase they produce 'anti-silence'. Let us see graphically how the two waves come alternately in 'phase' and 'anti-phase' due to their difference in frequencies.

Consider some one point in space through which the waves are passing. In the figure, we plot the displacement (y) produced at such a point by the two waves separately as a function of time. For simplicity we take the time-period of the first wave $2s$ and that of the second wave $3s$. In the figure

$AA'=A'A''=\ldots$
$=6s$

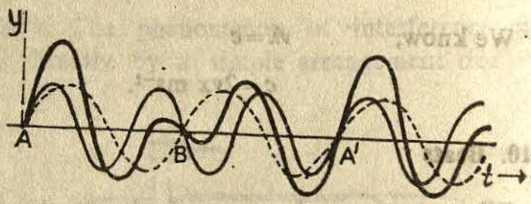


Fig. 1.8

and so within each of them there are traces of three waves of the first wave and traces of two waves of the second one. The resultant vibration at that point as a function of time is the algebraic sum (sum with proper sign) of the individual vibrations and is plotted in the same figure and shown by the thick line. We see that the amplitude of the resultant wave at the point is not constant, but varies with time. The loudness varies as the square of the amplitude. Thus due to the difference in frequencies, there is a variation of loudness of the wave at the same point with time. This is how beats are formed.

Let us now see analytically the formation of beats. Suppose that a sound wave of frequency n_1 is travelling to the right and another wave of frequency n_2 is also travelling along the same direction. Consider the displacements of the two waves at some point in space. Taking the point itself as the origin for measuring the distance ($\therefore \ x=0$), we can write for the displacement at that point due to

the first wave

$$y_1 = a \sin 2\pi n_1 \; t$$

and that due to the second wave

$$y_2 = b \sin 2\pi n_2 \, t.$$

By the principle of superposition the resultant vibration is given by

$$y = y_1 + y_2$$

or $$y = a \sin 2\pi n_1 t + b \sin 2\pi n_2 t.$$

Suppose $m = n_2 - n_1$.

$$\therefore \quad y = a \sin 2\pi n_1 t + b \sin (2\pi mt + 2\pi n_1 t)$$

$$= a \sin 2\pi n_1 t + b \sin 2\pi mt \cos 2\pi n_1 t + b \sin 2\pi n_1 t \cos 2\pi mt$$

$$= (a + b \cos 2\pi mt) \sin 2\pi n_1 \, t + b \sin 2\pi mt \cos 2\pi n_1 t.$$

Make 'sin – cos' substitution, i.e., put

$$a + b \cos 2\pi mt = c \cos\delta$$

$$b \sin 2\pi mt = c \sin\delta.$$

Squaring and adding we have

$$(a + b \cos 2\pi mt)^2 + b^2 \sin^2 2\pi mt = c^2 \cos^2 \delta + c^2 \sin^2 \delta$$

or $$a^2 + 2ab \cos 2\pi mt + b^2 \cos^2 2\pi mt + b^2 \sin^2 2\pi mt = c^2$$

$$(\because \quad \sin^2 \delta + \cos^2 \delta = 1)$$

or $$a^2 + b^2 + 2ab \cos 2\pi mt = c^2. \qquad \text{... (i)}$$

Dividing we have

$$\tan\delta = \frac{b \sin 2\pi mt}{a + b \cos 2\pi mt}. \qquad \text{... (ii)}$$

$$\therefore \quad y = c \cos \delta \sin 2\pi n_1 t + c \sin \delta \cos 2\pi n_1 t$$

$$y = c \sin (2\pi n_1 t + \delta).$$

The resulting vibration at the point under consideration is thus a vibration of amplitude c. The loudness of a sound wave is proportional to the square of the amplitude of the vibration.

Therefore, the loudness at the point of consideration is proportional to c^2.

or the loudness at the point $= kc^2 = k\,(a^2+b^2+2ab\cos 2\pi mt)$

where k is the constant of proportionality.

Obviously, the loudness changes periodically with time.

The loudness is maximum when $\cos 2\pi mt = +1$

or $\quad 2\pi mt = 0, 2\pi, 4\pi, 6\pi, \ldots$

or $\quad t=0, \dfrac{1}{m}, \dfrac{2}{m}, \dfrac{3}{m}, \ldots\ldots = 0, \dfrac{2}{2m}, \dfrac{4}{2m}, \dfrac{6}{2m}, \ldots\ldots$

and the loudness is minimum when $\cos 2\pi mt = -1$

or $\quad 2\pi mt = \pi, 3\pi, 5\pi, \ldots\ldots$

or $\quad t=\dfrac{1}{2m}, \dfrac{3}{2m}, \dfrac{5}{2m}, \ldots\ldots$

Examining the two series of values of time one can see easily that the loudness alternately becomes maximum and minimum.

Maximum loudness $= k(a^2+b^2+2ab) = k(a+b)^2$

and minimum loudness $= k(a^2+b^2-2ab) = k\,(a-b)^2$.

Further if the amplitudes of the waves be the same, then

maximum loudness $= 4ka^2$ (anti-silence)

and minimum loudness $= 0$ (complete silence).

The period of occurrence of beats (that is, a maximum loudness) $= 1/m$.

$\therefore$ The frequency of occurrence of beats, that is, the number of beats per second $= m = n_2 - n_1$.

$\therefore$ Number of beats per second = difference of frequencies. .. (1.8)

Uses of 'beats'. The fact that the frequency of beats is equal to the difference between the frequencies of the component vibrations provides an easy and accurate method to 'tuning' two musical instruments and finding an unknown frequency. When two musical instruments are to be tuned, say, a tuning fork and a sonometer the

frequency of one is adjusted till beats are distinctly heard and then fine adjustment is made to slow down the rate of beating and finally 'beats' are made to disappear completely. At this stage the frequency of one is exactly equal to that of the other when the two are said to be in **'tune'**.

To find the unknown frequency of a vibrator, it is sounded with tuning forks of known frequencies to hear 'beats'. If beats appear distinctly they are timed against a stop-watch and their number is counted and hence the number of beats per second is calculated. If N is the frequency of the tuning fork and n is the number of beats per second, then the frequency of the vibrator is either $N+n$ or $N-n$. The prong of the tuning fork is then loaded with a little wax; this lowers its frequency of vibration. The two are then sounded together and the number of beats per second is found in the above way. If the number of beats per second be less than the previous value, then the frequency of the vibrator is $N-n$. If the number of beats per second be greater than what it was before, then the frequency of vibrator is $(N+n)$.

Why should frequencies not differ by more than 16 Hz. When two sound waves of frequencies differing by more than 16 Hertz are produced, beats will be produced as usual, but it is the ear which will fail to recognise them as beats. When beats occur with sufficient rapidity, the ear recognises them as a continuous sound. This tone (sound of one frequency) is called beat tone. Thus beat tone and beats are physically identical. It is the ear which distinguishes between them recognising roughly more than 16 beats per second as a beat tone, and less than 16 as 'beats'. This is why to produce beats, the component waves must not differ by more than 16 Hz in frequencies.

1.11. Stationary Waves (Standing Waves)

When two identical progressive waves (waves having the same amplitude and frequency), but travelling in opposite directions along the same line with the same velocity, the result of superposition of such waves is the formation of a system of waves which alternately appear and disappear in the region where the two waves meet without advancing in either direction. Such waves are called stationary or standing waves due to the obvious reason that they do not progress in the medium. Such waves are very common in practice. In a taut string held between two clamps, travelling (or progressive) waves

in the string are reflected from the clamps. Each such reflection gives rise to a wave travelling in the string in the opposite direction. In an organ pipe the waves sent down the pipe get reflected from the other end and travel back. Thus within an organ pipe there are two identical travelling waves in opposite directions. We shall first show their formation graphically and then analytically. Let us consider two sources of the same amplitude and frequency at A and G. For simplicity we take A, B, C, D, E, F and G as points $\lambda/4$ apart. Take the graphical representation of the wave at the instant when the source at A passes through its mean position and take it as the initial time $t=0$. The graphical representation at such an instant will be the same as at the instants $t=T$, $2T$, $3T$, ... depending on how many oscillations it has already made before this instant. Similarly take the graphical representation of the wave from G in the backward direction. In the

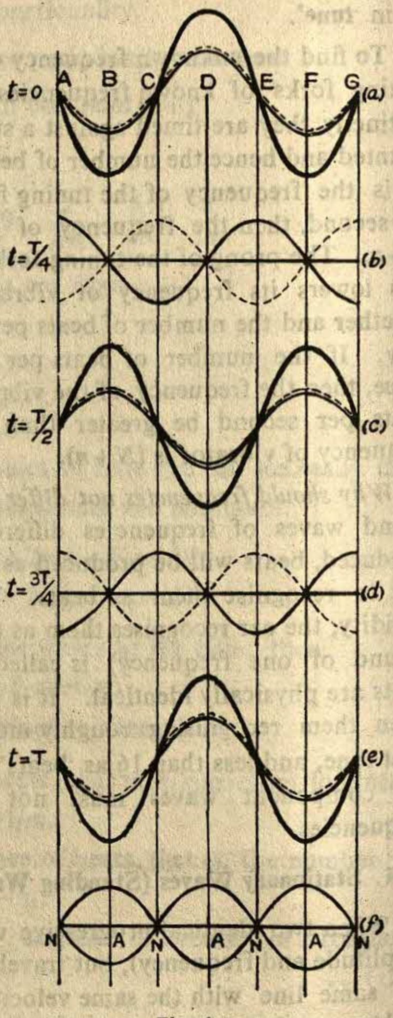


Fig. 1.9

figure we have taken the source at G to be in the same phase. The thick line shows the resultant wave obtained by adding two waves algebraically. After the lapse of $T/4$ each wave advances a distance

$\lambda/4$ in their respective direction of propagation. By taking the graphical representations of the two waves at this instant $t=T/4$ and compounding the two graphical representations, it is found that the resultant wave is a straight line. Similarly at the instant $t=T/2$, it is found that the wave appears with maximum amplitude. It disappears at $t=3T/4$ and reappears at $t=T$. After this the same sequence of changes takes place. Thus we see graphically that the resultant waves simply appear and disappear in the region AG and do not advance in the medium. In the last Fig. 1.9 (f) we have shown the resultant waves at the four instants $t=0$, $t=T/4$, $t=T/2$ and $t=3T/4$ together. We see from this figure that the displacements of the particles at the points A, C, E and G are always zero. These points are called *Nodes* or *places of no displacement*. Note that the distance between consecutive nodes is $\lambda/2$, where λ is the wavelength of either wave. At the points B, D and F which are points exactly midway between nodes, the particles of the medium vibrate with the maximum amplitude. These points are called *Antinodes* or *places of maximum displacement*.

Now let us see the formation of stationary waves analytically. Consider a progressive wave travelling along the positive direction of the x-axis. The equation of the wave can be written as

$$y_1=a\sin\left(\omega t-\frac{2\pi}{\lambda}x\right)$$

$=a\sin\omega(t-x/c)$ where c is the phase velocity of the wave

$$=a\sin(\omega t-kx) \quad \text{where} \quad k=\frac{2\pi}{\lambda}$$

is a constant of the wave called *wave-vector*.

The equation due to an identical wave moving in the negative direction of x will be the same with a negative sign for c. Therefore,

$$y_2=a\sin\omega\,(t+x/c)$$
$$=a\sin(\omega t+kx).$$

The resultanet displacement of the particle due to both the waves, will, by the principle of superposition, be given by

$$y=y_1+y_2$$
$$y=a\sin(\omega t-kx)+a\sin(\omega t+kx).$$

Making use of the trigonometric relation

$$\sin C+\sin D=2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$$

$$y=2a\sin\omega t.\cos kx$$

or
$$y=\left(2a\cos\frac{2\pi}{\lambda}x\right)\sin\omega t.$$

This is definitely not of the form of a progressive wave $y=a\sin(\omega t-kx)$. Therefore this equation represents non-progressive or stationary waves.

If $\cos\frac{2\pi}{\lambda}x=0$, $y=0$ for all t. This means that at the points where $\cos\frac{2\pi}{\lambda}x=0$, particles do not vibrate at all. These are called *Nodes*. We know that the cosine of an angle is zero, when the angle is an odd multiple of $\pi/2$. Hence nodes will occur at the positions

where, $\frac{2\pi}{\lambda}x=\pi/2,\ 3\pi/2,\ 5\pi/2\ ......$

or $x=\lambda/4,\ 3\lambda/4,\ 5\lambda/4\ ......$

Obviously, the distance between two consecutive nodes is $\lambda/2$.

If $\cos\frac{2\pi}{\lambda}x=\pm 1$, $y=\pm 2a\sin\omega t$. This means that at the points where $\cos\frac{2\pi}{\lambda}x=\pm 1$, particles vibrate with a maximum amplitude $2a$. These are called *Antinodes*. We know that the cosine of an angle is ± 1 when the angle is a multiple of π. Hence antinodes occur at the positions where, $\frac{2\pi}{\lambda}x=0,\ \pi,\ 2\pi,\ 3\pi,\ 4\pi,\ 5\pi......$

or $x=0,\ \lambda/2,\ 2\lambda/2,\ 3\lambda/2,\ 4\lambda/2,\ 5\lambda/2.....$

Obviously antinodes occur exactly midway between nodes. The distance between two consecutive antinodes is also $\lambda/2$.

Distinction between a progressive wave and a stationary wave.

1. A progressive wave advances onwards but a stationary wave is confined in the region where it is formed.

2. In a progressive wave all the particles of the medium successively go through the same cycle of changes of displacement, velocity, acceleration, pressure-variation etc. but in a stationary wave all the particles do not go through the same cycle of changes. For example, at some equidistant points $\lambda/2$ apart, particles do not vibrate at all, these are called nodes; at other points particles vibrate with varying amplitudes and exactly mid-way between nodes, particles vibrate with maximum amplitude. These are called anti-nodes.

3. In a progressive wave, particles vibrate with the same amplitude but their phases are different; but in a stationary wave particles vibrate with varying amplitude. In between two nodes all the particles vibrate in the same phase, but the particles on the opposite sides of a node are in antiphase.

4. In a progressive wave at no instant are particles simultaneously at their mean position or at their respective extreme positions. In a stationary wave there are two such instants in one cycle of change, when all the particles pass simultaneously through their mean position or they are at their respective extreme positions.

5. In a progressive wave there is a definite 'energy current', i.e., energy is carried by a progressive wave but in a stationary wave there is no such 'energy current' as the wave itself is non-progressive in character.

1.12. Energy Density in a Stationary Wave

Though there is no energy current in a stationary wave there is definitely a certain energy stored in the region where they are formed.

Consider a thin layer at a distance x and thickness dx. Since the thickness of the layer is small, all the particles are of the same kinetic and potential energy. Consider the layer to have unit area of cross-section.

The kinetic energy of a particle having instantaneous displacement

$y = 1/2\, m \left(\frac{dy}{dt}\right)^2$ where m = mass of the particle.

In a stationary wave,

$$y = 2a \cos kx \sin \omega t,$$

Differentiating we have

$$\frac{dy}{dt} = 2a \cos kx . \omega \cos \omega t.$$

$$= 2a\, \omega \cos kx \cos \omega t.$$

The kinetic energy contained in the layer

$$=1/2\,(\rho dx)\,4a^2\omega^2\cos^2 kx\cos^2 \omega t$$

where ρ = density of the medium

$$=2\rho a^2\omega^2\cos^2 kx\cos^2 \omega t\,dx.$$

The potential energy of the same particle = work done in producing the displacement y.

The instantaneous force on the particle $=-m\dfrac{d^2y}{dt^2}$, minus sign is put to account for the fact that force is opposite to displacement.

The elementary work done $=\left(-m\dfrac{d^2y}{dt^2}\right).dy$

($\because$ work done = force $\times$ displacement).

The total work done $=\int_0^y\left(-m\dfrac{d^2y}{dt^2}\right)dy=\int_0^y -m\left(-\omega^2y\right)dy$

$$\left(\because\ \frac{d^2y}{dt^2}=-\omega^2y\right)$$

$$=\int_0^y m\omega^2 y\,dy=\tfrac{1}{2}m\omega^2y^2$$

$$=\tfrac{1}{2}m\omega^2 4a^2\cos^2kx\sin^2\omega t.$$

The potential energy contained in the layer

$$=\tfrac{1}{2}(\rho dx)\omega^2.4a^2\cos^2kx\sin^2\omega t$$
$$=2a^2\omega^2\rho\cos^2kx\sin^2\omega t\,dx.$$

The total energy contained in layer

$$=\text{kinetic energy}+\text{potential energy}$$
$$=2a^2\omega^2\rho\cos^2kx\,dx\quad(\because\ \sin^2\omega t+\cos^2\omega t=1).$$

The total energy contained in a tube of length λ

$$=2a^2\omega^2\rho\int_0^\lambda\cos^2kx\,dx$$

$$=2a^2\omega^2\rho\,\lambda/2=a^2\omega^2\lambda\rho.$$

The volume of the tube is numerically equal to its length because its area of cross section is unity.

Energy density in a stationary wave $=\dfrac{a^2\omega^2\lambda\rho}{\lambda}=a^2\omega^2\rho$.

This is just double the energy density of either wave. This, of course, follows easily from the principle of conservation of energy. Due to one component wave the energy density is $\frac{1}{2}\rho a^2\omega^2$ and that due to an identical wave is also $\frac{1}{2}\rho a^2\omega^2$. Hence in a stationary wave which is a combination of the two, the energy density must be $\frac{1}{2}\rho a^2\omega^2+\frac{1}{2}\rho a^2\omega^2=\rho a^2\omega^2$.

1.13. To Show that Displacement Nodes and Velocity Nodes Co-exist, but Displacement Nodes Co-exist with Pressure Antinodes

We have as equation of a stationary wave

$$y=2a\cos\frac{2\pi}{\lambda}x\ \sin\omega t.$$

Now, $$v=\frac{dy}{dt}=2a\omega\cos\frac{2\pi}{\lambda}x\ \cos\omega t=2a\omega\cos\frac{2\pi}{\lambda}x\ \sin(\omega t+\pi/2).$$

This is also the equation of a standing simple harmonic motion of amplitude $\left(2a\omega\cos\frac{2\pi}{\lambda}x\right)$. Since $\cos\frac{2\pi}{\lambda}x$ occurs in the amplitude of both the displacement and velocity, wherever displacement amplitudes are zero, their velocity amplitudes are also zero. Wherever displacement amplitudes are maximum, their velocity amplitudes are also maximum. Thus we see that velocity nodes and antinodes co-exist with amplitude node and antinode respectively.

The excess of pressure in a wave is given by

$$p=E\frac{dy}{dx}\ (*)\text{ where } E=\text{bulk modulus of the medium}$$

or $$p=c^2\rho\frac{dy}{dx}\quad \left(\because\ c=\sqrt{\frac{E}{\rho}}\ (**)\text{ and }\rho=\text{density of the medium}\right)$$

or $$=c^2\rho\left(-2a\sin\frac{2\pi}{\lambda}x\frac{2\pi}{\lambda}\right)\sin\omega t.$$

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* For proof of this formula refer to Art. 3.1 Chapter 3.

** For proof of this formula refer to Art. 3.1 Chapter 3.

or $$p=-\frac{4\pi c^2\rho a}{\lambda}\sin\frac{2\pi}{\lambda}x\sin\omega t=\frac{4\pi c^2\rho a}{\lambda}\sin\frac{2\pi}{\lambda}x\sin(\omega t-\pi/2).$$

This is also the equation of a standing simple harmonic motion of amplitude $\left(\frac{4\pi c^2\rho a}{\lambda}\sin\frac{2\pi}{\lambda}x\right)$. Since $\sin\frac{2\pi}{\lambda}x$ occurs in the amplitude of the pressure and $\cos\frac{2\pi}{\lambda}x$ occurs in the amplitude of the displacement, where displacement amplitudes are zero, there pressure amplitudes will be maximum, because when $\cos\frac{2\pi}{\lambda}x=0$, $\sin\frac{2\pi}{\lambda}x$ is maximum. Wherever displacement amplitudes are maximum, pressure amplitude are zero. Thus we see that pressure nodes co-exist with displacement antinodes and pressure antinodes co-exist with displacement nodes. For an experimental demonstration of this fact see Chapter 5, Art. 5.5.

1.14. Reflection and Refraction of Sound Wave : Echo

Sound waves are reflected and refracted exactly in the same way as light waves. But there are certain peculiarities in reflection and refraction of sound waves. When reflection takes place from a rigid wall there is no change in the nature of the wave, that is, compression remains compression and rarefaction remains rarefaction after reflection. When reflection takes place from an yielding surface such as an open end of a pipe there is change in the nature of wave, that is, compression is reflected as rarefaction and rarefaction as compression. In refraction also there are some peculiarities in contrast to light. Light travels fastest in vacuum but sound is dead slow in vacuum. If μ represents the acoustic refractive index of a medium relative to air, then

$$\mu=\frac{\text{speed of sound in air}}{\text{speed of sound in medium}}.$$

Obviously acoustic refractive index of a medium is always less than 1 in contrast to its optical refractive index which is always greater than 1. The optical phenomenon of total internal reflection has its analogy in acoustics. But sound will be total internally reflected when it travels from air to any other medium, say, water. If C represents

the critical angle at air-water surface, then.

$$\frac{\sin C}{\sin \pi/2} = \frac{C_{air}}{C_{water}} = \mu$$

or in sound, $\mu = \sin C$ $\left(cf. \text{ in light } \mu = \frac{1}{\sin C} \right)$.

Echo is a natural phenomenon of reflection of sound. It is simply the repetition of a sound wave produced by reflection from an obstacle, such as the wall of a building or cliff. The essential condition for formation of echo is that the interval between arrivals of the direct wave and the reflected wave must be at least 1/10th of a second, because a human ear cannot distinguish between two sounds arriving within 1/10th of a second. The speed of sound in air at STP is 332 ms^{-1}. Hence the minimum distance between observer (source assumed to be by his side) and the obstacle must be at least 16·6 m.

A natural consequence of refraction of sound due to temperature gradient near the surface of the earth is the fact that voices are more clearly heard at dusk than during the daytime. In the daytime, the temperature of air is maximum near the ground and it diminishes upwards. Therefore, the speed of sound is the greatest near the earth ($c \propto \sqrt{T}$) and decreases upwards. So a ray of sound diverging upwards from a source on or near the earth's surface is refracted continuously towards the normal and hence less sound reaches the observer. At dusk the situation is just opposite. Now a ray diverging upwards from a source on or near the surface of the earth is refracted continuously away from the normal, and it is totally reflected when it begins to travel downwards with continuous refraction towards the vertical, only to reach the observer.

Examples

1. *A source of sound is placed at a perpendicular distance 3 m from a reflecting wall and a microphone is located at a point which is at a distance 27 m along the perpendicular line through the source and at a distance 15 m parallel to the wall from this line. The frequency of the source is variable. Find two different frequencies for which the sound intensity reaching the microphone will be a maximum. The speed of sound in air is 332 ms^{-1}.*

Sol. The direct path from S to $M = \sqrt{15^2 + 24^2} = 28{\cdot}3$ m.

The path of the reflected wave = the path from the image source S' to $M = \sqrt{15^2 + 30^2}$ $= 33{\cdot}54$m.

The path difference between the direct wave and the reflected wave reaching the microphone (neglecting inclination factor as it is small) $= 33{\cdot}54 - 28{\cdot}30$ $= 5{\cdot}24$ m. The condition for a maximum at M is that this path difference be a multiple of the wavelength. That is,

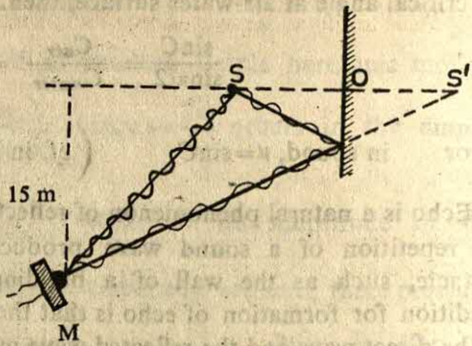


Fig. 1.10

$$5{\cdot}24 = \lambda,\ 2\lambda,\ 3\lambda\ldots\ldots$$

$$= \frac{332}{\nu}, 2\frac{332}{\nu}, 3\frac{332}{\nu}\ldots\ldots \quad (\because\ \nu\lambda = c)$$

or $$\nu = \frac{332}{5{\cdot}24} \text{ or } \frac{332 \times 2}{5{\cdot}24}$$

$$= 63{\cdot}5 \text{ or } 127 \text{ Hz. } \textbf{Ans.}$$

2. *In the Quincke's acoustic interferometer it is found that the sound intensity has a minimum of 100 units at one position of the sliding tube and continuously climbs to a maximum value of 900 units at a second position 1·65 cm from the first. Find (a) the frequency of the sound emitted from the source, and (b) the relative amplitude of the two waves arriving at the detector. Velocity of the wave in the air of the tube = 330 ms*$^{-1}$.

Sol. (*a*) Let l be the length of the fixed tube and l' be the length of the adjustable path in the first position. Then

$l' - l = (2s+1)\ \lambda/2$ where s is any integer including zero.
In the second position

$$(l' + 2 \times 1{\cdot}65) - l = (2s+1)\lambda/2 + \lambda/2.$$

Subtracting we have, $2 \times 1{\cdot}65 = \lambda/2$

or $\lambda = 6{\cdot}60$ cm $= 6{\cdot}6 \times 10^{-2}$ m

$\because \quad c = \nu\lambda,$

$$\therefore \quad \nu = \frac{330}{6 \cdot 6 \times 10^{-2}} = 5000 \text{ Hz. Ans.}$$

(*b*) Let a_1 be the amplitude of the wave travelling along the fixed path and a_2 be that of the wave travelling along the other path. Then the intensity of the resultant wave at the site of the detector

$$=(a_1 - a_2)^2 = 100 \text{ (given)}.$$

The intensity of the resultant wave at the site of the detector in the second position $=(a_1 + a_2)^2 = 900$ (given).

$$\therefore \quad a_1 - a_2 = 10$$

$$\text{and} \quad a_1 + a_2 = 30;$$

$$\therefore \quad a_1 = 20 \text{ and } a_2 = 10$$

$$\text{and} \quad \frac{a_1}{a_2} = \frac{2}{1}. \text{ Ans.}$$

3. *A tuning fork of unknown frequency makes three beats per second with another fork of frequency 288 Hz. The beat frequency decreases when a small piece of wax is put on a prong of the first fork. What is the frequency of this fork ?*

Sol. Beat frequency = difference of the frequencies of component waves. Hence the frequency of the first fork is either $288 + 3 = 291$ or $288 - 3 = 285$. On putting a small piece of wax, the frequency of the body on which wax is put **decreases**. Since after putting wax on the prong of the first fork, the beat frequency decreases, the actual frequency of the first fork is 291 Hz. **Ans.**

4. *Calculate the number of beats per second if there are three sources of equal intensity with frequencies 300 Hz, 304 Hz and 308 Hz.*

Sol. The three frequencies differ from each other by the same amount. Let us take the middle frequency as n and m for the difference of frequencies. Then we can write

$$y_1 = a \sin 2\pi(n-m)t$$
$$y_2 = a \sin 2\pi nt$$
$$y_3 = a \sin 2\pi(n+m)t.$$

By the principle of superposition

$$y = y_1 + y_2 + y_3 = a \sin(2\pi nt - 2\pi mt) + a \sin 2\pi nt + a \sin(2\pi nt + 2\pi mt)$$
$$= a \sin(\theta - 2\pi mt) + a \sin\theta + a \sin(\theta + 2\pi mt)$$

where $\quad \theta = 2\pi nt$

or $$y = a\sin\theta\cos2\pi mt - a\cos\theta\sin2\pi mt + \sin\theta + a\sin\theta\cos2\pi mt + a\cos\theta\sin2\pi mt$$

or $$y = a(1+2\cos2\pi mt)\sin\theta$$

or $$y = a(1+2\cos2\pi mt)\sin2\pi nt.$$

This is the equation of simple harmonic motion of frequencies n and amplitude $a(1+2\cos 2\pi mt)$.

Amplitude is maximum when $\cos 2\pi mt = 1$

or $$2\pi mt = 0,\ 2\pi,\ 4\pi,\ 6\pi\ldots\ldots$$

or $$t = 0,\ \frac{2}{2m},\ \frac{4}{2m},\ \frac{6}{2m},\ldots\ldots$$

Amplitude is minimum when $\cos 2\pi mt = -1$

or $$2\pi mt = \pi,\ 3\pi,\ 5\pi,\ \ldots\ldots$$

or $$t = \frac{1}{2m},\ \frac{3}{2m},\ 5/2m\ \ldots\ldots$$

Hence period of beats $= \frac{1}{m}$

and beat frequency $= m$.

In the above example $m = 4$.

Hence the number of beats per second $= 4$. **Ans.**

5. *Calculate the velocity of sound in a gas in which two waves of wavelengths 1 m and 1·01 m produce 20 beats in 6 seconds.*

Sol. Let c be the velocity of sound in the gas.

Then $$\nu_1 = \frac{c}{1} = c \text{ Hz and } \nu_2 = \frac{c}{1{\cdot}01} \text{ Hz.}$$

Obviously, $\nu_1 > \nu_2$. By the principle of 'beats',

$$\nu_1 - \nu_2 = \frac{20}{6} \quad \text{or} \quad \frac{c}{1} - \frac{c}{1{\cdot}01} = \frac{20}{6}$$

or $$c = \frac{20 \times 1{\cdot}01}{6 \times {\cdot}01} = 336{\cdot}7 \text{ ms}^{-1}. \quad \textbf{Ans.}$$

6. *Sixtyfour tuning forks are arranged in order of increasing frequency and any two successive forks give 4 beats per second when sounded together. If the last fork be an 'octave higher, than the first, calculate the frequency of the latter.*

Sol. Suppose that the frequency of the first tuning fork $= n$. The second fork is then of frequency $n+4$, that of the third $n+2\times4$, that of the fourth $n+3\times4$ ....

and lastly that of 64th is $n+63\times4$.

Since the last one is an 'octave higher' than the first,

$\therefore\quad n+63\times4=2n$ or $n=252$ Hz. **Ans.**

QUESTIONS

(A)

1. The particle velocity and the particle displacement at any point in a progressive wave are out of phase by (a) π, (b) $\pi/2$, (c) 0, (d) none of these.

2. The intensity of a wave is proportional to (a) the amplitude, (b) the square of the amplitude, (c) the cube of the amplitude, (d) the square root of the amplitude.

3. If two progressive waves of amplitudes a and b, but slightly differing in frequency meet in a medium, the loudness of the resultant wave fluctuates in between (a) $(a+b)$ and $(a-b)$, (b) $(a+b)^2$ and $(a-b)^2$, (c) $\sqrt{a+b}$ and $\sqrt{a-b}$ and (d) none of these.

4. In a stationary wave (a) displacement nodes and velocity nodes co-exist, (b) displacement nodes and pressure nodes co-exist, (c) velocity nodes and pressure nodes co-exist, (d) velocity nodes and pressure antinodes co-exist.

5. A tuning fork produces 'beats' 4 times per second with another tuning fork of frequency 256. When the former is loaded it again produces 'beats' four timess per second. The unloaded frequency of the fork is (a) 256, (b) 252, (c) 260, (d) none of these.

6. The minimum distance between an antinode and a node in a stationary wave is (a) $\lambda/2$, (b) λ, (c) $\lambda/4$, (d) $\lambda/3$.

7. The equation of a progressive wave in air is $y=\cdot001 \sin 300\pi\left(t-\frac{x}{300}\right)$, the maximum speed of a particle of the medium is (a) 300 ms^{-1}, (b) 330 ms^{-1}, (c) $\cdot3\pi$ ms^{-1}, (d) 300π ms^{-1}.

8. The frequency of the above wave is (a) 300 Hz, (b) 300π Hz, (c) 150π Hz, (d) 150 Hz.

9. The minimum distance between node and node or between antinode and antinode is (a) $\lambda/2$, (b) λ, (c) $\lambda/4$, (d) $\lambda/3$.

10. In a stationary wave (a) there is no energy current but there is energy density in the medium, (b) there is energy current as well as energy density, (c) there is neither energy current nor energy density.

11. Which one of the following a longitudinal wave will not have in common with a transverse wave ? : (a) reflection, (b) diffraction, (c) refraction, (d) polarisation.

12. A longitudinal wave can travel freely in (a) solids only, (b) in solids and liquids only, (c) in solids, liquids and gases, (d) in a mixture of the three.

Ans. : **1.** b, **2.** b, **3.** b, **4.** a+d, **5.** c; **6.** c, **7.** c, **8.** d, **9.** a, **10.** a, **11.** d, **12.** c)

(B)

1. Explain amplitude, phase and frequency of a wave.
2. Show that $c = \nu\lambda$ where c, ν and λ have their usual meaning.
3. What are beats ? How are they formed ? Explain graphically.
4. What are stationary waves ? Explain their formation graphically.
5. Distinguish between stationary and progressive waves.

(C)

1. Explain what is meant by wave motion. Deduce the equation of a progressive wave.

2. Distinguish between (a) transverse and longitudinal waves (b) stationary and progressive waves. Which of these waves cannot exist in a gas and why ?

3. Explain analytically the formation of stationary waves and deduce its characteristics.

4. Distinguish between interference and beats in sound. Explain analytically the formation of beats.

5. What is interference ? What are the conditions for interference ? Deduce analytically the conditions for interference.

6. Describe Quincke's acoustical interferometer and explain how it shows interference in sound.

(D)

1. The equation of a wave travelling in air is given by

$$y = \cdot 001 \sin \pi (500\, t - 1\cdot 43\, x),$$

where y and x are expressed in metre and t in seconds.

(a) Find the amplitude, frequency, velocity, wavelength and intensity of the wave. (b) Find the maximum speed of a particle of air. (Density of air $= 1\cdot 293$ kgm^{-3})

(Ans : (a) ·001 m; 250 Hz; 349·7 ms^{-1}; 1·4 m; ·558 Wm^{-2}. (b) 1·47 ms^{-1})

2. A line source emits a cylindrical wave. Assuming the medium absorbs no energy, find how the amplitude and intensity of the wave depend on the distance from the source.

(Ans : The intensity varies inversely as the distance and the amplitude varies inversely as the square root of the distance.)

3. What is the pressure amplitude of the wave in the 1 ?

[Hint. $p = -c^2\rho \dfrac{dy}{dx}$] (Ans. : 710 Nm^{-2})

4. Show that the intensity of a sound wave in terms of the pressure amplitude p_m, is given by $I = \dfrac{p_m^2}{2\rho c}$ where ρ is the normal density of the medium and c is the velocity of wave through that medium.

5. A note of frequency 500 Hz has an intensity of one microwatt per square metre. What is the amplitude of the air vibrations caused by this sound ? Velocity of sound in air $=350\ ms^{-1}$ and density of air $=1 \cdot 293\ kgm^{-3}$

(Ans. : 2×10^{-8}m)

(E)

1. When waves interfere, is there a loss of energy ?

2. A wave transmits energy. Does it transfer momentum ? Can it transfer angular momentum ?

3. The inverse square law does not apply exactly to the decrease of sounds with distance. Why not ?

4. Why don't we observe interference effects between the sound waves emitted by two violins ?

5. A wave transmits energy. Does a stationary wave really do so ?

6. If two waves differ only in amplitude and are propagated in opposite directions along the same line in a medium, will they produce standing waves ? Are there any nodes ?

7. Sound is reflected, refracted and diffracted exactly in the same way as light and according to the same laws of reflection and refraction. Air is optically rarer than water for light. Is it true for sound as well ?

8. Voices of bathers whose heads are naturally a little above the water surface can be plainly heard on the shore. This is due to.........(reflection, refraction, total internal reflection from the air-water interface).

9. Voices are more clearly heard from a distance at night than during the daytime. This is due to.........(reflection, total internal reflection from the upper layers of atmosphere).

10. The speed of sound in air at ordinary temperatures is 340 ms^{-1} and in water 1440 ms^{-1}, the critical angle at air-water interface is.........(13°, 60°, 27°).

11. When sound passes from air into water, the angle of incidence is ..... (greater, less) than angle of refraction.

(Ans. 1. No, there is simple redistribution of energy. 2. Yes. Yes, when two transverse waves are set up in a body in two mutually perpendicular planes, torsional waves will be set up and angular momentum will be transmitted. 3. Because of absorption of energy by the medium. 4. Because they are not coherent. 5. No. 6. Yes, No. 7. No, air is acoustically denser because sound travels faster in water and slower in air which is just opposite in light. 8. total internal reflection from the air-water interface. 9. total internal reflection from the upper layers of atmosphere at night. 10 13°. 11. less.)

CHAPTER 2

FREE VIBRATION: FORCED VIBRATION AND RESONANT VIBRATION

2.1. Free Vibration

When a body capable of vibrations is displaced from its position of rest and, then left to itself, it will vibrate with a time period or frequency characteristic of its own. Such vibrations of a body are called its free vibrations provided it is free from any kind of resisting force, external or internal. For example, the prong of a tuning fork, the stretched string of a sonometer etc. are bodies capable of vibrations. When they are disturbed from their position of rest they will vibrate. These vibrations are free vibrations, if they are not resisted by frictional forces such as air resistance, viscosity, frictional forces etc. This clause 'if not resisted by external or internal frictional force' makes free vibrations of a body something that cannot be realised. It can only be approximated to in practice but can never be fully realised. Even if a tuning fork is put in vacuum and then set to vibrations, its vibrations will not persist for an indefinite time. Though its vibrations are free from external forces, namely air resistance, they are not free from internal frictional forces, namely, viscous force. In the course of the vibrations, the different layers of the prong are moving relatively to one another like a pack of cards. Due to this relative motion between layers, a force is called into play which tries to destroy the relative motion. This force is called viscous force or internal frictional force. It is this internal force which is responsible for the ultimate subsidence of the vibrations of the tuning fork even in vacuum. In practice vibrations of any body are resisted by some kind of frictional force and hence resisted vibrations are the natural vibrations. The frequency of free vibrations of a body is called its natural frequency. The natural frequency of a stretched string of length l stretched by a force T is

$$n=\frac{1}{2l}\sqrt{\frac{T}{m}}$$ where m is the mass per

unit length of the string. The natural-frequency of vibration of an

air column closed at one end is

$$n=\frac{1}{4l}\sqrt{\frac{K}{\rho}}$$ where l is the length of the air column, K is the bulk modulus of air and ρ is the density of air. In this way every body capable of vibrations has its own natural frequency.

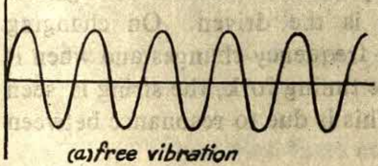


Fig. 2.1 (a)

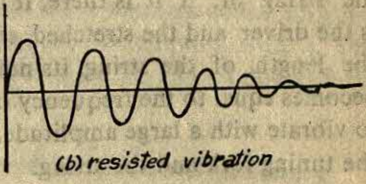


Fig. 2.1 (b)

2.2. Forced Vibration : Resonance

Consider a body capable of vibrations and subject it to a periodic force. A periodic force is provided by a body vibrating harmonically. This is called the 'driver' and the former is called the 'driven'. The driver is supposed to have sufficient energy to drive the driven. If the natural frequency of the driver is not the same as the frequency of the impressed force i.e., the frequency of the driver, the vibrations of the body (the driven) in the beginning are irregular for a short while, but soon its vibrations become regular and the body begins to vibrate with the frequency of the impressed periodic force, i.e., of the driver, with a very small amplitude. The moment the driver is coupled to the driven the resisted vibrations of the latter are mixed up with the vibrations enforced on it by the driver, and so its vibrations in the beginning are irregular. However, the damped vibrations (Fig. 2.1 b) of the body die out shortly and only the vibrations enforced on it by the driver continue. Such vibrations of the body with the frequency of the impressed force, irrespective of its natural frequency, are called *forced vibrations*.

The amplitude of forced vibrations of a body depends on the frequency of the externally impressed periodic force. It is generally very small. On changing the frequency of the periodic force, the amplitude of vibrations changes and at one stage, when it is nearly equal to the natural frequency of the driven, the amplitude suddenly becomes very large. This particular case of forced vibrations of a body when it vibrates with maximum possible amplitude is called *Resonance*.

ILLUSTRATION OF FORCED VIBRATION AND RESONANCE

(i) A piece of cotton thread is tied to one prong of a tuning fork, which can be driven electrically. The string is passed horizontally over a frictionless pulley and finally tied to a pan. Some weight is put on the pan to stretch up the string. Now the tuning fork is set to vibrations. It will be found that there is no visible vibration in the string or, if it is there, it is very small. Here the tuning fork is the driver and the stretched string is the driven. On changing the length of the string, its natural frequency changes and when it becomes equal to the frequency of the tuning fork, the string is seen to vibrate with a large amplitude. This is due to resonance between the tuning fork and the string.

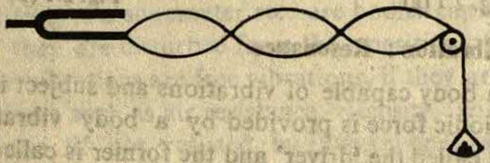


Fig. 2.2

(ii) A vibrating tuning fork is held above a vertical tube dipped in water contained in a tall jar. Here the tuning fork is the driver and the air column in the tube is the driven. On raising the tube slowly at one stage a very sharp sound is heard when the tube is said to 'speak'. This is due to resonance. When the tube is almost completely immersed, the air in the tube will be of very small mass and so its natural frequency is very large because lighter body has greater frequency. On slowly raising the tube, the natural frequency of the air column decreases. The natural frequency of the air column being different from the frequency of the tuning fork (the driver), there is forced vibration of the air column. In a forced vibration, the amplitude is very small. So the sound is not intensified by vibrations of the air column. But when the natural frequency becomes equal to the frequency of the tuning fork, the air column vibrates with a very large amplitude and so the sound is intensified considerably.

Fig. 2.3

(iii) A good mechanical illustration of forced vibration and resonance is provided by Barton's pendulum (Fig. 2.4). Three pendulums A, B and C are suspended from a horizontal rubber cord. The pendulums A and C have the same lengths while that of B is longer than that of A or C. Now set A to oscillations. Its vibrations provide a periodic force which is communicated to B and C through the rubber cord. It will be found that C vibrates with a very large amplitude. This is due to resonance. B vibrates with a small amplitude and with the time period of A or C. This is due to forced vibrations enforced on it by A.

2.3. Some Interesting Facts on Forced Vibration and Resonance

(i) Sound from a tuning fork is audible only at a short distance from the ear. However, if its stem is pressed against a table, it becomes audible even from a distance. This is due to the forced vibration of the table. The vibrating tuning fork acting as a driver sets up forced vibrations in the table and so the table vibrates with the frequency of the tuning fork with small amplitude. Though the amplitude is small, even then the sound is intensified because the amplification of sound depends not only on the amplitude, but also on the mass of air set in vibration. The table being a large body sets a large mass of air in vibration and so the sound is magnified on this account.

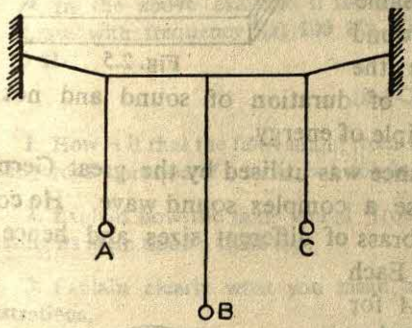


Fig. 2.4

(ii) Soldiers are ordered to break step when crossing a bridge. Soldiers marching in steps across a bridge can set it vibrating with a destructively large amplitude, if the frequency of their steps happens to be the natural frequency of the bridge. This is the reason why soldiers break step while crossing a bridge. In this connection, it is worth noting here how a good lesson was given by Nature to the engineers. In 1940 a new bridge was well constructed and declared open to traffic in the U. S. A. Just four months later

a mild gale completely damaged the bridge. This was due to resonant vibrations of the bridge. The wind produced a fluctuating force in resonance with the natural frequency of the bridge. This caused a steady increase in amplitude, until the main span broke up and crashed into the water below. After this all bridges are designed to make them aerodynamically stable.

(iii) Tuning-forks are often mounted on hollow wooden boxes, called sounding boxes. The boxes are cut to such sizes that the natural frequency of the mass of air enclosed is the same as that of the tuning-fork. When the fork is struck, the enclosed air readily picks up its vibrations and so the sound becomes louder. Here the energy of the tuning-fork is simply transferred quickly to the enclosed air at resonance and the same amount of energy is thrown around in a shorter interval. Thus the loudness is gained at the cost of duration of sound and not in violation of conservation principle of energy.

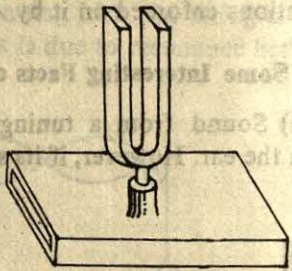


Fig. 2.5

(iv) The principle of resonance was utilised by the great German scientist Helmholtz to analyse a complex sound wave. He constructed a number of globes of brass of different sizes and hence of different natural frequencies. Each globe has a small aperture A for receiving the sound wave and a short nipple B at the other end for insertion in the ear of the observer. It is called a resonator. To analyse a complex wave the aperture of the resonator is turned towards the oncoming test wave. Loud sound is heard if the complex wave contains a tone of frequency that equals the natural frequency of the resonator; otherwise it remains 'dead'.

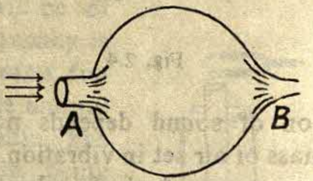


Fig. 2.6

(v) The phenomenon of resonant vibration is utilised in many musical instruments, specially in Sitar, to bring brilliance to the note emitted by it. If two strings are first tuned and then subsequently one of them is sounded, the other is automatically set to vibrations when placed closeby. These are called sympathetic vibrations. In a Sitar there are many auxiliary wires tuned to different notes of the

musical scale. These wires are responsible for brilliancy of the note emitted by the main strings on account of their sympathetic vibrations.

QUESTIONS

(A)

1. In the forced vibration of a body (a) the amplitude is small but the frequency is large, (b) the amplitude is small and the frequeney is equal to the frequency of the driver, (c) the amplitude is large and the frequency is its natural frequency, (d) the amplitude is large and the frequency is equal to the frequency of the driver.

2. In the resonant vibration of a body (a) amplitude is small but its frequency is large, (b) the amplitude is large and the frequency is equal to the frequency of its driver, (c) the amplitude is large and the frequency is any.

3. A tuning fork of frequency 100 Hz is tied to a stretched string of 120 Hz. When the tuning fork is set to vibrations, the thread is forced to vibrate with frequency (a) 100 Hz, (b) 60 Hz, (c) 220 Hz, (d) 120 Hz, (e) 20 Hz.

4. In the above example if the string is set to vibration, the fork is forced to vibrate with frequency (a) 100 Hz, (b) 60 Hz, (c) 220 Hz, (d) 120 Hz, (e) 20 Hz. (Ans. 1. b, 2. b, 3. a, 4. d)

(B+C)

1. How is it that the faint sound from a tuning fork becomes quite loud due to the forced vibration of a table, but not due to that of an air column in a tube ?

2. Explain how the faint sound from a tuning fork becomes loud on pressing its stem upon a table top.

3. Explain clearly what you mean by free and forced vibration. Give illustrations.

4. Discuss the phenomenon of resonance and give some of its practical applications.

5. Give reasons why soldiers are ordered to break step while walking over a bridge.

(E)

1. In the forced vibration of a body its frequency of vibration is............ (equal or greater) than the frequency of the forcing periodic force.

2. In the forced vibration of a body its amplitude is.........(large, small).

3. In the resonant vibration of a body its frequency is............(equal or greater) than the frequency of the forcing force.

4. In the resonant vibration of a body its amplitude of vibration is............ (large, small).

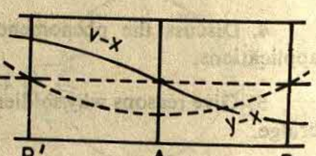
(Ans. 1. equal, 2. small, 3. equal, 4. large.)

CHAPTER 3

VELOCITY OF SOUND; NEWTON'S AND LAPLACE'S EXPRESSION; EFFECT OF TEMPERATURE, PRESSURE AND HUMIDITY ON THE VELOCITY OF SOUND

3.1. Velcoity of Longitudinal Waves through Gaseous and Liquids Media

Method 1 (*An elementary method*) : Let a plane sound wave travel through a tube of unit transverse section to the right with velocity C. For definiteness let A be the region of normal pressure (that is, region of zero particle velocity) and B that of rarefaction (that is, region of maximum particle velocity to the left). If now the medium is supposed to be moved to the left with the same velocity C, then the wave form will become stationary in the tube, that is, particle velocity, their displacement, density and pressure of medium, though different from section to section, will remain stationary relative to the ground. The particle velocity relative to the ground at B is $C+v$ where v is the particle velocity due to the wave at B. Let this be C'. The particle velocity at A relative to the ground is $C+0=C$. Since the pressure and density in the region between A and B remain constant from section to section, the mass within the chamber AB must remain constant. Hence mass flowing into the chamber AB through section B must be equal to the mass passing out through section A. This is called equation of continuity. So, if ρ and ρ' are densities at A and B,



$$C\rho = C'\rho' \qquad \text{(i).}$$

Momentum passing into the chamber in one second through section $B=(C'\rho')C'=C'^2\rho$.

Momentum passing out of the chamber in one second through section $A=(C\rho)C=C^2\rho$.

$\therefore$ Rate of gain of momentum by the body of the gas in the chamber $= C'^2\rho' - C^2\rho$.

By Newton's law of motion, this is the force on the body of the gas in the chamber. If P and P' are the pressures at A and B, then the same force is also $P - P'$.

$$\therefore \quad P - P' = (C'^2\rho' - C^2\rho) \qquad \text{.. (ii).}$$

Substituting the value of C' from (i) in (ii),

$$P - P' = \frac{C^2\rho^2}{\rho'} - C^2\rho = C^2\rho\left(\frac{\rho}{\rho'} - 1\right).$$

If u and u' be specific volumes (volume per unit mass) at A and B respectively, then $\rho = 1/u$ and $\rho' = 1/u'$.

Hence $$P - P' = C^2\rho\left(\frac{u'}{u} - 1\right) = C^2\rho\,\frac{u' - u}{u}.$$

or $$C^2 = \frac{1}{\rho} \cdot \frac{P - P'}{\dfrac{u' - u}{u}}.$$

But $\dfrac{P - P'}{\dfrac{u' - u}{u}}$ is the modulus of volume elasticity E of the medium.

Hence $$C^2 = E/\rho \quad \text{or} \quad C = \sqrt{E/\rho}.$$

Method II (*via Calculus*) : Consider a tube of the gaseous medium of unit sectional area and a thin layer of it bounded by transverse plane section A and B at distance x and $x + \delta x$ from an arbitrary origin O. The length of the layer AB is evidently δx and since the sectional area of the tube is unit, the volume of the layer is also δx.

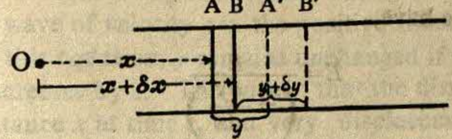


Fig. 3.1

Let a longitudinal disturbance (wave) be incident on the layer and displace A through y and B through $y + \delta y$. Since in a longitudinal wave, the displacements of particles and direction of propagation of the wave are along the same line, the new position (A' in the figure) of A is at a distance $(x + y)$ from the same fixed origin

and that of B (B' in the figure) is $(x+\delta x+y+\delta y)$.

$\therefore$ The length of the layer in displaced position

$$=(x+\delta x+y+\delta y)-(x+y)=\delta x+\delta y.$$

By calculus we have, $\delta y=\frac{dy}{dx}\,\delta x.$

$\therefore$ The length of the layer in the displaced position

$$=\delta x+\frac{dy}{dx}\delta x.$$

$\therefore$ The volume of the layer in the displaced position

$$=\delta x+\frac{dy}{dx}\delta x.$$

$\therefore$ Increase in volume produced by incident disturbance

$$=\left(\delta x+\frac{dy}{dx}\delta x\right)-\delta x=\frac{dy}{dx}\delta x.$$

$\therefore$ Volume strain produced in the layer

$$=\frac{\frac{dy}{dx}\delta x}{\delta x}=\frac{dy}{dx} \qquad \text{.. (i)}$$

If p be the excess of pressure (above normal) over the plane section A, then that over B will be $p+\delta p$ and the excess of pressure producing the above 'volume strain' of the layer is

$\frac{p+p+\delta p}{2}=p+\frac{\delta p}{2}=p$, since δp is negligible in comparison to p.

So if E is the modulus of volume elasticity called bulk modulus of the medium, we have,

$$E=\frac{p}{\left(-\frac{dy}{dx}\right)}$$

(the negative sign is put to account for the fact that an increase of pressure causes a decrease in the volume and vice-versa)

or $$p=-E\frac{dy}{dx}. \qquad \text{.. (ii)}$$

The excess of pressure over A is also the force on the layer to the right and that over B is the force on the layer to the left.

$\therefore$ The force on the layer to the right $= p - (p + \delta p)$

$$= -\delta p.$$

By calculus, we have $\quad \delta p = \frac{dp}{dx}\delta x.$

$\therefore$ The force on the layer to the right

$$= -\frac{dp}{dx}\delta x = -\frac{d}{dx}\left(-E\frac{dy}{dx}\right)\delta x$$

$$= E\frac{d^2y}{dx^2}\delta x.$$

This is the moving force on the layer at the instant, say, t. Then the acceleration of the layer at time t is $\frac{d^2y}{dt^2}$

Since by Newton's second law of motion, force = mass $\times$ acceleration

$$\therefore \quad E\frac{d^2y}{dx^2}\delta x = (\delta x \rho) \times \frac{d^2y}{dt^2} \text{ where } \rho = \text{density of the medium.}$$

or,
$$\frac{d^2y}{dt^2} = \frac{E}{\rho}\,\frac{d^2y}{dx^2}.$$

If we put c^2 for E/ρ then the equation becomes

$$\frac{d^2y}{dt^2} = c^2\frac{d^2y}{dx^2}. \qquad \ldots (3.1)$$

This equation is known as D'Alembert's wave equation. The general solution of this equation is

$$y = f(x - ct) + F(x + ct)$$

where f and F denote any functions. The first function represents a progressive wave of velocity c in the positive direction of the x-axis. According to this equation, y remains unchanged if when t increases by unity, x increases by c. This means that the displacement, which exists at a distance x at time t, that very displacement occurs after one second at a distance $x + c$ from the same origin. Thus $f(x - ct)$ represents a progressive wave of velocity c whatever be the form of the function. Similarly $F(x + ct)$ represents a progressive wave travelling in the negative direction of x-axis with the same speed c. Thus the speed of propagation of a longitudinal wave through a gaseous medium is given by

$$c = \sqrt{\frac{E}{\rho}}. \qquad \ldots (3.2)$$

3.2. Velocity of Longitudinal Waves through Solids

As considered above, here also consider a tube of an extended solid medium of unit sectional area and a thin layer of it bounded by transverse section A and B at distances x and $x+\delta x$ from an arbitrary origin. The length of the layer AB is evidently δx. Let a longitudinal disturbance (wave) be incident on the layer and displace A through y and B through $y+\delta y$. Since in a longitudinal wave, the displacements of particles and the direction of propagation of the wave are along the same line, the new position of A is a distance $(x+y)$ and that B is $(x+\delta x+y+\delta y)$.

$\therefore$ The length of the layer in the displaced position

$$=(x+\delta x+y+\delta y)-(x+y)$$
$$=\delta x+\delta y$$

By Calculus, $$\delta y=\frac{dy}{dx}\delta x$$

$\therefore$ The length of the layer in the displaced position

$$=\delta x+\frac{dy}{dx}\delta x.$$

$\therefore$ the change in length of layer $=\delta x+\dfrac{dy}{dx}\delta x-\delta x$

$$=\frac{dy}{dx}\delta x.$$

and the longitudinal strain produced

$$=\frac{\frac{dy}{dx}\delta x}{\delta x}=\frac{dy}{dx} \qquad \text{.. (i)}$$

If F represents the stretching force over the plane section A, then stretching force over B will be $F+\delta F$ and the stretching force stretching up the layer by $\left(\dfrac{dy}{dx}\delta x\right)$ is $\dfrac{F+F+\delta F}{2}=F+\dfrac{\delta F}{2}=F$, since δF is negligible in comparison to F.

So if y is the axial young's modulus* of the medium

$$Y=\frac{F}{\frac{dy}{dx}}$$ (no minus sign here, because with

* The axial young modulus is the modulus of longitudinal strain not accompanied by lateral strain. In an extended medium there is no lateral contraction, and hence one has to consider the axial young's modulus.

increase of the stretching force there is an increase of length).

or, $$F = Y\frac{dy}{dx} \qquad \text{.. (ii)}$$

The stretching force over A is also the motive force on the layer to the left and that over B to the right.

∴ The net motive force on the layer to the right

$$= F + \delta F - F = \delta F = \frac{dF}{dx}\,\delta x$$

$$= \frac{d}{dx}\left(Y\frac{dy}{dx}\right).\quad \delta x = Y\frac{d^2y}{dx^2}\,\delta x$$

If y be the displacement of the layer at time t then its acceleration at the same time is $\frac{d^2y}{dt^2}$. Since by Newton's second law of motion force = mass × acceleration we have

$$Y\frac{d^2y}{dx^2}\,\delta x = (\rho\delta x)\,\frac{d^2y}{dt^2}$$ where ρ is the density of the medium.

or $$\frac{d^2y}{dt^2} = \frac{Y}{\rho}\,\frac{d^2y}{dx^2}$$

If we put c^2 for $\frac{Y}{\rho}$ then the above equation becomes

$$\frac{d^2y}{dt^2} = c^2\,\frac{d^2y}{dx^2}$$

The rest is the same as in the above section (3.1.)

Then the velocity of propagation of a longitudinal wave through an extended solid medium is given by

$$c = \sqrt{\frac{Y}{\rho}} \qquad \text{... (3.2),}$$

3.3. Newton's Formula and Laplace's Correction

Sound is a longitudinal wave and hence the speed of a sound wave through a gaseous medium is also

$$C = \sqrt{\frac{E}{\rho}}$$ where E is the bulk modulus

of the medium and ρ is the density of the medium.

Newton then specially assumed for sound that the alternate compressions and rarefactions of a longitudinal wave took place in the longitudinal sound waves so slowly that the heat developed as a result of the compression of a layer could find enough time to be dissipated away into the body of the gas and the cold produced in a a rarefied layer was fully compensated for by the heat of the surrounding medium. That is, in other words, Newton assumed that there was no change in temperature of the medium during the propagation of sound waves.

Let P be the normal pressure of a layer of volume V before the arrival of a disturbance. Let a compression be incident on the layer, and compress the layer to a volume $(V-v)$ and a pressure to $P+p$.

Since the temperature remains constant, we have by Boyle's law,

$$PV=(P+p)(V-v)$$
$$=PV+pV-Pv-pv$$

Since p and v are very small, their product pv is negligible.

$$\therefore \quad pV=Pv \text{ or } P=\frac{pV}{v}=\frac{p}{\frac{v}{V}}=\frac{\text{stress}}{\text{strain}}=E$$

Thus the isothermal bulk modulus of a gas is equal to the normal pressure of the gas,

$$\therefore \quad C=\sqrt{\frac{P}{\rho}} \qquad .. (3.3)$$

This formula is due to Newton and hence it is called Newton's expression for the speed of sound wave through a gaseous medium.

To see to what extent Newton's formula is correct, let us use it to calculate the speed of sound in air at NTP and compare the result with the experimentally observed value of the speed of sound wave. At *NTP* the pressure is ·76 m of mercury and the density of air $1{\cdot}293\text{kgm}^{-3}$.

$$P=\cdot 76\times 13{\cdot}6\times 10^3\times 9{\cdot}81=1{\cdot}014\times 10^5\text{Nm}^{-2}$$

$$\therefore \quad C=\sqrt{\frac{1{\cdot}014\times 10^5}{1{\cdot}293}}\ \text{ms}^{-1}$$

or, $\quad C=280\text{ms}^{-1}$

The experimentally determined value of a sound wave at NTP is $332 ms^{-1}$. The great difference of $52 ms^{-1}$ cannot be attributed to an error in the experimental determination. Hence Newton's expression is not the correct one for the velocity of sound wave in a gaseous medium.

About 120 years later, Laplace found the correct expression for the velocity of sound waves through a gaseous medium. He observed that the alternate compressions and rarefactions took place so rapidly that the heat developed in the compressed layer did not have enough time to be dissipated into the surroundings, but remained fully lodged in that layer. Similarly the cold produced in the rarefied layer remained fully in it. Such change in which heat is neither allowed to go out nor allowed to come in from outside is called an adiabatic change. For such a change we have $PV^\gamma =$ a constant, where γ is a constant of the gas representing the ratio of the specific heat capacity of the gas at constant pressure (C_p) to the specific heat capacity of the same gas at constant volume (C_v) i.e., $\gamma = \dfrac{C_p}{C_v}$. The value of γ depends on the number of atoms (atomicity) a molecule of the gas. The value of γ is 1·67 for a monatomic gas, 1·4 for a diatomic gas, 1·33 for a triatomic gas.

Let a compression incident on a layer of initial pressure P and volume V compress it adiabatically to pressure $P+p$ and volume $V-v$. We have by the above relation,

$$PV^\gamma = (P+p)\,(V-v)^\gamma$$

$$= (P+p)\,V^\gamma \left(1-\frac{v}{V}\right)^\gamma$$

or $$P = (P+p)\left[1-\gamma\frac{v}{V}+\frac{\gamma(\gamma-1)}{2}\frac{v^2}{V^2}\cdots\cdots\right]$$

Neglecting terms containing higher powers of v/V we have

$$P = (P+p)\left(1-\gamma\frac{v}{V}\right)$$

$$= P+p-\frac{\gamma P v}{V}-\frac{\gamma p v}{V}$$

The term $\frac{\gamma p v}{V}$ is small compared to other terms.

$$\therefore \qquad p=\frac{\gamma P v}{V} \quad \text{or,} \quad \gamma P=\frac{pV}{v}$$

$$=\frac{p}{\frac{v}{V}}=\frac{\text{stress}}{\text{strain}}=E.$$

Thus adiabatic bulk modulus of a gas is γ times the normal pressure of the gas,

$$\therefore \qquad C=\sqrt{\frac{\gamma P}{\rho}} \qquad .. \ (3.3)$$

Now let us use this formula to calculate the speed of sound in air at NTP. Air is diatomic and hence γ for it is 1·4.

$$\therefore \quad \text{Speed of sound at NTP in air}=\sqrt{\frac{1{\cdot}4\times1{\cdot}014\times10^5}{1{\cdot}293}}\,\text{ms}^{-1}$$

$$=332\text{ms}^{-1}.$$

This value is in excellent agreement with experimental results. Hence Laplace's formula is the correct expression for the speed of sound in a gaseous medium.

3.4. Effect of Pressure, Temperature and Humidity on the Speed of Sound

(*a*) *Effect of pressure.* Let C represent the speed of sound in a gas when its pressure is P and density is ρ and C' is the speed at pressure P' and density ρ'.

$$\text{Then,} \qquad C=\sqrt{\frac{\gamma P}{\rho}} \quad \text{and} \quad C'=\sqrt{\frac{\gamma P'}{\rho'}}$$

Since the temperature remains constant and also the mass of the medium considered as a whole body of a gas remains constant, we have by Boyle's law

$PV=$a constant. This constant depends on the temperature and mass of the gas.

$$\text{or} \qquad \frac{P}{\rho}=\text{a constant.} \quad \left(\because \ \rho=\frac{M}{V}\right)$$

$$\therefore \qquad \frac{P}{\rho}=\frac{P'}{\rho'}$$

$$\therefore \quad C = C'$$

Thus the speed of sound is independent of change of pressure.

(*b*) *Effect of temperature.* Let C be the speed at pressure P and temperature $t°C$ and C_o is the speed at pressure P and temperature $0°C$. Then, $C=\sqrt{\dfrac{\gamma P}{\rho_t}}$ and $C_o=\sqrt{\dfrac{\gamma P}{\rho_o}}$ where ρ_t and ρ_o are the densities at $t°C$ and $O°C$ respectively.

Dividing $$\frac{C}{C_o}=\sqrt{\frac{\gamma P}{\rho_t}\times\frac{\rho_o}{\gamma P}}=\sqrt{\frac{\rho_o}{\rho_t}}$$

If α is the cubical expansivity of the gas at constant pressure, then

$$\rho_o=\rho_t(1+\alpha t)$$

$$\therefore \quad \frac{C}{C_o}=\sqrt{1+\alpha t}=\sqrt{1+\frac{t}{\left(\frac{1}{\alpha}\right)}}=\sqrt{\frac{1/\alpha+t}{1/\alpha}}$$

Now according to the perfect gas scale the absolute zero is $(-1/\alpha)°C$ and hence $(1/\alpha)°$ absolute corresponds to $0°C$ and $(t+1/\alpha)°$ absolute corresponds to $t°C$. If we denote the temperatures on absolute scale by T then $T=t+1/\alpha$ and $T_o=0+1/\alpha=\dfrac{1}{\alpha}$

$$\therefore \quad \frac{C}{C_o}=\sqrt{\frac{T}{T_o}}$$

$$\therefore \quad C\propto\sqrt{T} \qquad \text{.. (3.4)}$$

Thus the velocity of sound depends on the change of temperature and is directly proportional to the square root of the absolute temperature of the gas. The accepted value of α is $1/273{\cdot}16$

$$\therefore \quad \frac{C}{C_o}=\sqrt{\frac{273{\cdot}15+t}{273{\cdot}15}}=\left(1+\frac{t}{273{\cdot}15}\right)^{\frac{1}{2}}$$

If t is is not very large, we may expand and neglect higher powers in the expansion, then,

$$\frac{C}{C_o}=\left(1+\tfrac{1}{2}\frac{t}{273{\cdot}15}\right)=\left(1+\frac{t}{546{\cdot}32}\right)$$

$$=(1+{\cdot}001831t)$$

$$\text{or} \quad C=C_o(1+{\cdot}001831t) \qquad \text{.. (3.4a)}$$

(c) *Effect of simultaneous change of temperature and pressure.* Let C_1 be the velocity of sound at pressure P_1 and temperature T_1 and C_2 be that at pressure P_2 and temperature T_2.

Then $$C_1=\sqrt{\frac{\gamma P_1}{\rho_1}} \quad \text{and} \quad C_2=\sqrt{\frac{\gamma P_2}{\rho_2}}$$

We know from the perfect gas laws

$$PV=RT \quad \text{or} \quad \frac{P}{\rho T}=\text{a constant} \quad (\because \rho=M/V)$$

Here also the constant depends on the mass of the gas. Hence if mass of the medium considered as a body of a gas remains constant,

$$\frac{P_1}{\rho_1 T_1}=\frac{P_2}{\rho_2 T_2}$$

$$\therefore \quad C_1=\sqrt{\frac{\gamma P_1}{\rho_1}}=\sqrt{\frac{\gamma P_2 T_1}{\rho_2 T_2}}$$

$$=\sqrt{\frac{\gamma P_2}{\rho_2}}\sqrt{\frac{T_1}{T_2}}$$

or $$C_1=C_2\sqrt{\frac{T_1}{T_2}}$$

$$\therefore \quad \frac{C_1}{\sqrt{T_1}}=\frac{C_2}{\sqrt{T_2}}$$

Thus when both temperature and pressure vary, the velocity of sound in a gas is not dependent on the change of pressure but depends on the change of temperature.

(d) *Effect of Humidity.* The presence of moisture (water vapour) in air lowers its density and hence sound travels faster in moist air than in dry air. Let C_d represent the velocity of sound in dry air at pressure P and temperature t and C_m the velocity in moist air at the same pressure and temperature. If ρ_d and ρ_m be the densities of dry and moist air at the same temperature and pressure, we have

$$\therefore \quad C_d=\sqrt{\frac{\gamma P}{\rho_d}} \quad \text{and} \quad C_m=\sqrt{\frac{\gamma P}{\rho_m}}$$

$$\therefore \quad \frac{C_d}{C_m}=\sqrt{\frac{\rho_m}{\rho_d}}$$

Let B be the barometric height corresponding to the pressure P in absolute units, namely, Nm^{-2} and f be the vapour pressure in terms of the height of mercury. The partial pressure of dry air is $(B-f)$ and f is the partial pressure of water vapour in terms of the height of mercury.

Now, ρ_m = mass of unit volume of moist air at pressure B and temperature t.

= mass of unit volume of dry air at pressure $(B-f)$ and temperature t + mass of unit volume of water vapour at pressure f and temperature t.

($\because$ moist air is a mixture of dry air and water vapour)

Let us change the pressure of both dry air and water vapour to the common pressure B using the formula PV = a constant at a constant temperature (Boyle's law). Let v' be the volume of the mass of dry air at pressure B' which occupied unit volume at pressure $B-f$.

Then
$$v' \times B = 1 \times (B-f)$$

$$\therefore \qquad v' = \frac{B-f}{B}.$$

Thus the mass of dry air which occupies unit volume at pressure $(B-f)$ and temperature t would occupy $\left(\frac{B-f}{B}\right)$ volume at pressure B and temperature t. Similarly the mass of water vapour that occupies unit volume at pressure f and temperature t would occupy $\left(\frac{f}{B}\right)$ volume at pressure B and temperature t.

$\therefore$ ρ_m = mass of $\left(\frac{B-f}{B}\right)$ volume of dry air at pressure B and temperature t + mass of f/B volume of water vapour at pressure B and temperature t.

or $\rho_m = \frac{B-f}{B}\rho_d + \frac{f}{B}\,\frac{5}{8}\rho_d$ ($\because$ it is a fact that the density of water vapour at a given temperature and pressure bears a constant ratio to the density of dry air at the same temperature and pressure and this ratio is nearly 5/8)

$$\text{or} \qquad \frac{\rho_m}{\rho_d}=\frac{B-f}{B}+\frac{5}{8}\frac{f}{B}=\frac{B-f+\frac{5}{8}f}{B}=\frac{B-3/8f}{B}$$

$$=\frac{B-\cdot 375f}{B}$$

$$\therefore \qquad \frac{C_d}{C_m}=\sqrt{\frac{B-\cdot 375f}{B}}$$

$$\text{or} \qquad C_d=C_m\sqrt{\frac{B-\cdot 375f}{B}} \qquad \text{.. (3.5)}$$

(*e*) *Effect of wind.* The velocity of sound in still air is affected by the wind. If w is the velocity of wind in the direction of wave propagation, then the effective speed of sound wave in the direction of the wind is $C+w$ and that against it is $C-w$. If the direction of the wind makes an angle θ with the direction of wave propagation then the effective speed of the sound wave is $(C+w\cos\theta)$.

3.5. Experimental Determination of the Velocity of Sound

(*i*) *Regnault's method.* This is an open-air method of determining the velocity of sound in air. Here the departure and arrival of a sound wave are automatically registered electrically.

The apparatus consists of a drum D capable of being rotated at a constant speed by a cl ck work or an electric motor. A style S is arranged to trace a line on the drum when attracted by an electro-magnet M. The electromagnet and a fine wire W, placed at a large distance from the drum, are connected to a battery B through line wires L, L. The receiver R of sound wave is a membrane placed close to the drum and connected to one of the line wires. Behind the membrane there is a metallic stud C connected to the other line wire. In the experimental procedure, the drum D is set in rotation and the key K of the circuit is closed. The wire W is then broken by firing a

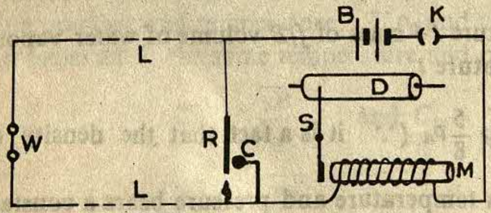


Fig. 3.2

gun at it. This opens the circuit and the styles ceases to make its usual mark on the drum. When the compressional pulse of the report of the gun reaches the receiver, it pushes the membrane establishing contact with C momentarily. The style being attracted by the electromagnet traces a mark on the drum. The interval between the instants when the style discontinued tracing a mark on the drum and when it again made a momentary trace is obviously the time taken by the report of the gun to travel from the wire to the receiver from the knowledge of the speed of rotation of the drum, the time of travel is determined and hence the velocity of sound in open air.

(*ii*) *Hebb's method.* In 1904 Hebb devised a method in which sounds of normal intensity are used. The experimental arrangement consists of two large parabollic reflectors placed at a large distance (30 metre) apart. A high pitched whistle S is placed at the focus of one of the reflectors. A parallel beam of sound is thus sent across the distance to the other reflector, which focusses the sound at its focus. Two microphones are placed at the focii of the reflectors. Each microphone is connected in series with a suitable battery to the primary winding of a transformer. The common secondary winding is connected to a headphone. The sound heard in the headphone is the resultant effect of two alternating currents in the secondary. When the path of the sound from S to the focus of the

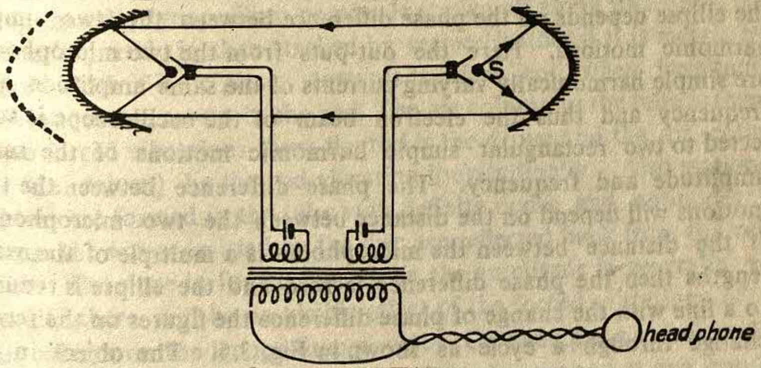


Fig. 3.3

mirror is an even number of half-wave lengths, the two microphones receive waves in the same phase and so a loud sound is heard in the headphone. When the distance is an odd number of half wavelengths, the sound waves received are out of phase and the sound heard is

minimum. The second mirror and the second microphone are moved as a compact unit through a known distance and the number of maximum and minimum is counted. The distance from a maximum to a maximum, or a minimum to a minimum is the wavelength of the wave. Knowing the wavelength in this way the velocity of sound is calculated from the simple formula, $C = \nu\lambda$ where ν is the frequency of the source.

(*iii*) *Cathode ray Oscilloscope method.* This method is simply a modification of Hebb's method. This is a very simple and accurate method. It uses a loud speaker excited by an oscillator as source of sound. Two microphones are placed in front of the loudspeaker in

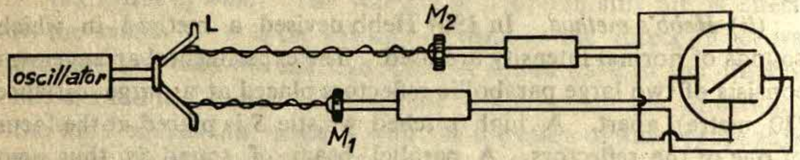


Fig. 3.4

such a way that both of them may receive sound waves from the speaker. The out-put from the first microphone after being amplified by a suitable amplifier is fed to the X-plates of an oscilloscope and that from the other microphone to the Y-plates. The principle of the method is that the resultant of two S. H. M's of the same frequency and amplitude is in general an elliptical motion. The shape of the ellipse depends on the phase difference between the two simple harmonic motions. Here the out-puts from the two microphones are simple harmonically varying currents of the same amplitude and frequency and thus the electron beam of the oscilloscope is subjected to two rectangular simple harmomic motions of the same amplitude and frequency. The phase difference between the two motions will depend on the distance between the two microphones. If the distance between the microphones is a multiple of the wave lengths then the phase difference is zero and the ellipse is reduced to a line with the change of phase difference the figures on the screen change through a cycle as shown in Fig. 3.5. The object in the

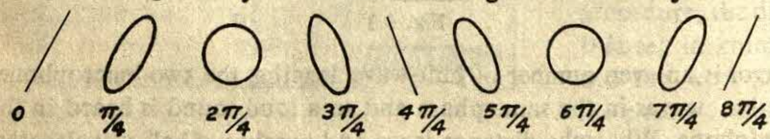


Fig. 3.5

experimental procedure is to study these cycle of changes of the figures on the screen of the oscilloscope by moving the second microphone. The distance through which it is to be moved for a complete cycle of changes of the figures is the wavelength of the sound wave emitted by the speaker. Knowing λ in this way, the velocity of sound is calculated from the formula $C = \nu\lambda$ where ν is the frequency of the oscillator.

(*iv*) ***Kundt's dust tube method.*** This is a method to determine velocity of sound in solids, liquids and gases in a closed tube.

The apparatus consists of a long, wide glass tube closed at one end by a tightly fitted piston. This is called the *wave tube*. The experimental solid is taken in the form of a rod and is provided with a disc having its diameter a little less than the diameter of the wave tube. The disc is introduced into the wave tube and the rod is clamped rigidly exactly at its centre. This rod is called the *sounding rod*.

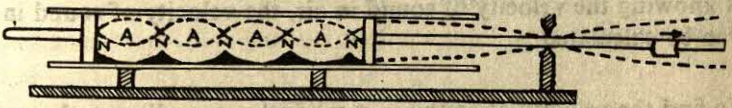


Fig. 3.6

In the experimental procedure the wave-tube is first well dried and then dry lycopodium powder is strewn all along its length. The sounding rod is then stroked by resined leather if it is metal, by a moist cloth if it is glass, towards the free end when longitudinal vibrations are set up in the rod. The air column in the tube is thrown into forced vibrations (here the rod is the driver and the air column is the driven) by the rod vibrating longitudinally. Also stationary waves are formed in the tube with definite nodes and antinodes because the waves emitted by the disc are reflected back from the other disc. But since in forced vibrations the amplitude is generally very small, there will be no visible effect of the forced vibrations of the air column on the lycopodium powder. On sliding the movable disc, the frequency of air or gas whatever it is changes and at one length is thrown into resonant vibrations when the lycopodium powder is seen to be agitated vigorously. It flies away from the positions of antinodes where the displacement of the air particles is maximum to the

nodes, the places of minimum displacements, and collects in heaps at the positions of the nodes. Thus the positions of the nodes are marked clearly by lycopodium powder enabling one to estimate the wavelength of the wave (λ), because the distance from node to node is $\lambda/2$. The tube is then filled with the experimental gas, say CO_2, and the above process is repeated.

Suppose l = distance between nodes in air.

l' = distance between nodes in experimental gas.

Now, $C(\text{air}) = \nu\lambda(\text{air})$ where ν = frequency of the sounding rod and $C(\text{gas}) = \nu\lambda\ (\text{gas})$

$$\therefore \qquad \frac{C(\text{air})}{C(\text{gas})} = \frac{\lambda(\text{air})}{\lambda(\text{gas})}$$

But, $\lambda\ (\text{air}) = 2l$ and $\lambda\ (\text{gas}) = 2l'$

$$\therefore \qquad \frac{C(\text{air})}{C(\text{gas})} = \frac{l}{l'} \quad \text{or} \quad C(\text{gas}) = C(\text{air})\,\frac{l'}{l}$$

Thus knowing the velocity of sound in air, the velocity of sound in a gas can be calculated.

To find the velocity of sound in a solid, the sounding rod made of that solid is taken, and used in the Kundt's tube apparatus. The sounding rod in its simplest mode of vibration has a node at the middle, where it is clamped, and two antinodes at the two ends. Hence if L is the length of the rod then

$$L = \frac{\lambda\ (\text{solid})}{2} \quad \text{or} \quad \lambda\ (\text{solid}) = 2L$$

$$\therefore \qquad \frac{C\ (\text{solid})}{C\ (\text{air})} = \frac{2L}{2l} = \frac{L}{l}$$

$$\text{or} \qquad C\ (\text{solid}) = C\ (\text{air}) \times \frac{L}{l}$$

Kundt's tube provides a ready method for finding the velocity of sound in a gas under controlled conditions of temperature and pressure. This apparatus can be used to show that the velocity of sound in a gas is independent of pressure and is proportional to the square root of the absolute temperature of the gas.

Examples

1. *Show that the adiabatic bulk modulus of a perfect gas is γ times its isothermal bulk modulus.*

Sol. In an adiabatic change

$$PV^{\gamma} = \text{a constant}$$

Differentiating we have,

$$V^{\gamma}dP + P\gamma V^{\gamma-1}\,dV = 0 \quad \text{or} \quad VdP + \gamma PdV = 0$$

or
$$\gamma P = -\frac{dP}{\dfrac{dV}{V}}$$

$$= \frac{\text{stress}}{\text{strain}} = E_S \text{ (adiabatic bulk modulus)}$$

In an isothermal change

$$PV = \text{a constant}$$

$$\therefore \quad VdP + PdV = 0 \quad \text{or} \quad P = -\frac{dP}{\dfrac{dV}{V}} = \frac{\text{Stress}}{\text{Strain}} = E_T \quad \text{(isothermal bulk modulus)}$$

$$\therefore \quad \frac{E_S}{E_T} = \frac{\gamma P}{P} = \gamma$$

$$\therefore \quad E_S = \gamma E_T \textbf{ Proved.}$$

2. *Find at what temperature the speed of sound in air is double its speed at 0°C.*

Sol. We have,
$$\frac{C}{C_0} = \sqrt{\frac{T}{T_0}}$$

Here
$$C = 2C_0$$

$$\therefore \quad 2 = \sqrt{\frac{T}{T_0}} \quad \text{or} \quad T = 4T_0 = 4 \times 273 = 1092°K.$$

or
$$T = 1092 - 273 = 819°C. \quad \textbf{Ans.}$$

3. *One gramme of hydrogen is sealed in a tube of volume $10^{-3}m^3$ at 27°C. Calculate the velocity of sound in the sealed tube. (γ for hydrogen$=1.4$.)*

Sol. We have $\quad PV = mRT$

Here $\quad m$ is $\frac{1}{2}$. $\quad \therefore \quad P \times 10^{-3} = \frac{1}{2} \times 8{\cdot}31 \times 300$

or $\quad P = 4{\cdot}155 \times 3 \times 10^5 Nm^{-2}$

$$D=\frac{10^{-3}}{10^{-3}}=1 \text{ kg } m^{-3}$$

$$\therefore \qquad C=\sqrt{\frac{\gamma P}{D}}=\sqrt{1{\cdot}4\times 4{\cdot}155\times 3\times 10^5}$$

$$=1321 \text{ ms}^{-1}. \quad \textbf{Ans.}$$

4. ***The velocity of sound in hydrogen at 0°C is 1284 ms^{-1}. What is the root mean square velocity of hydrogen at 0°C ? (γ for hydrogen =1.41.)***

Sol. We have, $C_{sound}=\sqrt{\dfrac{\gamma P}{\rho}}$

and $\qquad P=\frac{1}{3}\rho C^2$ from the kinetic theory of gases

$$\therefore \qquad C_{sound}=\sqrt{\frac{\gamma C^2}{3}}$$

or $$\sqrt{C^2}=\sqrt{\frac{3}{\gamma}}\times C_{sound}=\sqrt{\frac{3}{1{\cdot}41}}\times 1284$$

$$=1{\cdot}87\times 10^3 \text{ms}^{-1}. \quad \textbf{Ans.}$$

5. *The maximum pressure variation that the ear can tolerate is about 28 Nm^{-2}. Find the maximum displacement for a sound wave in air having a frequency of 1000 Hz. (Density of air 1·293 kg m^{-3} and velocity of sound in air 332 ms^{-1}).*

Sol. $P=-E\dfrac{dy}{dx}=-C^2\rho\dfrac{dy}{dx}$ $\quad (\because \quad C=\sqrt{E/\rho})$

We have $$y=a\sin\left(wt-\frac{2\pi}{\lambda}x\right)$$

$$\therefore \qquad \frac{dy}{dx}=-\frac{2\pi a}{\lambda}\cos\left(wt-\frac{2\pi}{\lambda}x\right)$$

$$\therefore \qquad p=\frac{2\pi a}{\lambda}C^2\rho\cos\left(wt-\frac{2\pi}{\lambda}x\right)$$

$$\therefore \qquad p_m=\frac{2\pi aC^2\rho}{\lambda}=2\pi a v C\rho \quad (\because \quad v\lambda=C)$$

or $$a=\frac{p_m}{2\pi v.C\rho}=\frac{28}{2\pi\times 1000\times 332\times 1{\cdot}293}$$

$$=1{\cdot}037\times 10^{-5} \text{ m. } \textbf{Ans.}$$

QUESTIONS

(A)

1. The velocity of sound through a gas is proportional to (a) the absolute temperature of gas, (b) the square of the absolute temperature of gas, (c) the square root of the absolute temperature, (d) is independent of the temperature.

2. The velocity of sound through a gas of given density is proportional to (a) the pressure, (b) the square of the pressure, (c) the square root of the pressure, (d) none of these.

3. The presence of water vapour in air, (a) lowers the velocity of sound, (b) raises the velocity of sound, (c) has no effect on velocity of sound, (d) may raise or lower it depending on the temperature of the air.

4. When a sound wave travels through a gaseous medium, the changes of pressure and volume of the layers take place (a) adiabatically, (b) isothermally, (c) isobarically, (d) isochorically.

5. The correct expression for the velocity of a sound wave through a gaseous medium is (a) $\sqrt{\frac{P}{\rho}}$, (b) $\sqrt{\frac{P}{\gamma\rho}}$, (c) $\sqrt{\frac{\gamma P}{\rho}}$, (d) $\sqrt{\frac{\rho}{P}}$.

(Ans. 1. c. 2. c. 3. b. 4. a. 5. c.)

(B)

1. Show that in a longitudinal wave strain of a layer at a distance x is $\frac{dy}{dx}$ where y is the instantaneous displacement of the layer.

2. Give Newton's formula for the velocity of sound in air. In what respect was it defective, and what correction was made by Laplace ?

3. Discuss the effect of temperature, pressure and humidity on the velocity of sound through a gas.

(C)

1. Derive a general expression for the velocity of sound in a gas and discuss its modification due to Newton and Laplace.

2. Describe any method of determining the velocity of sound in air.

3. Describe a method of determining the velocity of sound in a gas under controlled conditions of temperature and pressure.

(D)

1. Calculate the speed of a sound wave in brass, given that Young's modulus of brass is 10×10^{10} Nm^{-2} and its density is 8500 kg m^{-3}. (Ans. 3430 ms^{-1})

2. In a dust tube experiment, the sounding rod of length 120 cm produced nodal heaps 12 cm apart in air. Calculate the velocity of sound in the material of the rod given that the velocity of sound in air is 350 ms^{-1}. (Ans. 3500 ms^{-1}),

3. If the velocity of sound in hydrogen at 0°C is 1284 ms^{-1}, what will be the velocity of sound in a mixture of two parts by volume of hydrogen to one part of oxygen ? (Ans. 524·3 ms^{-1})

4. The velocity of sound through an unknown gas at S. T. P. is 258·4 ms^{-1} and its density 1·977 kg m^{-3}. What is the atomicity of the molecules of the gas ? (Ans. 3)

(E)

1. The velocity of sound in moist air is (less, greater) than that in dry air.

2. The adiabatic bulk modulus is......times the isothermal bulk modulus.

3. Sound travels fastest in......(solid, liquid, gases).

4. The speed of sound is proportional to the......(square, square root) of the absolute temperature.

5. The density of saturated water vapour at a given temperature and pressure is 3/8, 5/8, 4/9 times the density of dry air of the same temperature and pressure.
sure. (Ans. 1. greater, 2. γ times, 3. solid, 4. square root, 5. 5/8)

CHAPTER 4

VELOCITY OF TRANSVERSE WAVE ALONG A STRETCHED STRING: LAWS OF TRANSVERSE VIBRATION OF STRING: SONOMETER AND MELDE'S APPARATUS

4.1. The String

Strings are very important in the world of music.. They have been used from time immemorial as a source of musical sound in many musical instruments such as the bina, the setar, the esraj, the violin etc. For musicians, a string is simply a brass or steel wire. In the version of physicists, a string is a solid body infinitely **thin**, perfectly **uniform** all over its length and perfectly **flexible** i.e. having no stiffness whatsoever. A string differs from a rod in respect of thickness and flexibility (or stiffness). A rod has a finite diameter and limited stiffness. A string can vibrate only when it is put under a certain tension.

4.2. Velocity of Transverse Waves along a Stretched String

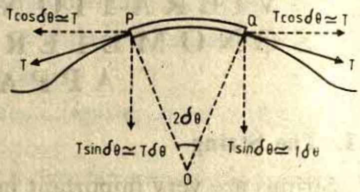
When a string is stretched under a tension and small portion of it is pulled laterally a little and then released, transverse waves are set up in the string which travel along the string with a velocity depending upon the tension of the string and its mass per unit length called its *linear density*. To deduce an expression for the velocity of the transverse waves, consider a string of linear density 'm' and stretched with a tension T.

Method I (*an elementary method*) : Let C be the velocity of the wave along the string to the right. Now if the string is imagined to be moved to the left with the same velocity, then the form of the wave will become stationary and every element of the string will move along the form of the wave to the left. In other words the elements will move along a curved path. When the displacement of the string is small, each part of the path may be supposed to be an arc of a circle. Consider an element PQ of length δx. The forces acting on the element are T, T at the ends along the tangent in the

directions shown in the figure. Draw normals at P and Q and let them meet at O. Then O is the centre of the circle of which PQ is a part. Let $\angle POQ$ be $2\delta\theta$. Then $\delta x = (2\delta\theta \times r)$ where $r = OP = OQ$.

Resolving forces along and perpendicular to PQ, we have centripetal force $= 2T\delta\theta$.

But centripetal force $= \dfrac{(m\delta x)C^2}{r}$.



$$\therefore \quad \frac{(m\delta x)C^2}{r} = 2T\delta\theta$$

$$= 2T.\frac{\delta x}{2r}$$

or $$C = \sqrt{\frac{T}{m}}.$$

Method II (*via Calculus*) : Consider an element $PQ = \delta x$ of the string at a distance x from the left end of the string. Let y be the displacement of the left end of the element at time t. Then displacement of the other end at the same time will be $y + \delta y$. Thus at time t,

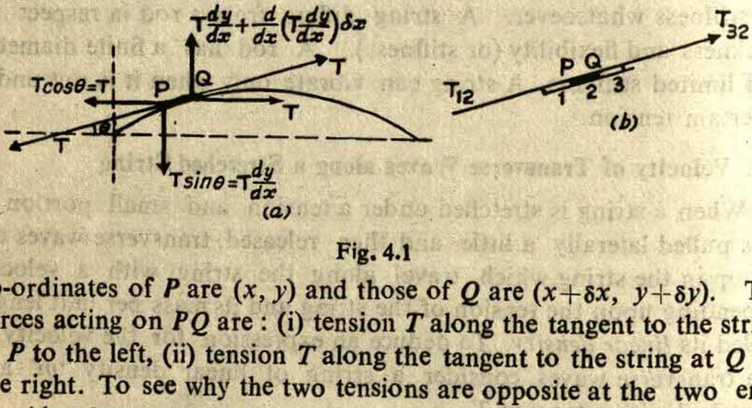


Fig. 4.1

co-ordinates of P are (x, y) and those of Q are $(x+\delta x, y+\delta y)$. The forces acting on PQ are : (i) tension T along the tangent to the string at P to the left, (ii) tension T along the tangent to the string at Q to the right. To see why the two tensions are opposite at the two ends consider three consecutive elements 1, 2 and 3. Number 2 is the element PQ under consideration. Over the section P number 2 exerts a force on 1 to the right, because the string itself is stretched from left to right. Let us denote it by T_{21} and read it as force by 2 on 1. By Newton's third law number 1 exerts the same force on 2 in the opposite direction. This force will be denoted by T_{12}. Thus the force on the left end of the string is to the left. The same consideration will show that the force on the right end of the element is to the right. Let the tangent at P make an angle θ with the undisplaced position of the string i.e. x-axis.

Resolving the tension at P along and perpendicular to the x-axis we have

$T\cos\theta$ along the negative direction of x and $T\sin\theta$ along the negative direction of y.

Since θ is small, $\cos\theta=1$ and $\sin\theta=\theta=\tan\theta=\frac{dy}{dx}$. Therefore the above components of the tension at P are respectively equal to

T along the negative direction of x .. (i)

and $T\frac{dy}{dx}$ along the negative direction of y. .. (ii)

Let us denote the magnitudes of the components of tension at P along the x-axis and the y-axis by X and Y respectively. Then the magnitudes of the corresponding components of the tension at Q are $X+\delta X$ and $Y+\delta Y$. By calculus $\delta X=\frac{dX}{dx}\delta x$ and $\delta Y=\frac{dY}{dx}\delta x$.

Here $X=T$ and $Y=T\frac{dy}{dx}$.

Hence $\delta X=\frac{dT}{dx}\delta x=0$ ($\because$ T is a constant)

and $\delta Y=\frac{d}{dx}\left(T\frac{dy}{dx}\right)\delta x=T\frac{d^2y}{dx^2}\delta x$.

Thus the components of the tension at Q are T along the positive direction of x and $T\frac{dy}{dx}+T\frac{d^2y}{dx^2}\delta x$ along the positive direction of y.

Obviously the forces along the x-direction are equal and opposite and so they cancel each other. The resultant force on the element at time t is the difference of the forces along the y-axis.

$\therefore$ The resultant force on the element at time t

$$=T\frac{dy}{dx}+T\frac{d^2y}{dx^2}\delta x-T\frac{dy}{dx}=T\frac{d^2y}{dx^2}\delta x.$$

If m represents the mass per unit length of the string then $m\delta x$ is the mass of the element and by formula,

force = mass $\times$ acceleration.

we have $$T\frac{d^2y}{dx^2}\,\delta x=(m\delta x).\ \frac{d^2y}{dt^2}$$

or $$\frac{d^2y}{dt^2}=T/m\ \frac{d^2y}{dx^2}$$

If we put C^2 for T/m, then this equation is reduced to the form,

$$\frac{d^2y}{dt^2}=C^2\frac{d^2y}{dx^2}.$$

This is the differential equation of a progressive wave of velocity C. Thus the velocity of transverse waves along a stretched string is

$$C=\sqrt{\frac{T}{m}} \quad \cdots \quad (4.1).$$

4.3. Modes of Vibration of a Stretched String of Finite Length

When a string is stretched between two supports, the transverse waves, set up by striking or plucking, travel in either direction with a velocity $C=\sqrt{\dfrac{T}{m}}$ and get reflected from the fixed ends. Thus there is superposition of two identical progressive waves travelling in opposite directions in the string. The result of superposition of two such waves is always the formation of stationary waves with definite nodes and antinodes. As the string is clamped at the ends, there must be nodes at the two fixed ends and any number of nodes and antinodes in between. This means that a string may vibrate in different ways. These different ways in which stationary waves may be formed, represent different **modes of vibration of the string.** Obviously the simplest mode of vibration of a string will be that one in which there are two nodes at the fixed ends and an antinode in the middle. This is called the fundamental mode of vibration. In the next mode of vibration there are two nodes at the fixed ends and two antinodes and a node in between. In the third mode of vibration there are three antinodes

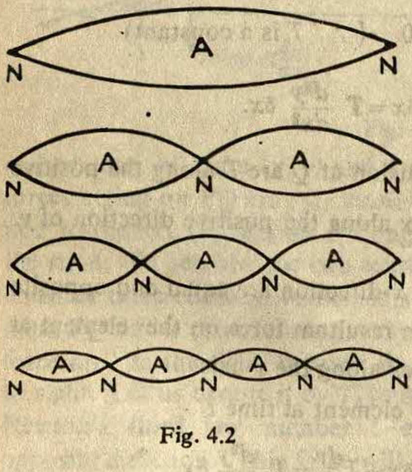


Fig. 4.2

and two nodes besides the two extreme nodes. In the fourth mode there are four antinodes and three nodes besides the two nodes at the extremities.

If l is the length of the string then in the fundamental mode of vibration we have $l=\lambda/2$, because node to node distance is half a wave, or $\lambda=2l$.

We have, $n\lambda=C$. Therefore if n_1 is the frequency in this mode then

$$n_1=\frac{C}{2l} \quad \text{or} \quad n_1=\frac{1}{2l}\sqrt{\frac{T}{m}} \qquad .. \quad (4.2)$$

$$\left(\because \quad C=\sqrt{\frac{T}{m}}\right).$$

Note here that there is only one loop in this mode of vibration.

In the second mode of vibration $l=\lambda$, and the frequency of vibration is

$$n_2=\frac{C}{\lambda}=\frac{C}{l}=\frac{1}{l}\sqrt{\frac{T}{m}}$$

or
$$n_2=\frac{2}{2l}\sqrt{\frac{T}{m}}=2n_1.$$

Note here that there are two loops in this mode of vibration.

In the third mode of vibration $l=3\lambda/2$, or $\lambda=2l/3$ and

$$n_3=\frac{3}{2l}\sqrt{\frac{T}{m}}=3n_1.$$

In general if the string vibrates in s loops there will be s antinodes separated by $(s+1)$ nodes and the length of the string $l=s\ \lambda/2$ or $\lambda=\dfrac{2l}{s}$. The frequency of vibration of the sth mode is then

$$n_s=\frac{s}{2l}\sqrt{\frac{T}{m}}. \qquad .. \quad (4.3)$$

4.4. Laws of Transverse Vibration of String

The laws of vibration of a stretched string are incorporated in equation $n_s=\dfrac{s}{2l}\sqrt{\dfrac{T}{m}}$.

There are three laws of transverse vibration of a string :

(*i*) *Law of length*. The frequency of vibration of a stretched string is inversely proportional to its length when its tension and linear density remain constant.

i.e. $n \propto \dfrac{1}{l}$ when T and m remain constant.

(*ii*). *Law of tension.* The frequency of vibration of a stretched string is directly proportional to the square root of its tension when its length and linear density remain constant

i.e. $n \propto \sqrt{T}$ when l and m remain constant.

(*iii*) *Law of mass.* The frequency of vibration of a stretched string is inversely proportional to the square root of its linear density i.e. mass per unit length, when its length and tension remain constant

i.e. $n \propto \frac{1}{\sqrt{m}}$, when l and T remain constant.

The 'law of mass' may be split into two other laws, namely, the law of diameter and the law of density, because $m = \frac{1}{4}\pi d^2 \rho$ where d is the diameter of the string and ρ is the density of the material of the string. Therefore,

$$n \propto \frac{1}{d} \text{ when } l, T \text{ and } \rho \text{ are constant.}$$

This is *law of diameter.*

$$n \propto \frac{1}{\sqrt{\rho}} \text{ where } l, T \text{ and } d \text{ remain constant.}$$

This is *law of density.*

4.5. Note : Tone : Overtones and Harmonics

A sound wave of only one frequency is called a tone. A tone is hardly emitted by a musical instrument. When a musical instrument is excited, it emits a complex wave consisting of a number of tones corresponding to the different modes of vibration of the string. Such a sound is called a note. A note emitted by a musical instrument consists of, theoretically speaking, infinite series of tones corresponding to the infinite number of modes of vibration of the string. All of them are not present always. As for example if the string is excited at its centre, the tones corresponding to the modes of vibration needing the mid-point as node will be absent. The frequency of the note emitted by a string is always the same as the frequency of vibration of its fundamental whatever be the way of exciting it. The presence of other tones corresponding to its higher modes of vibration simply modifies the form of the wave. The quality of the note depends on the form of the wave and the form, in its turn, depends on the upper tones (or overtones). The greater the number of overtones present, the smoother is the shape of the wave form and the better is the quality of the note. A tuning fork, when struck gently, almost emits

a smooth sine wave of one frequency i.e. it emits almost a tone. When the frequencies of overtones or uppertones are exact multiples of the frequency of the fundamental, the overtones or upper tones are specially called *harmonics.* Thus overtones of a stretched string are harmonic overtones. But in case of a fixed-free bar, a membrane etc. overtones are not in exact multiples of their fundamental frequencies. Such overtones are called *inharmonic overtones.*

The harmonics are counted from the fundamental but overtones are counted from next to the fundamental.

4.6. Verification of the Laws of Vibration of a Stretched String

The laws of transverse vibrations of a stretched string are verified either by a sonometer or by Melde's experiment.

(a) *Sonometer.* A sonometer consists of a rectangular hollow wooden box called the sounding box on the top of which two fine wires of brass or steel are stretched. At one end they are tied to pegs P_1 and P_2 provided at the end of the box for this purpose and at the other end one of the wires passes over a smooth pulley fixed at the other end of the sounding box. This wire is called the experimental wire. It carries a hanger at its free end. By putting slotted weights on the hanger its tension can be fixed to any known value. The other wire called the auxiliary wire is clamped to a third peg P_3 at the other end. By turning the peg P_3, its tension can be adjusted. Both the wires pass over two bridges fixed at the ends of the box. Besides these two end bridges there are two movable bridges, one below each. The sounding box intensifies the note emitted by the string by executing forced vibrations enforced on it by the vibrating strings. There are a few side holes for communication of the vibrations of the air inside the sonometer box with the atmospheric air.

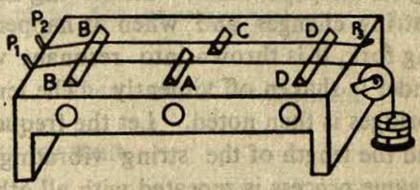


Fig. 4.3

To verify the law of length, a few tuning forks of known frequencies are taken and the experimental wire is stretched by placing a suitable load (say, 4 kg) on the hanger. One of the tuning forks is taken and it is lightly struck by a rubber hammer. Its stem is gently pressed on the top of the sounding box when a loud sound is heard. The string is then plucked gently at the middle point and

it is examined carefully whether the two notes emitted by the string and the tuning fork are in unison. If not, the movable bridge below the wire is adjusted so that the notes are exactly in unison. The unison is tested either by beats or a paper rider. When the notes are nearly equal in frequency they will produce distinct beats. The frequency of the string is then carefully adjusted by moving the bridge slowly so that beats disappear completely. By the principle of formation of beats, that is, the number of beats per second is equal to the difference of frequencies of the component waves, the two notes are now exactly in unison. In the paper rider method, a small piece of paper is put on the experimental wire. The tuning fork when gently placed on the top of the sonometer box will drive the string and force it to vibrate. But as the amplitude of vibration is small, there will be no visible effect of forced vibration of the string on the paper rider. On moving the bridge the frequency of the string changes and when it happens to be equal to that of the tuning fork, it is thrown into resonant vibration and consequently the rider is shaken off violently. The length of the string between the bridges is then noted. Let the frequency of the tuning fork be n_1 and the length of the string vibrating in resonance with it be l_1. The same process is repeated with all other tuning forks. It is found that

$$n_1l_1 = n_2l_2 = n_3l_3 = \ldots\ldots \text{when } T \text{ and } m \text{ are constant}$$

$$\text{or} \quad nl = \text{a constant;}$$

$$\therefore \quad n \propto \frac{1}{l} \text{ when } T \text{ and } m \text{ are constant is verified.}$$

To verify the law of tension, we make use of the auxiliary wire. As per requirement of the law, the length of the experimental wire is fixed at a certain length. Changing the tension of the experimental wire, the length of the auxiliary wire vibrating in resonance with the fixed length of the experimental wire is found. Suppose T_1, T_2, T_3....... are the tensions of the experimental wire and the corresponding lengths of the auxiliary wire vibrating in resonance with the fixed length of the experimental wire are α_1, α_2, α_3........ It is found that

$$\alpha_1\sqrt{T_1} = \alpha_2\sqrt{T_2} = \alpha_3\sqrt{T_3} \ldots\ldots \text{when } l \text{ and } m \text{ are constant}$$

$$\text{or} \quad \alpha\sqrt{T} = \text{a constant, when } l \text{ and } m \text{ are constant}$$

$$\text{or} \quad \alpha \propto \frac{1}{\sqrt{T}}, \text{ when } l \text{ and } m \text{ are constant.}$$

Since the tension and the mass per unit length of the auxiliary wire remain constant, the first law, which has already been tested,

is applicable to it. So if n is its frequency, then $n \propto \frac{1}{\alpha}$. The frequency of the auxiliary wire is also the frequency of the experimental wire because the two are in unison.

$\therefore$ $n \propto \sqrt{T}$, when l and m are constant.

To verify the law of mass, the length and tension of the experimental wire are fixed at certain suitable values as per requirements of the law. The length of the auxiliary wire vibrating in resonance with the experimental wire is noted. A certain length of the wire is then cut out and weighed on a balance. The mass divided by the length of the wire gives mass per unit length of the wire. The process is repeated by changing the diameter and the material of the experimental wire. Suppose $\alpha_1, \alpha_2, \alpha_3$........are the observed resonant lengths of the auxiliary wire, and m_1, m_2, m_3........are the corresponding linear densities of the experimental wire. It is found that

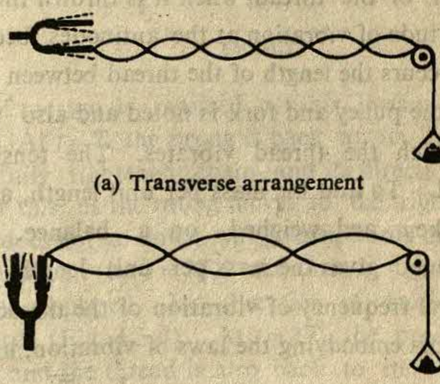
$$\frac{\alpha_1}{\sqrt{m_1}} = \frac{\alpha_2}{\sqrt{m_2}} = \frac{\alpha_3}{\sqrt{m_3}} \ \ldots\ldots, \text{ when } l \text{ and } T \text{ are constant}$$

or $\alpha \propto \sqrt{m}$ when l and T are constant.

But $n \propto \frac{1}{\alpha}$ by the first law.

$\therefore$ $n \propto \frac{1}{\sqrt{m}}$, when l and T are constant, is verified.

(b) *Melde's experiment*. A very simple method of verifying the laws of transverse vibrations of a stretched string is due to Melde (1859). In Melde's experiment an electrically driven tuning fork or a vibrator is coupled to a cotton thread passing horizontally over a frictionless pulley and carrying a scale pan at its other free end. By putting known weights on the pan the thread can be stretched by a



(a) Transverse arrangement

(b) Longitudinal arrangement

Fig. 4.4

known tension. The experiment is usually carried out in two ways : (i) **Transverse arrangement.** In this arrangement the tuning fork or vibrator and the thread are arranged in such a way that the line of vibration of the fork is perpendicular to the thread.

(ii) **Longitudinal arrangement.** In this arrangement the line of vibration of the tuning fork is arranged along the length of the thread.

To verify the laws of vibration of stretched string in either arrangement, a certain weight is put on the pan and the vibrating length of the thread is adjusted by drawing the tuning fork, or vibrator whatever it is, on the table till the thread is thrown into resonant vibrations. When the tuning fork or vibrator, is set in vibrations, it drives the thread and forces it to vibrate. The thread is thus subjected to forced vibration. Also the disturbance (wave) created by the fork at this end travels to the other end and gets reflected from there and travels backwards. Thus, in the thread there will be superposition of two identical waves, the result of which is always the formation of stationary waves with definite nodes and antinodes. In general, the frequency of the fork being different from the natural frequency of the thread, there will be forced vibrations of the thread and hence the amplitude of vibration at the antinodes will be very small. By proper adjustment of the length of the thread, when it is thrown into resonant vibration, the amplitude of vibration at the antinodes becomes maximum. When this occurs the length of the thread between its points of attachment with the pulley and fork is noted and also the number of segments in which the thread vibrates. The tension of the thread is also noted. To find the mass per unit length, a large length of the thread is taken and weighed on a balance. The mass divided by the length gives the mass per unit length of the string. Then the natural frequency of vibration of the thread is calculated from the formula embodying the laws of vibration, namely

$$n=\frac{1}{2l}\sqrt{\frac{T}{m}}.$$

If this formula is correct, the calculated frequency must be equal to the frequency (N) of the fork in the transverse arrangement, because resonance takes place when the frequency of the thread (the driver) is equal to the frequency of the fork (the driven).

That is, in the transverse arrangement resonance occurs when, $N=n$ ...., in the transverse arrangement.

But in the longitudinal arrangement, resonance occurs when the frequency of the fork is just double the frequency of the thread. The full mathematical treatment is beyond the scope of the book. An easier explanation is given below Suppose that the prong of the fork to which the thread is attached is at the extreme left end and the thread is completely tightened and its direction of motion is downwards. After half the oscillation i.e. at $t=T/2$, the same prong goes to its extreme right end and the thread becomes slack and gets fully displaced. Its direction of motion is reversed at this instant and goes upwards (Fig. 4.5 b). At $t=T$, the prong is back at its initial position, the thread is fully tightened again and its direction of motion is upwards. After this, as the prong moves to the right to let the thread loose it does not go down, but goes upwards due to its upward inertia of motion. At $t=3T/2$ the thread is let loose completely and gets fully displaced. Its direction of motion is reversed and goes downwards (Fig. 4.5 d). At $t=2T$, the prong is back to its initial position and the thread is also back to its initial conditions. Thus during the time when the fork makes two vibrations, the thread makes one vibration. Hence the frequency of the fork is double the frequency of the thread at resonance.

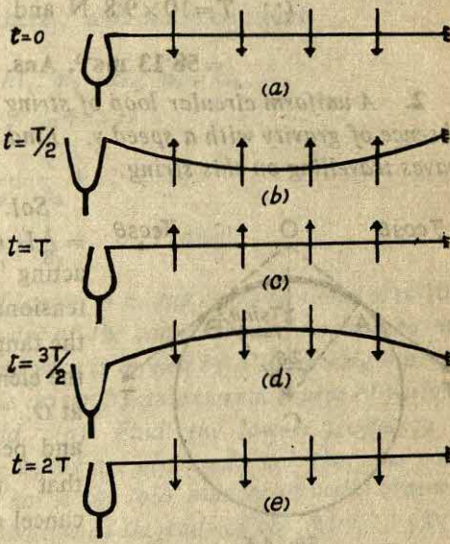


Fig. 4.5

i.e. $N=2n$........in the longitudinal arrangement.

So in the longitudinal arrangement, if the calculated frequency of the thread is found to be just half the frequency of the fork, the laws of vibration of strings will be verified.

Examples :

1. *What is the speed of a transverse wave in a string of length* 2m *and mass* ·058 *kg, when stretched by a* 10 *kg weight.*

Sol. We have $C=\sqrt{\frac{T}{m}}=\sqrt{\frac{10\times 9{\cdot}8\times 2}{{\cdot}058}}$

$$(\because \quad T=10\times 9{\cdot}8 \text{ N and } m=\frac{{\cdot}058}{2}\text{ kgm}^{-1})$$

$$=\mathbf{58{\cdot}13 \ ms^{-1}.\ Ans.}$$

2. *A uniform circular loop of string is rotating clockwise in the absence of gravity with a speed v. Find the speed of the transverse waves travelling on this string.*

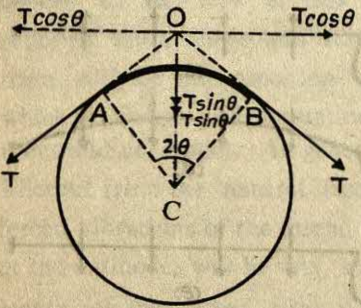


Fig. 4.6

Sol. Consider an element $AB=\triangle l$ of the string. The forces acting on the element are the tensions T and T at A and B along the tangents to the string away from the element. Let the tensions meet at O. Resolving the tensions along and perpendicular to OC, we see that the tangential component cancel out, but the radial components (towards the centre C) are added up.

$\therefore$ The force towards the centre effective on the element $=2T\sin\theta=2T\theta$ ($\because$ θ small)

$=T.\ \frac{\triangle l}{r}$ where r is the radius of the ring. The acceleration of the element towards the centre is $\frac{v^2}{r}$ because the string is rotating with a speed v.

$$\therefore \qquad T\frac{\triangle l}{r}=(m\triangle l)\times\frac{v^2}{r} \text{ where } m \text{ is the mass per unit length of the string}$$

($\because$ force = mass × acceleration)

or $\quad \dfrac{T}{m} = \nu^2.$

Now $\quad C_{wave} = \sqrt{\dfrac{T}{m}}.$

$\therefore$ Here $\quad C_{wave} = \sqrt{\nu^2} = \nu.$

3. *Show that if $n_1, n_2, n_3 \ldots$ are the frequencies of the segments of a stretched string, the frequency (n) of the string itself is given by* $\quad \dfrac{1}{n} = \dfrac{1}{n_1} + \dfrac{1}{n_2} + \dfrac{1}{n_3} + \ldots.$

Sol. We have $n \propto \dfrac{1}{l}$, where T and m are constant.

Hence $\quad n = kl;\ n_1 = kl_1;\ n_2 = kl_2;\ n_3 = kl_3 \ldots.$

But $\quad l = l_1 + l_2 + l_3 + \ldots.$

$$\therefore \quad \frac{k}{n} = \frac{k}{n_1} + \frac{k}{n_2} + \frac{k}{n_3} + \ldots.$$

or $$\frac{1}{n} = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \ldots. \text{ Proved.}$$

4. *An aluminium wire of length ·6 m and cross-sectional area 10^{-6} m^2 is connected to a steel wire of the same cross-sectional area and length ·866 m. The compound wire is loaded by a 10 kg weight at one end (steel wire end) and is driven by an external source of variable frequency at the other end. (a) Find the lowest frequency of excitation for which standing waves are observed, such that the joint in the wire is a node. (b) What is the total number of nodes observed at this frequency, excluding the two at the ends of the wire? (Density of aluminium = 2600 kgm^{-3}, and that of steel = 7800 kgm^{-3})*

Sol. $$C_{Al} = \sqrt{\frac{T}{m}} = \sqrt{\frac{T}{\pi r^2 \rho}} = \sqrt{\frac{10 \times 9{\cdot}8}{10^{-6} \times 2600}} = \sqrt{\frac{49}{13}} \times 10^2$$

$$C_{steel} = \sqrt{\frac{10 \times 9{\cdot}8}{10^{-6} \times 7800}} = \sqrt{\frac{49}{39}} \times 10^2.$$

Now the frequency of vibration of a stretched string is given by

$$n = \frac{s}{2l}\sqrt{\frac{T}{m}}, \quad \text{where } s \text{ is the number of loops.}$$

or $$n = \frac{s}{2l} \cdot C.$$

Here the frequency of both the wires is the same as they are driven by the same source.

$$\therefore \quad \frac{s}{2\times\cdot 6}\sqrt{\frac{49}{13}}\times 10^2 = \frac{s'}{2\times\cdot 866}\sqrt{\frac{49}{39}}\times 10^2$$

or $\qquad s'=2\cdot 5s$

where s is the number of loops in the aluminium wire, and s' that in steel.

Since s and s' are both integers, hence in the minimum s may be 2 and $s'=5$.

Fig. 4.7

There are in all eight nodes. Hence excluding the nodes at the end there are 6 nodes.

$$n=\frac{2}{2\times\cdot 6}\times\sqrt{\frac{49}{13}}\times 10^2=\mathbf{323\ Hz.\ Ans.}$$

5. *In Melde's experiment, an ordinary cotton thread of length* 150 *cm vibrates in* 3 *loops in the longitudinal arrangement. If the frequency of the vibrator is* 100 *Hz and the stretching force* 20 *gm wt, what is the mass of* 1 *km of ordinary thread ?*

Sol. We have, $n=\frac{s}{2l}\sqrt{\frac{T}{m}}$.

In the longitudinal arrangement $N=2n$.

$$\therefore \qquad \frac{N}{2}=\frac{s}{2l}\sqrt{\frac{T}{m}}$$

or $$\frac{100}{2}=\frac{3}{2\times 1\cdot 5}\sqrt{\frac{20\times 10^{-3}\times 9\cdot 8}{m}}$$

or $$m=\frac{196\times 10^{-3}}{50^2}\ \mathrm{kg\ m^{-1}}$$

$$\therefore \qquad \text{Mass of 1 km}=\frac{196\times 10^{-3}}{50^2}\times 10^3=\mathbf{\cdot 0784\ kg.\ Ans.}$$

6. *A brass wire of radius* ·5 *mm and length* 2*m is stretched by a* 8 *kg weight, when it is found to extend by* 2 *mm. What is the velocity*

of longitudinal waves and transverse wave in the wire ?
(density of brass = 7800 *kgm*$^{-3}$)

Sol. Stress $= \dfrac{8 \times 9{\cdot}8}{\pi({\cdot}5 \times 10^{-3})^2} = 9{\cdot}98 \times 10^7 \text{ Nm}^{-2}$

$$\text{Strain} = \frac{2 \times 10^{-3}}{2} = 10^{-3}$$

$$Y = \frac{9{\cdot}98 \times 10^7}{10^{-3}} = 9{\cdot}98 \times 10^{10} \text{ Nm}^{-2}$$

$$C_{longitudinal} = \sqrt{\frac{Y}{\rho}} = \sqrt{\frac{9{\cdot}98 \times 10^{10}}{7800}} = \mathbf{3577 \text{ ms}^{-1}}. \textbf{ Ans.}$$

$$C_{transverse} = \sqrt{\frac{T}{m}} = \sqrt{\frac{T}{\pi r^2 \rho}} = \sqrt{\frac{8 \times 9{\cdot}8}{\pi({\cdot}5 \times 10^{-3})^2 \times 7800}}$$

$$= \mathbf{113 \text{ ms}^{-1}}. \textbf{ Ans.}$$

QUESTIONS

(A)

1. If the tension of a wire is increased four times and its length is reduced to half the original length, the frequency of the wire (a) remains unchanged, (b) doubled, (c) halved, (d) increased four times.

2. The quality (sweetness of a note) of a note emitted by a stringed instrument depends on (a) the length of the string, (b) the body of the instrument, (c) the mode of excitation, (d) both on the body of the instrument and the mode of excitation.

3. In the longitudinal arrangement of Melde's apparatus, if N be the frequency of the vibrator and n be the frequency of the string, then (a) $n = 2N$, (b) $n = N$, (c) $N = 2n$, (d) none of these.

4. The maximum wavelength of a transverse wave, that can be set up in a string of length l is (a) l, (b) $2l$, (c) $l/2$, (d) $4l$.

5. In Melde's experiment if the tension is increased four times and the length is also simultaneously increased four times, the number of loops (a) remains unchanged, (b) doubled, (c) halved, (d) increased four times.

[Ans : **1.** d, **2.** d, **3.** c, **4.** b, **5.** *b*]

(B)

1. Explain the production of nodes and loops in the case of strings.

2. Explain carefully note, tone, overtones and harmonics.

3. State the laws of transverse vibration of a string.

4. Define a 'string in sound'.

(C)

1. Deduce an expression for the velocity of transverse waves in a stretched string.

2. State the laws of transverse vibration of a stretched string and describe experiments to verify them by a sonometer.

3. Establish the laws which govern the vibration of strings. Describe how you would verify them by Melde's apparatus.

4. Describe Melde's method of determining the frequency of a vibrator.

(D)

1. Two identical sonometer wires have a fundamental frequancy of 540 Hz when kept under the same tension. What fractional increase in the tension of one wire will lead to the occurrence of six beats per second when both wires vibrate simultaneously ? [Ans. 2·23%]

2. What is the maximum speed of a transverse wave in a brass wire, if the breaking stress of brass is 34×10^{6} kg m^{-2} and the density of brass is 8500 kg m^{-3}. [Ans. 198 ms^{-1}]

3. Show that the speed of a transverse wave along a loaded string is $\sqrt{\frac{T}{\rho}}$, where T is the stress of the string and ρ is the density of the material of the wire.

4. A copper wire is held at the two ends by rigid supports. At 30°C the wire is just taut, with negligible tension. Find the speed of transverse waves in this wire at 10°C (Young's modulus of copper$=1{\cdot}3\times10^{11}$ Nm^{-2} and density of copper$=9000$ kg m^{-3}, Coefficient of linear expansion$=1{\cdot}7\times10^{-5}K^{-1}$) (I. I. T. 1979) [Ans. 70 ms^{-1})

5. A wire of density 9000 kgm^{-3} is stretched between clamps 1 m apart while subjected to an extension of ·05 cm. What is the lowest frequency of transverse vibrations in the wire, assuming Young's modulus of the material to be 9×10^{10} Nm^{-2} ? (I. I. T. 1975) [Ans. 36·35 Hz]

6. A sonometer wire fixed at one end has a solid mass M hanging from its other end to produce tension in it. It is found that 70 cm length of the wire produces a certain fundamental frequency when plucked. When the same mass M is hanging in water, completely submerged in it, it is found that the length of the wire has to be changed 5 cm in order that it may produce the same fundamental frequency. Calculate the density of the material of mass M hanging from the wire. [Ans. 7270 kgm^{-3}]

7. The length of a stretched string is 40 cm, its mass is ·24 gm If the tension of the string is 1 kg wt, calculate the frequency of its fundamental note. [Ans. 160 Hz]

8. When the wire of a sonometer is 75 cm long, it is in tune with a tuning fork. On decreasing the length of the wire by 5 mm, it makes 2 beats per second with the fork. What is the frequency of the fork ? [Ans. 298 Hz]

9. On sounding two tuning forks 4 beats per second are heard. One tuning fork is in unison with 86·5 cm of a sonometer wire when stretched by a certain tension, and the other with 87·4 cm of the same wire under the same tension. Find the frequency of the tuning forks. [Ans. 388·4, 384·4 Hz]

10. The two parts of a sonometer wire, divided by a movable bridge differ by 2 mm and produce 1 beat per second when sounded together. Find their frequencies if the whole length of the wire is 1 m.

[Ans. 249·5 and 250·5 Hz]

11. A stretched string 1 m long is divided by two bridges into three parts so as to give notes whose frequencies are in the ratio 4 : 5 : 6. Find the distance between the bridges.

[Ans. 32·4 cm]

12. One end of a string in Melde's experiment is attached to a vibrating tuning fork while its other end carries a piece of stone. The string shows 8 vibrating loops. When the stone is immersed in water 10 loops are formed. Calculate the density of the stone. [Ans. 2770 kgm^{-3}]

(E)

1. The frequency of a note emitted by a string is the frequency of its......................(fundamental, second harmonic).

2. The pitch of the note emitted by a stretched string of a stringed instrument is determined by...................(the body of the instrument, the length of the string).

3. The loudness of the note emitted by a sonometer wire is determined by..............................(the body of the instrument, the length of the string).

4. If the string of a sonometer is bowed instead of being plucked, the quality of the note will not change. True or False ?

5. The paper rider method of testing unison works on the principle...... (forced vibration, beats, resonance).

6. When a string is plucked in middle all the.........(odd, even) harmonics will be absent.

[Ans : 1. fundamental. 2. the length of the wire. 3. the body of the instrument. 4. false. 5. resonance. 6. even (2nd, 4th.........].

CHAPTER 5

VIBRATION OF AIR COLUMNS: ORGAN PIPES

5.1. Vibration of Air Columns

Just as the vibration of a string is used as source of sound in many musical instruments from time immemorial, the vibrations of air in a closed pipe is also used as a source of sound in many musical instruments, such as the clarionet, the *sahnai*, the *bansuri*, the organ pipe etc. The former are called *stringed instruments* and the latter are called *wind instruments*.

To consider the modes of vibration of air in a pipe, let us take a tuning fork as exciter of vibrations in the air. The vibrating tuning fork will simultaneously do two things : first it will throw the air particles inside into 'forced vibrations' and second it will send disturbances (waves) down the tube. These waves are reflected back from the other end, whether it is open or closed. There are certain peculiarities with the reflection of longitudinal waves of which sound waves are one example. The first peculiarity is that a sound wave may be reflected from a closed end as well as from an open end. The second peculiarity is that there is a phase change of π when reflection takes place from an open end. The mechanism of reflection of a longitudinal wave from a closed end is easy to grasp. When a compression reaches the closed end, it checks the forward compression as it, being rigid, will not yield to the compression. Due to elasticity, the compressed layer of air compresses the layer above it, and it itself comes to a normal state after being relieved of the state of compression. That layer behaves exactly in the same way. Thus a pulse of compression is reflected as a pulse of compression and similarly a pulse of rarefaction is reflected as a pulse of rarefaction. So there is no phase change on reflection from a closed end.

When a compression reaches an open end, it is instantaneously converted into rarefaction because there being nothing to prevent forward compression, the compressed air expands suddenly into the vast open space, creating a partial vacuum at the open end. This rarefaction then starts travelling up the tube in the usual manner i.e., the layer of air above it expands to make up the fall of pressure leaving a partial vacuum (drop of pressure) in its own place. The layer of air above it behaves in the same way. Thus a pulse of compression is reflected as a pulse of rarefaction. Similarly, a pulse of rarefaction on reaching the open end is returned as a pulse of compression. In other words reflections from the open ends take place with a phase change of π, that is, the phase of the reflected wave is opposite to that of the incident wave.

Thus whether the air column is open or closed at the other end, there is superposition of two identical waves in the column of air. The result of superposition of two such waves is always the formation of stationary waves with definite nodes and antinodes.

Ordinarily, the air column will not give a loud sound, because ordinarily it will be instigated to execute forced vibration by the fork. In forced vibrations the amplitude is very small and so sound is not magnified. On changing the length of the column of air, the natural frequency of it changes and at one particular length it is thrown into resonant vibrations, when suddenly a very loud sound is heard.

5.2. Modes of Vibrations of Air Column in a Closed Pipe

In a closed pipe the fixed end is essentially the seat of a 'node' and the open end is essentially that of an 'antinode'. Therefore, in the fundamental mode of vibration there is a node at the closed end and an antinode at the top. Hence l (length of the column) $= \lambda/4$, because the distance from a node to the next antinode in a stationary wave is one quarter of a wavelength. This, however, requires a correction, because an antinode is not formed exactly at the open end of the pipe, but a little outside the pipe. By how much it is outside the pipe that depends on the diameter of the tube. This

distance by which an antinode is away from the open end is called the end correction. Let it be e. Then

$$l+e=\lambda/4, \text{ or } \lambda=4(l+e)$$

The frequency of the fundamental mode is given by,

$$C=n\lambda$$

$$\text{or, } n=\frac{C}{\lambda}=\frac{C}{4(l+e)} \quad \ldots (1)$$

$$\text{or, } n=\frac{1}{4(l+e)}\sqrt{\frac{E}{\rho}}$$

$$\left(\because \quad C=\sqrt{\frac{E}{\rho}}\right).$$

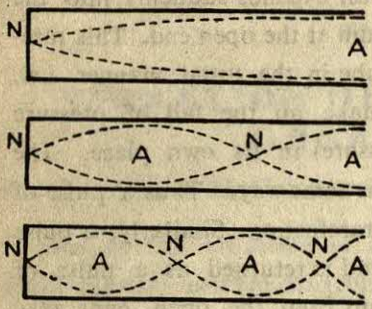


Fig. 5.1

Here E is the bulk modulus and ρ is the density of the gas in the pipe. Without end correction the frequency of the fundamental mode is,

$$n=\frac{C}{4l}=\frac{1}{4l}\sqrt{\frac{E}{\rho}}. \quad \ldots \quad (5.1)$$

In the next higher mode of vibration there are one additional node and one additional antinode within the pipe. The length of the pipe is related to the wavelength of stationary wave formed is

$$l+e=\frac{3\lambda}{4}, \quad \text{or} \quad \lambda=\frac{4}{3}(l+e).$$

The frequency of vibration of the air column is given by

$$n_1=\frac{C}{\lambda}=\frac{C}{4/3(l+e)}=\frac{3C}{4(l+e)}=3n \quad \ldots \text{ from (i).}$$

In the next higher mode there are two intermediate nodes and two intermediate antinodes, besides the node at the closed end and the antinode at the open end. The column of air contains five quarter-waves.

Hence, $\quad l+e=\dfrac{5\lambda}{4}$, or $\lambda=\dfrac{4(l+e)}{5}$

and the frequency of vibration n_2 is

$$n_2=\frac{C}{\lambda}=\frac{5C}{4(l+e)}=5n.$$

It can be similarly shown that the higher modes of vibration will produce tones of frequencies $7n$, $9n$, $11n$, ... Thus the

possible tones of a closed pipe have frequencies in the ratio of odd integers. The note emitted by the air column in a closed pipe will have only alternate overtones and hence is poor in harmonics.

5.3. Modes of Vibration of Air Column in an Open Pipe

When the air column is contained in an open pipe, the two ends are essentially seats of antinodes and hence in the fundamental mode of vibration of the air column, there are two antinodes and an intermediate node. The length of the pipe contains one half-wave.

$\therefore\ l+2e=\lambda/2$ or $\lambda=2(l+2e)$ and the frequency of the fundamental mode is

$$n=\frac{C}{\lambda}=\frac{C}{2(l+2e)}$$

$$=\frac{1}{2(l+2e)}\sqrt{\frac{E}{\rho}}.$$

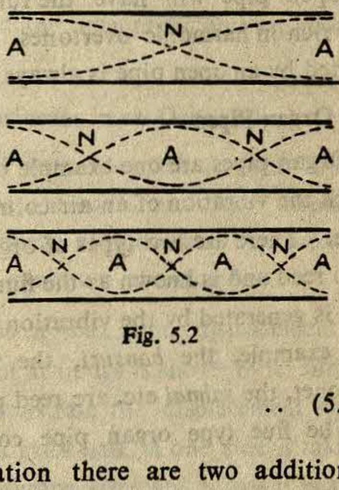


Fig. 5.2

Without end-correction the frequency of the fundamental mode is given by

$$n=\frac{C}{2l}=\frac{1}{2l}\sqrt{\frac{E}{\rho}}. \qquad .. \quad (5.2)$$

In the next higher mode of vibration there are two additional nodes and one additional antinode. The length of the pipe contains two half-waves.

$$\therefore \qquad l+2e=2\frac{\lambda}{2}, \quad \text{or} \quad \lambda=l+2e$$

and the frequency of vibration of the air column is

$$n_1=\frac{C}{\lambda}=\frac{C}{l+2e}=2.\frac{C}{2(l+2e)}=2n.$$

In the next higher mode, there are three intermediate nodes and two antinodes besides the two antinodes at the open ends. The pipe contains three half-waves.

$$\therefore \qquad l+2e=3\frac{\lambda}{2}$$

or $$\lambda = \frac{2}{3}(l+2e)$$

and the frequency of vibration n_2 is

$$\therefore \quad n_2 = \frac{C}{\lambda} = \frac{3C}{2(l+2e)} = 3n.$$

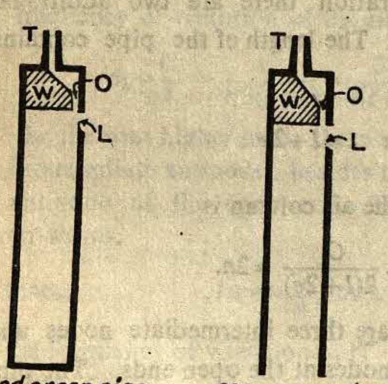
It can be similarly shown that the higher modes of vibration will produce tones of frequencies $3n$, $4n$, $5n$ ....

Thus the possible tones of an open pipe have frequencies in the ratio of natural numbers. The note emitted by the air column in an open pipe will have the full series of overtones and hence it is very rich in harmonic overtones. This is why the quality of a note emitted by an open pipe is always better than that of a closed pipe.

5.4. Organ Pipes

Organ pipes are one example of the various wind instruments in which the vibration of an air column is used as a source of sound wave. There are two types of organ pipes : the one in which there is no reed and is known as the flue pipe, and the other in which the note is generated by the vibration of reeds and is called the reed pipe. For example, the *bansuri*, the flute etc. are flue pipes and the clarionet, the *sahnai* etc. are reed pipes.

The flue type organ pipe consists of a pipe of wood or metal having a short tube T at one end and an aperture in the body of the pipe near this end. The lower edge of the aperture is chisel-shaped and is called the lip (L) of the pipe. There is a wedge (W) fixed near the lip to deflect the air blown through the tube. The pipe is termed closed or open, according as the other end is closed or open.



closed organ pipe open organ pipe

Fig. 5.3

The air stream blown under pressure is forced to flow through the narrow opening O and is directed on to the sharp

edge (lip) which is set into vibration of various frequencies. The air stream alternately passes on the two sides of the lip forming equispaced vortices on either sides of it; these initiate the vibrations of the whole air column. Out of the numerous frequencies of vibration of the edge, only those frequencies are responded to by the air column which correspond to its own fundamental and upper harmonics. Thus, though the sound originates at the edge, the frequencies and quality of the note emitted by an organ pipe are determined by the length of the air coulumn in the pipe.

The modes of vibration of closed and open organ pipe are already discussed in the previous article.

5.5. Demonstration of Nodes and Antinodes in an Organ Pipe

In stationary waves nodes and antinodes are formed with a maximum amplitude of vibration at antinodes and zero displacement or at least a minimum displacement at nodes but the pressure variation is maximum at nodes and minimum at antinodes.

The former fact that there is a maximum displacement at antinodes and minimum displacement at nodes is shown by a simple method devised by Savart. To demonstrate the displacement node and antinode an organ pipe having a glass wall on one side is taken. A light metal ring having a thin membrane stretched across it is also taken and very fine sand particles are sprinkled on the membrane, which are capable of vibrating on the slightest instigation. It is then lowered into the pipe and vibrations are set up by blowing air through the inlet pipe. At an antinode the sand particles are violently agitated. The motion of the sand particles can be seen through the glass wall and the rattling sound produced by the striking of the sand particles on the membrane distinctly heard. At a node the sand particles remain motionless and no rattling sound is heard (Fig. 5.4).

To show pressure nodes and antinodes a model organ pipe is provided with three Konig's manometric capsules at distances $\frac{\lambda}{4}$, $\frac{\lambda}{2}$ and $\frac{3\lambda}{4}$ of the way up the pipe (Fig. 5.5). The three capsules

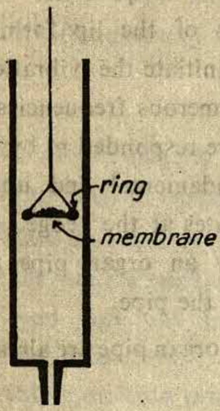


Fig. 5.4

are small hollow chambers having a flexible wall on one side and a pin hole burner. All the capsules are fed from the same gas supply and their burners are lighted. When the pressure on the flexible wall remains steady the length of the flame will not change. The variation of pressure on the flexible wall causes the wall to vibrate and this in its turn causes a vibration in the pressure of the gas supply. The flame thus jumps up and down with the frequency of the incident sound wave. The movements of the flame are, however, too rapid to be followed by the eye. The vibrations of the flame are studied by reflection in a rotating mirror. The reflection consists of a stretch of light having a toothed appearance, when there is a flickering of the flame. The reflection of a steady flame is simply a stretch of light of constant width since the greatest change in air pressure (i.e.,pressure antinode) occurs at the displacement nodes, the flame should show the greatest amount of flicker at the points where displacement nodes occur, and should not flicker at all at the positions of displacement antinodes.

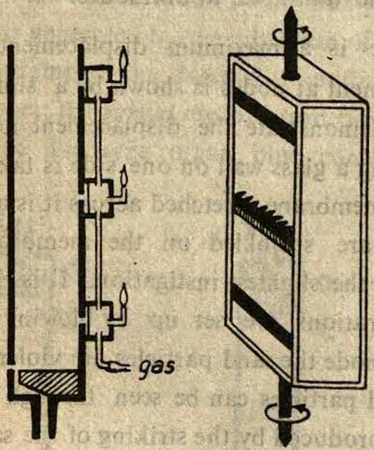


Fig. 5.5

The organ pipe is excited to emit its fundamental. Flame at the centre will flicker more than the other two. Because the vibration of the air column is not entirely in the fundamental node, the other two flames will also flicker slightly. Since the pipe is open at both ends, there is a displacement-node at the centre. The flame at the centre flickers to the greatest extent and hence at the centre there is a maximum pressure amplitude.

5.6. Determination of Velocity of Sound by Resonant Air Column

The resonant vibration of an air column in a pipe immersed in water in a tall jar provides an easy but not very precise method of determining the velocity of sound in air. A tuning fork of known frequency is taken and it is struck gently by a rubber hammer to excite its fundamental tone and then it is held above the mouth of a metal tube almost completely immersed in water in a tall jar. The air inside of it is thrown into forced vibrations. On raising the tube slowly the length of the air column increases and the natural frequency of air column decreases and at one particular length it is thrown into resonant vibration when suddenly a very loud sound is heared. The length of the tube above the level of the water is measured. Due to the reflection of the disturbance (wave) sent down the tube by the fork, stationary waves are also formed with definite nodes and antinodes. The lower end (water level) is essentially the seat of a 'node' as the particles there are not free to vibrate and the open end is essentially the seat of an antinode, because there air particles have the maximum freedom to vibrate. Therefore, if the length of the air column in the tube is l then

$$l=\frac{\lambda}{4}.$$

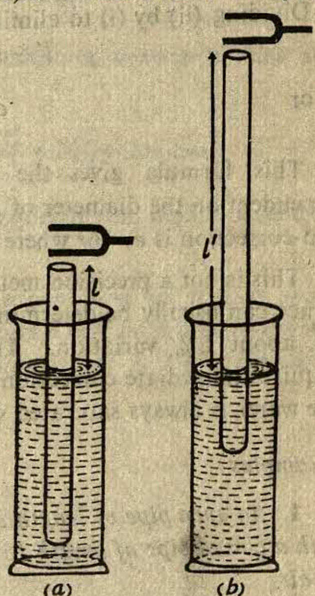


Fig. 5.6

This, however, requires a correction, because the antinode is not formed exactly at the open end, but a little above it. Let this distance be e. This is called the end-correction.

$$\therefore \quad l+e=\frac{\lambda}{4}. \qquad \text{.. (i)}$$

On raising the tube further to about three times its previous length again a loud sound (not so strong as before) is heard. This time the tuning fork resonates with the first harmonic of the air column of this length (i.e. $3l$). The first harmonic of the air column

of length $3l$ will have same frequency as the fundamental of the air column of length l. This is why the tuning fork resonates for the second time when the length of the air column is increased three times. This length of the tube is also measured. Let it be l'.

Then $$l'+e=\frac{3\lambda}{4}. \quad \cdots \text{(ii)}$$

Subtracting (i) from (ii) to eliminate e and calculate λ we have,

$$l'-l=\frac{\lambda}{2}, \quad \text{or} \quad \lambda=2(l'-l).$$

We have $$C=n\lambda$$

$$\therefore \quad C=2n(l'-l).$$

Knowing n, l and l', C is calculated from this formula.

Dividing (ii) by (i) to eliminate λ and calculate 'e' we have

$$l'+e=3(l+e)$$

or $$e=\frac{l'-3l}{2}.$$

This formula gives the end-correction. The end-correction is dependent on the diameter of the tube. According to Laplace the end-correction is $e=\cdot 6r$ where r is the radius of the tube.

This is not a precision method, because the best length for resonance can usually be determined only to within about 3 cm in 50 cm i.e. about 6% variation. The result obtained from this method requires immediate correction for humidity, because, the air above the water is always saturated with water vapour.

Examples

1. *An open pipe of length 30 cm makes 18 beats in 8·3 seconds with a closed pipe of length 15·05 cm. Calculate the velocity of sound in air.*

Sol. For an open pipe

$$n=\frac{C}{2l}=\frac{C}{2\times\cdot 3}=\frac{C}{\cdot 6}$$

and $$n' \text{ (closed pipe)}=\frac{C}{4l}=\frac{C}{4\times\cdot 1505}=\frac{C}{\cdot 602}$$

$$\therefore \quad \frac{C}{\cdot 6}-\frac{C}{\cdot 602}=\frac{18}{8\cdot 3}.$$

or $$C=\frac{\cdot 6\times \cdot 602\times 18}{8\cdot 3\times \cdot 002}=391\cdot 7\ \text{ms}^{-1}.\ \textbf{Ans.}$$

2. *A hollow cylindrical wooden vessel of length* 1 *m is closed lightly by a wooden plank at one end and the other end is closed by a flexible membrane. If the membrane is set to oscillations of small amplitude, find the modes of vibration of the air enclosed in the vessel.*

Sol. Since both the ends are closed, there must be two nodes at the ends and any number of nodes and antinodes in between. In the fundamental mode there is only one antinode besides the two nodes at the ends.

$$\therefore\quad \lambda/2=l \quad \text{or} \quad \lambda=2l$$

$$\therefore\quad n\ (\text{fundamental})=\frac{C}{\lambda}=\frac{C}{2l}.$$

In the next mode there are two antinodes and a node besides the two nodes at the ends.

$$\therefore\quad \lambda=l \qquad \therefore\quad n_1=\frac{C}{\lambda}=\frac{C}{l}=2\frac{C}{2l}=2n.\ \textbf{Ans.}$$

This way the next mode will have frequency $3n$, next to it $4n$ and so on.

3. *Calculate the length of a closed organ pipe so that it may sound the keynote A (440 Hz.). Where must a hole be drilled to excite a C (528 Hz) ? The adiabatic bulk modulus of air is $1\cdot 47\times 10^5 Nm^{-2}$ and density of air $1\cdot 29\ kgm^{-3}$.*

Sol. We have $$n=\frac{C}{4l}=\frac{1}{4l}\sqrt{\frac{E}{\rho}}$$

$$\therefore\quad l=\frac{1}{4n}\sqrt{\frac{E}{\rho}}$$

$$=\frac{1}{4\times 440}\sqrt{\frac{1\cdot 47\times 10^5}{1\cdot 29}}=\cdot 19\ \text{m. }\textbf{Ans.}$$

and $$l'=\frac{1}{4\times 528}\sqrt{\frac{1\cdot 47\times 10^5}{1\cdot 29}}=\cdot 16\ \text{m. }\textbf{Ans.}$$

4. *A tuning fork is held a little above the open end of a resonance tube. Resonance is heard when the lengths of the air column are respectively 24 cm and 74·1 cm. Calculate the end-correction and the diameter of the tube.*

Sol. We have $l_1+e=\lambda/4$ and $l_2+e=3\lambda/4$

$$\therefore \quad l_2+e=3(l_1+e) \text{ or } e=\frac{l_2-3l_1}{2}$$

or, $$e=\frac{74\cdot1-3\times24}{2}=\frac{2\cdot1}{2}=1\cdot05 \text{ cm.}$$

According to Laplace $e=\cdot3d$

$$\therefore \quad d=\frac{1\cdot05}{\cdot3}=3\cdot5 \text{ cm. } \textbf{Ans.}$$

QUESTIONS

(A)

1. When the reflection of sound wave takes place from a rigidly closed end of a pipe, there is a phase change by—(a) 0, (b) $\pi/2$, (c) π, (d) 2π.

2. When the reflection of a sound wave takes place from the open end of a pipe, there is a phase change by (a) 0, (b) $\pi/2$, (c) π. (d) 2π.

3. If the equation of a direct wave is $y=a\sin\left(wt-\frac{2\pi}{\lambda}x\right)$ the equation of the reflected wave, reflected from an open end is

(a) $y=a\sin\left(wt-\frac{2\pi}{\lambda}x\right)$

(b) $y=a\sin\left(wt-\frac{2\pi}{\lambda}x-\pi/2\right)$,

(c) $y=a\sin\left(wt+\frac{2\pi}{\lambda}x-\pi\right)$.

(d) none of these.

4. The longest wave that travels up and down an open pipe is of length (a) l, (b) $l/2$, (c) $2l$, (d) $4l$ where l is the length of the tube.

5. The longest wave that travels up and down a closed pipe, is of the wave length (a) l, (b) $l/2$, (c) $2l$, (d) $4l$, where l is the length of the tube.

6. If an open pipe is suddenly closed its frequency is (a) doubled, (b) halved, (c) trippled. (d) quadrupled.

7. If the closed end of a closed pipe is suddenly opened, its frequency is (a) doubled, (b) halved, (c) trippled, (d) quadrupled.

8. The end-correction in a closed pipe is (a) ·3d, (b) ·6d, (c) ·4d, (d) ·2d.

(Ans. : 1. a, 2. c, 3. c, 4. c, 5. d, 6. b, 7. a, 8. a.)

(B)

1. Explain how reflection takes place from the open end of a pipe with a reversal of the nature of the wave.

2. Explain with neat diagrams, the positions of nodes and antinodes in the different modes of vibration in a closed organ pipe.

3. Explain with neat diagrams, the positions of nodes and antinodes in the different modes of vibration of an open organ pipe.

4. Show that the quality of note emitted by an open organ pipe is better than that of a closed pipe.

(C)

1. Explain how stationary waves are produced in a closed pipe. Indicate, with neat diagrams, the positions of nodes and antinodes in the pipe. Calculate the frequency of each mode of vibration.

2. Explain how stationary waves are produced in an organ pipe. Indicate, with neat diagrams, the positions of nodes and antinodes in the different modes of vibration. Calculate the frequency of each mode.

3. Explain the formation of nodes and antinodes in an organ pipe. How do you demonstrate them ?

4. Describe, with necessary theory, the resonance air column method of determining velocity of sound in air. Indicate the corrections necessary.

(D)

1. A pipe of length 1·5 m closed at one end is filled with a gas and it resonates in its fundamental with a tuning fork. Another pipe of the same length but open at both ends is filled with air and it resonates in its fundamental with the same tuning fork. Calculate the velocity of sound at 0°C in the gas given that the velocity of sound in air is 360 ms^{-1} at 30°C where the experiments is performed. (I. I. T. 1974) (Ans. 683·4 ms^{-1}).

2. A tuning fork having a frequency of 340 Hz is vibrating just above a cylindrical tube. The height of the tube is 1·2 m. Water is slowly poured in. What is the minimum height of water required for resonance ? Velocity of sound in air = 340 ms^{-1}. (I. I. T. 1975) (Ans. 45 cm)

3. An open organ pipe emits a note of frequency 256 when the temperature of air is 40°C. What will be the frequency of the note when the temperature is 20°C ? (Ans. 247)

4. Two open organ pipes of length 1 m and 1·01 m are sounded together and 17 beats are heard in 10 seconds. Calculate the frequencies of the emitted notes and the velocity of sound in air. (Ans. 170 and 171·7 Hz; 343·4 ms^{-1})

5. Two tuning forks *A* and *B* give 6 beats per second. *A* resounds with a closed column of air 15 cm long and *B* with an open column 30·5 cm long. Calculate their frequencies. (Ans. 360 and 366 Hz)

6. Two open organ pipes, when sounded together, produce 4 beats per second at 15°C. If the temperature be 40°C, find how many beats will be heard. (Velocity of sound in air at 0°C = 332 ms^{-1}) (Ans. 4·17)

7. A certain tuning fork first produced resonance in a glass tube with an air column of 33 cm and it could resonate again with a column of 100·5 cm in the same tube. Calculate the end-correction. (Ans. ·75 cm)

8. When a fork of 512 Hz is sounded, the difference in level of water in a tube between two successive positions of resonance is found to be 33 cm. What is the velocity of sound in air ? (Ans. 338 ms^{-1})

9. The frequency of a note emitted by an organ pipe is 288 Hz at 15°C. At what temperature will the frequency be 300 Hz supposing the pipe to remain unchanged in length. (Ans. 39·5°C)

10. The pitch of the fundamental note of an open organ pipe 100 cm long is the same as that of a sonometer wire 2 m long having linear density ·1 kgm^{-1}. Find the tension of the wire. Velocity of sound = 330 ms^{-1}. (Ans. $4{\cdot}356 \times 10^4$N)

11. A closed pipe is filled with a gas whose density is 1·26 kgm^{-3}. If the length of the pipe is ·5 m, find the frequency of the note emitted. (The velocity of sound in air = 332 ms^{-1} and density of air = 1·293 kgm^{-3} and assume elasticity of of gases to be the same.) (Ans. 168 Hz)

12. A string 25 cm long and having a mass of 2·5 gm is under tension. A pipe closed at one end is 40 cm long. When the string is set vibrating in its first overtone and the air in the pipe in its fundamental frequency, 8 beats per second are heard. It is observed that decreasing the tension in the string decreases the beat frequency. If the speed of sound in air is 320 ms^{-1}, find the tension in the string. (I. I. T. 1982) (Ans. 2·76 kg)

(E)

1. When reflection takes place from a rigid wall, compression is reflected as .... (compression, rarefaction).

2. When reflection of sound wave takes place from an yielding surface compression is reflected as......... (compression, rarefaction).

3. The frequencies of harmonics of a closed organ pipe are in odd multiples of the fundamental. (True or false)

4. A flame will flicker maximum inside a pipe where air particles in the pipe are motionless. (True or false)

5. What is frequency of the fundamental mode of vibration of a closed organ pipe of length $\frac{l}{3}$, that of first harmonic of length l ?

(Ans. : 1. Compression. 2. rarefaction. 3. true. 4. true. 5. $\frac{3C}{4l}$, $\frac{3C}{4l}$.)

CHAPTER 6

CHARACTERISTICS OF MUSICAL SOUND: DETERMINATION OF PITCH: MUSICAL INTERVAL AND MUSICAL SCALE

6.1. Musical Sound and Noise

There is no clearcut demarcation between musical sound and noise. Any sound which appears pleasant to the ear is generally termed as musical sound and sound which is unpleasant to the ear is termed as noise. This way of distinguishing between noise and musical sound is more artificial than real. This is so because a sound which is pleasant to the ear of some person may not be pleasant on the ear of some other person. Moreover, the same sound may appear pleasant and unpleasant to the same person under different circumstances. However, we may broadly distinguish between noise and musical sound in this way. Musical sound is produced by regular periodic vibrations of material bodies, whereas noise is produced by irregular vibrations of material bodies. A musical sound is pleasant to the human ear and noise is unpleasant to the human ear.

6.2. Characteristics of Musical Sound

Musical notes differ from each other in respect of at least one of three properties, namely, Intensity, Pitch and Quality. These three are called *characteristics* of a musical sound. A musical sound is bound to differ from another musical sound in at least one of these three and hence they provide a means to distinguish one musical note from another.

6.3. Intensity and Loudness : Bel : Phon

Intensity. The intensity of a sound (musical or unmusical) is defined as the rate of flow of energy per unit area of a plane perpendicular to the direction of wave propagation. The intensity of a simple harmonic wave is given by

$I = 2\pi^2\rho a^2 \nu^2 c$ watt per square metre (Wm^{-2})

(For deduction of this formula see Art.-5, Chapter 1) and so the intensity of a musical sound wave is proportional to :

(i) the square of the amplitude of the wave,

(ii) the density of the medium,

(iii) the square of the frequency of the wave,

(iv) elasticity of the medium.

Apart from these factors it also depends on the distance from the source and the size of the source. The intensity of a wave varies inversely as the square of distance.

To establish this fact consider a sound-source at O sending out Q joules of sound energy per second equally in all directions. Suppose P_1 and P_2 are two points in the medium at distances r_1 and r_2 from O. Draw spheres of radii r_1 and r_2 about O. Obviously a flow of energy will take place normally to these surfaces. Since the intensity

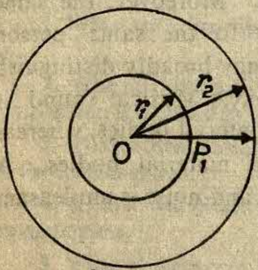


Fig. 6.1 (a)

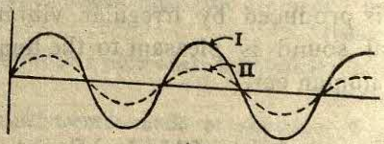


Fig. 6.1 (b)

of the wave is the rate of flow of energy per unit area,

$\therefore$ I_1 (intensity at P_1) $= \dfrac{Q}{4\pi r_1^2}$ and I_2 (intensity at P_2) $= \dfrac{Q}{4\pi r_2^2}$.

Dividing, $\dfrac{I_1}{I_2} = \dfrac{r_2^2}{r_1^2}$

or $I \propto 1/r^2$

That is, the intensity $\propto \dfrac{1}{\text{distance}^2}$. .. (6.1)

This is true when there is no absorbtion of energy in the medium.

In fact there is lot of absorption of energy by air. This is why, this law does not apply exactly.

The intensity of a sound wave depends on the size of the source. A heavy body will emit a sound wave of larger intensity than light bodies. It also depends on the presence of other vibrating bodies. A tuning fork mounted on a sound-box emits sound of greater intensity than the one without a sound-box. Graphically the intensity of a wave is shown by the amplitudes of the wave. In Fig. 6.1(b) wave I represents a wave of larger intensity than wave II.

Though the absolute unit of intensity is watt per square metre (Wm^{-2}), an alternative arbitrary unit on the logarithmic scale is used, because in actual practice the absolute intensity of a sound wave is not of much importance. It is the relative intensity which is of much practical value. A logarithmic scale is chosen because

(i) there is a very large range of intensity to be covered right from zero up to 10^9 watt per square metre;

(ii) the response of the ear follows approximately a logarithmic relation. According to Weber, loudness is proportional to the log of the intensity. This is known as Weber's law.

Bel. The arbitray unit selected for expressing intensity on the logarithmic scale is a 'bel', after the name of the inventor of the telephone Graham Bel. If I_1 is the absolute intensity in Wm^{-2} of a sound wave and I_2 is that of the other wave, then it is said that the intensity level of the first relative to the second is $\log_{10}\frac{I_1}{I_2}$ bels. The 'bel' proves to be rather too large a unit, therefore one-tenth of a bel, that is, a decibel is used. Thus the above relative intensity of the first relative to the second is also $10 \log_{10} \frac{I_1}{I_2}$ decibels (db). The standard intensity level selected for expressing relative intensity is 10^{-12} watt per square metre (Wm^{-2}). On the background of this choice a sound wave of absolute intensity 10^{-11} Wm^{-2} will be said to have an intensity level of 10 db relative to the standard level, a wave of absolute intensity 10^{-6} Wm^{-2} will have an intensity level of 60 db relative to the standard level and so on. The 'absolute intensity level' of a wave of intensity I Wm^{-2} is defined as log 1000 I bels. Thus

$$\text{the relative intensity level} = 12 + \log I \text{ bels} \quad \cdots \quad (6.2)$$

and the absolute intensity level $= 3 + \log I$ bels. .. (6.2a)

Loudness and Phon. The intensity of a sound wave is an absolute physical quantity and it stands for the rate of flow of energy across unit area. Loudness, on the other hand, is the judgement of the sensation produced on the ear of the observer. Hence loudness is something that is not absolute. It may vary from person to person. However, it is true that, for an individual, loudness is dependent on the intensity. Thus intensity is the physical cause of loudness. Although loudness is related to intensity, there is, unfortunately, no simple relation between them over the entire range of audible frequencies (20 Hz to 20,000 Hz). Weber's logarithmic relation between intensity and loudness holds good only over a limited range of intermediate audible frequencies. Loudness depends strongly on frequencies. At infra-frequencies (less than 20 Hz) we cannot hear anything though the wave has a definite intensity, and so the loudness of waves at infra-frequencies is zero. At ultra-frequencies (more than 20,000 Hz) also, we cannot hear any thing and so we would judge their loudness to be zero. In the audible range of frequencies loudness is measured by the intensity level of a standard source relative to a stated reference level and of intermediate frequency so that Weber's law may be applicable to it. This standard is a wave of frequency of 1000 Hz and initial energy current (absolute intensity) 10^{-12} Wm^{-2} The reason for selecting this particular frequency is that at this frequency the sensitivity of the human ear is maximum. To find the equivalent loudness of a sound wave of any frequency, the intensity level of the standard wave is changed by adjusting its amplitude of vibration till it appears to be equally loud as the sound wave under test to a normal observer. The intensity level to which the standard wave is needed to be raised from its initial stated energy level is called its equivalent loudness. The unit of equivalent loudness or simply loudness is the Phon. If the intensity level of standard sound wave has to be raised by n decibels from its stated initial intensity level (10^{-12} Wm^{-2}), then its loudness is said to be n phons, phon simply taking the place of decibel.

6.4. Pitch

The pitch of a musical note is that physical cause which distinguishes a shrill note from a bass (grave) note of the same intensity and coming from the same instrument. In short, the degree of shrillness of a musical note is its pitch The note of the first key of a harmonium is a musical sound of low pitch and that of the last key is a note of high pitch. The. pitch of a note emitted by a source depends upon its frequency of vibration. The greater the frequency of vibration of a source, the higher the pitch of the note emitted by it. This is why the pitch of a note is expressed by its frequency. The pitch of a note also depends on the relative motion between the observer and the source (Doppler's effect). Graphically pitch is shown by the wavelength of a wave. Fig. 6.2a represents a wave of low pitch and Fig. 6.2b represents a wave of very high pitch.

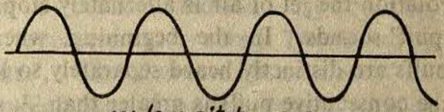

(a) low pitch wave

Fig. 6.2 a

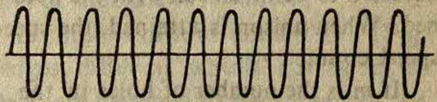

(b) high pitch wave

Fig. 6.2 b

6.5. Determination of Pitch

There are several methods available for determination of pitch (frequency) of a note. Some of these are comparison methods and some are absolute methods. Among the comparison methods we have: (i) The siren method, (ii) The Savart's toothed wheel method, (iii) The sonometer and among the absolute methods we have (i) The stroboscopic method, (ii) The falling plate method, and (iii) The digital electronic counter method.

COMPARISON METHODS

(i) The Siren. There are two types of Siren : one is the disc siren due to Seebeck and the other is Cogniard de la Tour's siren. Seebeck's disc siren (Fig. 6.3 a) consists of a metal disc having equally spaced concentric holes drilled along several concentric circles, different circles having different number of holes. The disc is clamped to the armature of an electric motor and rotated at a uniform speed, which is

controlled by a resistance in series with the motor. A fine nozzle is held in front of a suitable circular row of holes and air from a blower is passed through it under constant pressure. When the disc is set in rotation the jet of air is alternately stopped and allowed to pass with 'puff' sounds. In the beginning, when the speed of rotation is low puffs are distinctly heard separately so long as the interval between the consecutive puffs is greater than $\frac{1}{10}$ second. When the speed of rotation is increased and puffs occur at a rate more than 10 per second, due to persistence of hearing, a continuous sound is heard. By adjusting the speed of the disc, the note constituted by puffs is adjusted to be in unison with the note whose pitch is to be determined. When unison is attained, the speed of rotation is measured by a speed counter.

If m is the number of holes in the row and n is the number of revolutions per second at the time of unison, the pitch of the note constituted by puffs = number of puffs per second.

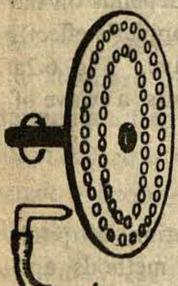


Fig. 6.3 a

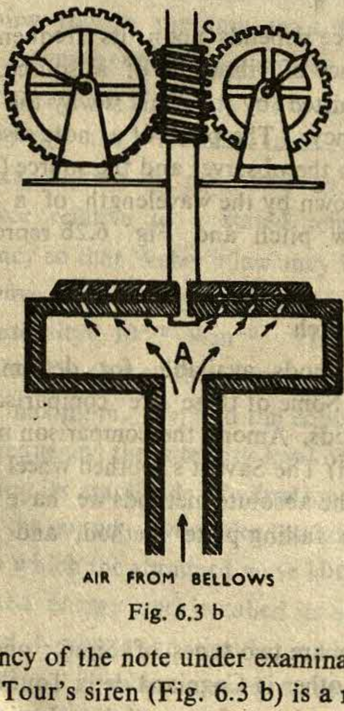


Fig. 6.3 b

Fig. 6.3 c

∴ The frequency of the note under examination $= m \times n$.

Cagniard de la Tour's siren (Fig. 6.3 b) is a more convenient form of siren. In this siren instead of a single jet there are a number of jets arranged in rings. It consists of a vertical spindle attached to a circular disc which is free to rotate about the spindle over another

fixed disc. Each disc is pierced with identical holes with obliquities turned oppositely. Below the fixed disc, there is a cylindrical wind chest, in which air is blown under pressure. The air issuing through the holes in the upper disc sets it in rotational motion. By adjusting the pressure of air, the disc can be rotated at any speed. The thread on the spindle drives a 'speed counter'. The alternate obstruction of the air produces a note, the pitch of which depends on the speed of rotation and the number of holes. By adjusting the speed of rotation, the note emitted by the siren is brought in unison with the note under examination. If m is the number of holes and n is the number of revolutions per second, the frequency (N) of the note is $N = m \times n$.

(*ii*) *Savart's Toothed Wheel.* It consists of four brass wheels (Fig. 6.3 c) having teeth cut on their edges, different wheels having different number of teeth. The wheels are clamped to the spindle of an electric motor. A hard card or a metal plate is held lightly against the teeth. When the wheel is set in rotation the plate strikes against the teeth producing a series of tap sound. When the speed of rotation is sufficiently large the taps constitute a continuous sound. By adjusting the speed of rotation, the pitch of the note constituted by the taps is brought to unison with the note under examination. The speed of rotation of the wheel at this stage is noted by a speed counter. If m is the number of teeth and n is the number of revolutions per second of the wheel at unison, then the frequency (N) of the note is

$$N = m \times n.$$

(*iii*) *Sonometer.* The experimental wire of a sonometer (for description and figure of a sonometer see Chapter 4) is put under tension by putting a known weight on the hanger. It is then excited to emit a note by plucking gently at the centre and by adjusting its length it is brought in unison with the note under test. The wire is, then, taken off from the sonometer and weighed on a balance. The mass divided by the length of the wire gives the linear density of the wire. If W is the weight on the hanger in kg and w is the weight of hanger in kg then the tension (T) of the string is $(W+w)\,g$ newton

and if M is the mass of the wire then $m = \dfrac{M}{L}$ kgm^{-1}.

The frequency of the resonant length of the wire is

$$n = \frac{1}{2l}\sqrt{\frac{T}{m}}.$$

Here l is the length of the wire vibrating in unison with the note under test. Since the note of the wire is in unison with the note under

test, hence whatever is the frequency of the wire, that is also the frequency of the note

$$\therefore\quad N = n = \frac{1}{2l}\sqrt{\frac{T}{m}}\ \text{Hz.}$$

ABSOLUTE METHODS

(*i*) *Stroboscopic Method.* If a body is in periodic motion and it is illuminated intermittently or viewed through an aperture alternately closing and opening, then the body will appear motionless if the period of intermittence or the period of opening and closing coincides with the period of the body. This is the principle of stroboscope method. In this device, a rapidly vibrating body is first made to appear as a slowly vibrating body and finally made to appear motionless. If the frequency of the vibrating shutter or frequency of intermittence is known, then that is the frequency of the periodically vibrating body.

A stroboscope suitable for measuring the frequency of a tuning fork consists of a mettalic disc D having a number of equally spaced dots arranged in a ring concentric with the disc. The disc is mounted on to the armature of an electric motor by which it is rotated in the experiment at a controllable speed. The prongs of the tuning fork of which the frequency is to be determined are provided with light aluminium plates, one facing the other without touching each other. Each plate has a slit cut in it, such that when the tuning fork is not vibrating, the two face each other and form an aperture. The disc and the tuning fork are arranged in such a way that the dots are visible to the eye peeping through the aperture. When the fork is set in vibration, the aperture will open and close alternately. In one oscillation the aperture will open fully twice and remain completely closed for the rest of the period. The disc is also then set in rotation and its speed is gradually increased from a low value until the dots appear stationary. This will happen when the aperture opens exactly at the right moment, when a dot comes in the line of sight. The speed of the disc is noted, when the dots appear stationary for the first time. At double this speed they will again appear

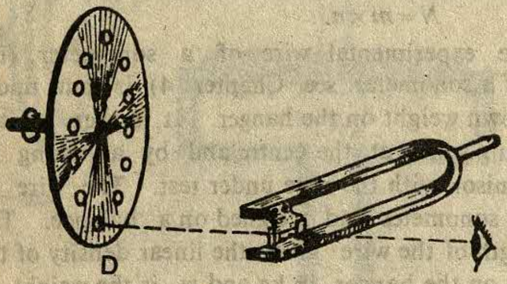


Fig. 6.4

motionless, also at three times, four times and so on. Let n be the number of revolutions per second when the dots appear motionless for the first time and there are m dots, then the interval between arrivals of two consecutive dots on the line of sight is $\frac{1}{mn}$ second. This must be equal to half the period of the fork, because the aperture opens twice in one oscillation. Therefore if T is the time period of the fork, then

$$\frac{T}{2}=\frac{1}{mn} \quad \text{or} \quad T=\frac{2}{mn}$$

or $$N=1/T=\tfrac{1}{2}\times mn$$

or $$N=\tfrac{1}{2}\,mn \text{ Hz.}$$

(ii) The Falling Plate Method (a graphical method). In this method a smoked glass plate is arranged to fall freely under gravity and the tuning fork of which the frequency is to be determined is so arranged that the light style, attached to one of its prongs, just touches the surface of the plate near its lower edge. The tuning fork is set to oscillations and the plate is allowed to fall freely under gravity. The style of the fork traces a wavy curve of gradually increasing wavelengths. Two lengths AB and BC containing an equal number of waves are selected on the plate (Fig. 6.6).

Let u be the velocity of the plate at A and the time of fall through AB or BC be t. The time of fall through AB or BC is the same because they contain an equal number of waves.

If $AB=l$ and $BC=l'$ then v, velocity at $B=u+gt$ and $l=ut+\frac{1}{2}gt^2$;

$$l'=(u+gt)\,t+\tfrac{1}{2}\,gt^2=ut+3/2gt^2.$$

$$\therefore \quad l'-l=ut+3/2gt^2-ut-1/2gt^2=gt^2$$

or $$t=\sqrt{\frac{l'-l}{g}}.$$

If there are m waves in AB or BC then N (frequency of the fork) $=\frac{m}{t}=m\sqrt{\frac{g}{l'-l}}$.

$$N=m\times\sqrt{\frac{g}{l'-l}} \text{ hertz (Hz).}$$

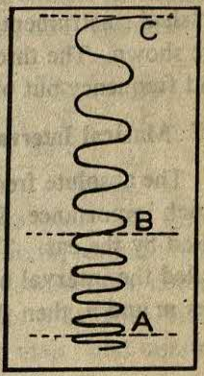


Fig. 6.6

(iii) The Digital Electronic Counter Method. The incident sound wave on a receiver is amplified linearly and made to pass through electronic gates which count the waves one by one. There is an automatic timing device which stops the gate

counting the waves exactly after one second.

6.6. Quality or Timbre

Quality or timbre is a third feature of a musical note which distinguishes between two notes of the same intensity and pitch but produced on two different musical instruments. The quality or timbre of a note is due to the presence of different harmonics. The presence of harmonics affects the form of the wave emitted by a musical instrument. For example, in closed organ pipes the full series of harmonics is present. In a stretched string, the number of harmonics present depends on the way of exciting it, whether it is excited by plucking, bowing, hammering etc. and on the point at which it is excited. If it is excited at the mid-point all the harmonics needing this point as a node will be absent, that is, second, fourth, sixth,...... harmonics will be absent. The harmonics present do not affect the frequency of the fundamental tone, but they simply reshape the form of the fundamental wave. On account of the different shaping of the wave form by the harmonics, the quality of different notes becomes different. In the Fig. 6.7 the wave forms of three different musical instruments—a tuning fork, a violin, and a clarionet—are shown. The three curves represent notes of the same intensity and frequency but of different quality.

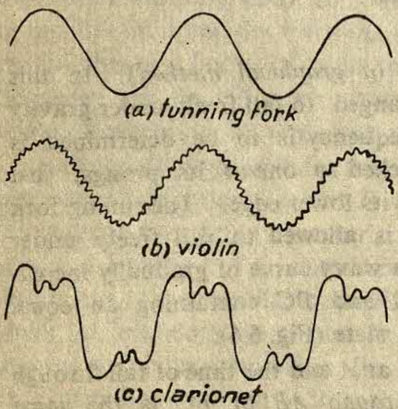


Fig. 6.7

6.7. Musical Interval : Consonance and Dissonance

The absolute frequencies of notes in the world of music are not of much importance. It is the ratio of the frequencies which is recognised by the ear. The ratio between the frequencies of two notes is called the interval between the two. If the two notes have frequencies m and n, then the musical interval between them is $m : n$.

$$\text{Musical interval} = \frac{m}{n}.$$

As it is simply the ratio between two frequencies, it has no unit. However, when it is expressed on logarithmic scale, it is assigned an

arbitrary unit 'octave', one octave being log 2. Thus

$$\text{Musical Interval} = \frac{\log \frac{m}{n}}{\log 2} \text{ octave} = 100 \times \frac{\log \frac{m}{n}}{\log 2} \text{ centioctave} \quad .. \quad (6.3).$$

In passing from one musical note to another, the ear recognises spontaneously the ratio (interval here) in which their frequencies alter. In music, whether vocal or instrumental, there is a change of frequencies and we enjoy in our ear the musical intervals in the music and not the actual frequencies. Now, all intervals do not produce harmoniusness or a pleasant sensation on the human ear. Only intervals in a simple ratio produce a pleasant sensation called 'consonance' and those which produce an unpleasant sensation called 'dissonance' are in the ratio of the higher numbers having no common factor such as 64 : 81. It is needless to say that the unison 1 : 1 is the most consonant interval, next comes 1 : 2 called the octave. The following is the list of the intervals with names arranged in decreasing order of harmoniusness :

unison	.. 1 : 1	major third	.. 5 : 4
octave	.. 2 : 1	minor third	.. 6 : 5
fifth	.. 3 : 2	major sixth	.. 5 : 3
fourth	.. 4 : 3	minor sixth	.. 8 : 5

Diatonic Scale. The earliest musical scale is the diatonic scale, which consists of eight notes, beginning with a note of lower frequency called the tonic or key note and ending in its higher octave. In between, there are six other notes having the best possible consonant intervals with the keynote. In European music they are named as *Do, Re, Mi, Fa, Sol, La, Si, Do* and in Indian music they are named *Sa, Re, Ga, Ma, Pa, Dha, Ni, Saa.* Helmholtz denoted them by *C, D, E, F, G, A, B, c.*

Do	*Re*	*Mi*	*Fa*	*Sol*	*La*	*Si*	*Do*
C	*D*	*E*	*F*	*G*	*A*	*B*	*c*
1	9/8	5/4	4/3	3/2	5/3	15/8	2

$$\underbrace{1 \quad 9/8}_{9/8}\ \underbrace{\quad 5/4}_{10/9}\ \underbrace{\quad 4/3}_{16/15}\ \underbrace{\quad 3/2}_{9/8}\ \underbrace{\quad 5/3}_{10/9}\ \underbrace{\quad 15/8}_{9/8}\ \underbrace{\quad 2}_{16/15}$$

Commencing from the keynote the notes in the scale are also termed as the first, second, third, fourth, fifth, sixth, seventh and eighth (octave) respectively. The interval between consecutive notes is not the same. There are three distinct intervals 9/8, 10/9 and 16/15. The intervals 9/8 and 10/9 respectively called major and minor tone and the interval 16/15 is called semitone or limma.

Tempered Scale. The serious disadvantage of the diatonic scale is that it lacks 'modulation'. Modern music needs frequent change

of the keynote. The capability of allowing a change of the keynote is called *modulation of the scale*. Let us see whether a music composed on the diatonic scale with C (having frequency 256) as the keynote can be composed with E as the keynote. When C has a frequency 256, the frequencies of the other notes of the scale will be as follows :

C	D	E	F	G	A	B	c	d	e ....
256	288	320	341·3	384	426·7	480	512	576	640

With E as the keynote the music requires the following frequencies :

E	F	G	A	B	c	d	e ....
320	360	400	426·7	480	533·3	600	640....

It is found that the music composed with C as the keynote cannot be composed with E as the keynote because it requires a few more notes which are not present in the scale. In this way the diatonic scale restricts the selection of the keynote. This is called *'lack of modulation'*.

To avoid this difficulty, a new scale was devised. The fixing of a suitable number of notes between a keynote and its octave must arise out of compromise between the power of modulation of the scale and the number of notes. The number of notes in an instrument cannot be too many, because that will bring a difficulty in playing it. A scale arising out of such a compromise is called a tempered scale. Among the tempered scale, the most common scale is the equitempered scale, in which the interval between a tonic and its octave is divided into twelve equal intervals. If x be the interval in the equitempered scale, then

$$x.\ x.\ x.\,.\,.\,.\,.\,.\,.\,.\,(12\text{ factors}) = 2$$

$$\therefore \qquad x^{12} = 2 \quad \text{or} \quad x = 2^{1/12} = 1{\cdot}059.$$

One additional black key is placed in a piano or harmonium dividing the five tones in the diatonic scale given by the white keys. All modern instruments (provided with keys such as harmonium, piano etc.) make use of this equitempered scale. In such instruments a vocal musician can start with any note as keynote to suit the up and down of his or her voice.

Examples

1. *A siren having 200 holes is making 132 revolutions per minute. The frequency of the note produced by the siren is found to be an octave higher than that of the given tuning fork. Determine the frequency of the tuning fork.* **(Pat 1973 A)**

Sol. The frequency of the siren $= m \times n = 200 \times \frac{132}{60} = 440.$

Since it is an octave higher than that of the tuning fork,

$\therefore$ the frequency of the fork $= 440 \div 2 = 220$ Hz. **Ans.**

2. *A light pointer fixed to one prong of a tuning fork touches a vertical plate. The fork is set vibrating and the plate is allowed to fall freely. Eight complete oscillations are counted when the plate falls through 10 cm. What is the frequency of the tuning fork ?*

(I. I. T. 1977)

Sol. The time of fall through 10 cm is equal to the time in which the tuning fork makes 8 complete oscillations. Let it be t.

Then $\cdot 1 = \frac{1}{2} \times 9{\cdot}8\, t^2$ or $t = \sqrt{\frac{\cdot 2}{9{\cdot}8}} = \sqrt{\frac{1}{49}} = 1/7.$

$\therefore$ The frequency of the fork $= \frac{8}{1/7} = 56$ Hz. **Ans.**

3. *A line source emits a cylindrical expanding wave. Assuming that the medium absorbs no energy, find how the amplitude and the intensity of the wave depend on the distance from the source.*

(I. I. T. 1981)

Sol. Let the source emit Q units of energy every second. Consider a cylindrical wave-front of radius r. The area of the wave-front is $2\pi rl$ where l is the length of the source.

$\therefore$ I (intensity of the wave at a distance r) $= \frac{Q}{2\pi rl}$.

$\therefore$ $I \propto \frac{1}{r}$,

Since Intensity $\propto$ amplitude^2,

$\therefore$ amplitude $\propto \frac{1}{\sqrt{r}}$.

Thus the intensity is proportional to r^{-1} and the amplitude is proportional to $r^{-\frac{1}{2}}$. **Ans.**

4. *Two notes are of frequencies 386 and 480 Hz. What is the absolute musical interval between the two ? In centioctaves ?*

Sol. The absolute interval $= \frac{384}{480} = \cdot 8.$

The interval on the logarithmic scale $= \log_{10} \cdot 8$.

The interval 2 : 1 is called an octave. On the logarithmic scale $\log_{10} 2$ is called an octave interval. One hundredth of an octave is called a centioctave.

$$\therefore \quad \text{The interval in centioctaves} = \frac{100 \log \cdot 8}{\log 2} = -32 \cdot 2. \textbf{ Ans.}$$

5. *A note of frequency 300 Hz has an intensity of one microwatt per square metre. What is its absolute intensity in decibels ? What is its relative intensity in decibels ?*

Sol. If I be the absolute intensity (in Wm^{-2}), then log 1000 I is called the absolute intensity level in bels.

$$\therefore \quad \text{The absolute intensity level} = \log 1000 \times 10^{-6} \text{ bel}$$
$$= \log_{10} 10^{-3} = -3 = -30 \text{ decibel. } \textbf{Ans.}$$

The relative intensity level relative to the standard (10^{-12} Wm^{-2})

$$= \log_{10} \frac{10^{-6}}{10^{-12}} = \log 10^{6} = 6 \text{ bel} = 60 \text{ db.}$$

QUESTIONS

(A)

1. A wave of 1000 Hz is considered as a standard for comparing the intensity level of other waves (a) simply because 1000 is a round figure, (b) because the frequency sensitivity of the human ear is maximum at this frequency, (c) because the frequency response of the human ear is minimum at this frequency, (d) because frequency response of the human ear is zero at this frequency.

2. If m and n are the frequencies of two musical notes then the musical interval between them is (a) $m \sim n$, (b) $m \times n$, (c) m/n, (d) $m+n$.

3. The interval 5 : 4 is called (a) octave, (b) unison, (c) major third,, (d) minor third.

4. The interval 3 : 2 is called (a) octave, (b) fifth, (c) fourth, (d) minor third.

5. The loudness of a sound wave is (a) proportional to the intensity of the wave, (b) proportional to the square root of the intensity, (c) proportional to the square of the intensity, (d) proportional to the log of the intensity.

6. Phon is the unit of the (a) intensity, (b) pitch, (c) quality, (d) equivalent loudness.

7. The relative intensity level of a sound wave of absolute intensity I watt/metre^2 is (a) $\log I$, (b) $3+\log I$, (c) $12+\log I$, (d) $6+\log I$.

(**Ans. 1.** b, **2.** c, **3.** c, **4.** b, **5.** d, **6. d**, **7.** c.)

(B)

1. Distinguish between musical sound and noise. (**Mag. 1974; Bih. '75**)
2. Explain 'consonance' and 'dissonance'.

3. What is the musical interval between two notes ? Mention some consonant musical intervals.

(C)

1. What are the characteristics of a musical note ? Explain them.

2. What is a stroboscope ? How is it used to determine the frequency of a tuning fork ?

3. Describe the stroboscopic method for the absolute determination of the frequency of a tuning fork.

4. What is a musical scale ? Write notes on the 'diatonic scale' and 'the tempered scale'.

(D)

1. The disc of a siren has 48 holes. The note produced by the siren is in unison with the note produced by a tuning fork of frequency 480 Hz. How many rotations per minute are being made by the disc of the siren ?

(Ans 600)

2. A column of air at 51°C and a tuning fork produce 4 beats per second when sounded together. As the temperature of the air column is decreased, the number of beats per second tends to decrease and when the temperature is 16°C the two produce one beat per second. Find the frequency of the tuning fork.

(I. I. T. 1977) (Ans. 50 Hz)

3. Calculate the intensity of the wave $y = \cdot 001 \sin (400t - 1\cdot 25x)$. Density of the medium = 1·29 kgm^{-3}. (Ans. 33 Wm^{-2}; 13·52 bel relative to the standard)

(E)

1. The shrill note from siren can be heard from a distance. Why ?

2. When a fan is switched on in a room well illuminated by a florescent tube the blades of the fan are found to have some change in motion. Is it an optical illusion or there is some truth in it ?

3. What experimental evidence is there for assuming that the speed of sound is the same for all wavelengths ?

4. The inverse square law does not apply exactly to the decrease of sounds with distance. Why not ?

5. Why some effort has to be made to make oneself heard by another in aeroplanes or balloons when flying high up from the surface of the earth ?

6. The musical scale used in harmonium or piano is ....(diatonic, equitempered).

(Ans. 1. Because of high frequency of the note. A note of high frequency has greater intensity and also greater penetrating power. 2. This is not an optical illusion. There is, in fact, strobosopic effect due to intermittent illumination of the blades by the fluorescent tube. 3. Because all the notes from a source, say a transistor set, reach us simultaneously. 4. Due to absorption of sound wave by air. 5. Because of low density of air at high altitudes. 6. equitempered.)

CHAPTER 7

DOPPLER'S PRINCIPLE (EFFECT)

7.1. Doppler's Effect

Whenever there is a relative motion between an observer and a source, the pitch of the note emitted by the source appears to be changed to the observer. This apparent change in pitch due to relative motion between observer and source and also due to motion of the medium is called *Doppler's effect in sound.* Actually Doppler first observed this effect with light waves. The spectral lines of certain stars were found to be shifted towards the red or the violet end of the spectrum from their normal position by a very small distance. Doppler explained this spectral shift on the assumption of the relative motion between the observer and the sounce along the line of sight. In fact, Doppler's effect is a basic characteristic of a wave and hence it is to be found in all types of waves and, therefore, in sound as well.

In sound Doppler's effect can be observed from a railway platform, when a whistling locomotive engine passes past the platform at a very high speed. Before passing the platform the pitch of the whistling appears higher and after passing past the platform, its pitch appears lower.

EXPLANATION OF DOPPLER'S EFFECT

(*a*) ***When the source is in motion, the observer stationary.*** Let us suppose that the source S is stationary for the time being and n is its frequency of vibration. This means the source emits n waves in every second and these n waves will be in a row forming a train of waves extending from the source S up to the point A (say) such that $SA = V$, velocity of the wave. In one second the wave train formed has its 'head' at A and 'tail' at S. By repetition of this wave train of

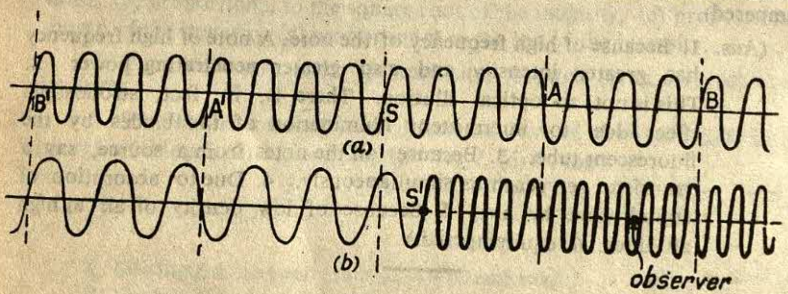


Fig. 7.1

one second both to the right and left, we get the whole wave train travelling to the right and to the left. Let us now suppose that the source is also moving to the right with the velocity V_s, a velocity less than the velocity of sound. Let S move to S' in one second so that $V_s = SS'$. Now the 'head' of those n waves will again be at A but its 'tail' will be at S'. Thus the waves are now accomodated in a shorter space in the direction of the source and hence the waves are crowded in $S'A$. By the repetition of the wave train in $S'A$ we get the whole wave train travelling to the right and by a repetition of the wave train between $S'A'$, we get the whole wave train travelling to the left. Obviously, the wavelength of the wave in the direction of the source is shortened and hence an observer stationed somewhere on the right hand side will receive waves of shorter wavelength and he will hear sound of higher pitch. The wavelength of the wave to the left is lengthened and hence an observer stationed somewhere to the left will hear a sound of lower pitch. Thus, due to the motion of the source the wavelength of the wave is actually changed and hence the pitch appears changed to the observer.

Calculation of the apparent pitch. To calculate the altered wavelength λ', we note that n waves emitted by the source in one second are contained in a tube of length $S'A$ in the direction of the source.

$$\therefore \quad \lambda' = \frac{S'A}{n} = \frac{SA - SS'}{n} = \frac{V - V_s}{n}. \qquad .. \quad (7.1)$$

If n' is the apparent pitch (frequency), then $n'\lambda' = V$.

$$\therefore \quad n' = \frac{V}{\lambda'} = \frac{V}{V - V_s} \times n. \qquad .. \quad (7.2)$$

In the direction opposite to that of the source,

$$n' = \frac{V}{V + V_s} \times n.$$

(*b*) *When the observer is in motion, the source stationary.* Let the source be stationary at S and the observer also stationary for the time being at O. As the source remains stationary the wavelength of the sound wave emitted by it is not altered. The observer receives waves of wavelength of real magnitude λ and these waves received by him in one second are contained in a tube of length $OA = V$ after

passing past the observer. He hears in every second only those waves

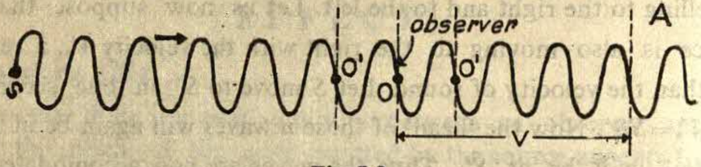


Fig. 7.2

which pass past him in that second. Let us now suppose that the observer is also moving to the right with velocity V_o, a velocity less than the velocity of sound. Let the observer move from O to O' in one second so that $OO' = V_o$. Now obviously the waves contained in $O'A$ will pass past him in one second and hence he will hear only the waves contained in $O'A$ and miss the waves contained in OO'. As the observer will receive a smaller number of waves in every second, the pitch of the note will appear to have 'dropped' by a certain amount. On the other hand if the observer moves towards the source from O to O' to the left, then he will receive additional waves contained in OO' to the left and so the pitch of the note will appear to have gone 'up'. Thus, due to the motion of the observer, the pitch appears to be changed, i.e., Doppler's effect takes place due to the fact that the observer either actually misses some of the waves when he is moving away from the source or gains waves when he is moving towards the source.

Calculation of the apparent pitch. To calculate the apparent pitch we note that the observer hears the waves contained in $O'A$ in one second. Hence the apparent pitch, $n' = \frac{O'A}{\lambda} = \frac{OA - OO'}{\lambda} = \frac{V - V_o}{\lambda}$.

We have, $V = n\lambda$ where n = real pitch and λ = real wavelength.

$$\therefore \qquad n' = \frac{V - V_o}{V} \times n. \qquad .. \quad (7.3)$$

When the observer moves towards the source,

$$n' = \frac{V + V_o}{V} \times n. \qquad .. \quad (7.3a)$$

(*c*) *Source and observer both in motion.* We have seen that due to the motion of the source the wavelength of the wave is changed from λ to λ' in the direction of the source and is given by

$$\lambda' = \frac{V - V_s}{n}.$$

When the observer is stationary, the observer will receive waves of wavelength λ' and these waves received by him in one second are contained in a tube of length $OA = V$ after passing past him. But he is also moving away from the source with a velocity V_o from O to O' in one second so that $OO' = V_o$, obviously he will hear only the waves contained in $O'A$ and miss the waves contained in OO'

$\therefore$ The apparent frequency, $n' = \dfrac{O'A}{\lambda'} = \dfrac{OA - OO'}{\lambda'}$

or
$$n' = \frac{V - V_o}{\dfrac{V - V_s}{n}}, \text{ or } n' = \frac{V - V_o}{V - V_s} \times n. \quad \quad (7.4)$$

(d) Source, Observer and Medium all are in motion. In all the above discussions we have assumed V to be the velocity of sound in still air. Let us suppose that the source, the observer and the medium all are in motion with velocity V_s, V_o and ω respectively. Then the velocity of sound in the direction of the wind is $V + \omega$ and opposite to the wind is $V - \omega$. Hence when the medium is also in motion in the direction from the source to the observer, in place of V we have to use $V + \omega$.

$\therefore$
$$n' = \frac{V + \omega - V_o}{V + \omega - V_s} \times n. \quad \quad (7.5)$$

7.2. Demonstration of Doppler's Effect in Sound

A laboratory method of demonstrating Doppler's effect in sound was devised by Mach. A long tube about 2 metre long is provided with a whistle at one end, and it is arranged to rotate it about a vertical axis passing through its centre of gravity. The whistle is blown by the wind forced along the axis of the tube. When the tube is in motion with high speed, an observer situated in the horizontal plane of rotation of the tube will clearly hear a 'rise' and a 'drop' in pitch. In the course of rotation, the whistle approaches the observer and then recedes away from the observer. When it approaches the pitch of the whistle appears higher and as it recedes, the pitch appears lower.

Fig. 7.3

7.3. Mach Number

When the velocity of the source exceeds the phase velocity of the wave, the Doppler's effect has no meaning. There are many instances in which the source moves through a medium at a speed greater than the phase velocity of the wave in that medium. A speed boat moves on the surface of water at speed greater than the velocity of water waves. A jet plane or a ballistic missiles may move through the air at a speed greater than the velocity of the sound. In such cases, the wave-front takes the shape of a cone with the moving object at its apex. For water waves, the cone reduces to a pair of intersecting lines with the source at the point of intersection. The semi-vertical angle of the conical wave-front is a constant for a

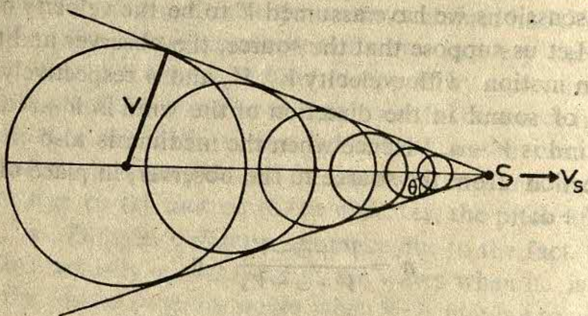


Fig. 7.4

given speed of the source. It is given by $\sin\theta = \frac{V}{V_s}$ where V = velocity of the wave and V_s = velocity of the source. In aerodynamics the ratio $\frac{V_s}{V}$ is called a Mach Number.

Examples :

1. *What is the apparent frequency of sound of frequency 600 Hz from the whistle of an engine approaching an observer at rest at the speed of 10 ms^{-1} ? The velocity of sound in air = 332 ms^{-1}.*

Sol. We have in general

$$n' = \frac{V - V_o}{V - V_s} \times n.$$

Here $V_o = 0$, $V_s = 10$ ms^{-1} and $V = 332$ ms^{-1}

$\therefore$
$$n' = \frac{332}{332 - 10} \times 600 = 618{\cdot}6 \text{ Hz. } \textbf{Ans.}$$

2. *Two aeroplanes are approaching each other and their velocities are 175 and 250 kph. The frequency of a note emitted by the first as heard by the passenger in the other is 1000 Hz. Calculate the true frequency of the note as heard by its own passengers. Velocity of sound* $=340\ ms^{-1}$.

Sol. We have in general $n'=\dfrac{V-V_o}{V-V_s}\times n.$

Here $V=340\ \text{ms}^{-1},\ n'=1000\ \text{Hz}$

$$V_s=\frac{175\times10^3}{3600}=48{\cdot}6\ \text{ms}^{-1}$$

$V_0=-\dfrac{250\times10^3}{3600}=-69{\cdot}6\ \text{ms}^{-1}$ (minus because the observer is moving opposite to the wave propagation).

$$\therefore \qquad 1000=\frac{340-(-69{\cdot}6)}{340-48{\cdot}6}\times n$$

or $$n=1000\times\frac{291{\cdot}4}{409{\cdot}6}=711{\cdot}4\ \text{Hz. } \textbf{Ans.}$$

3. *A car is moving along a straight road at 100 kph sounding its horn of frequency 200 Hz. Another car approaches the first along a straight side road at right angles to it at 60 kph. Determine the frequency of the note heard by a passenger in the second car, when the line joining the two cars is at* 30° *with the first road. Velocity of sound* $=350\ ms^{-1}$.

Sol. The component of the velocity of the first car (source) along the line joining the two is $\dfrac{100\times10^3}{3600}\cos30°=24\ \text{ms}^{-1}$ and, that of the second car along the same line is $\dfrac{60\times10^3}{3600}\cos 60°=8{\cdot}3\ \text{ms}^{-1}$

Now the problem becomes the same as Ex. 2.

$$\therefore \qquad n'=\frac{350-(-8{\cdot}3)}{350-24}\times200=219{\cdot}8\ \text{Hz. } \textbf{Ans.}$$

QUESTIONS

(A)

1. A source emitting sound waves moves away from the observer with a velocity equal to that of sound, the frequency of the note will be (a) halved, (b) doubled, (c) unchanged, (d) squared.

2. If V_s be the velocity of source, V_0 velocity of the observer in the direction of the velocity of sound in a stationary medium and n is the actual frequency of the source, then the apparent frequency is given by

(a) $\frac{V-V_0}{V-V_s}\times n$, (b) $\frac{V-V_s}{V-V_0}\times n$, (c) $\frac{V_0}{V-V_s}\times n$, (d) $\left(\frac{V-V_0}{V_s}\right)\times n$.

3. To an observer moving towards a stationary source, the note emitted by the source appears to be an octave higher. If V is the velocity of sound wave. the speed of the observer is (a) V, (b) $V/2$, (c) $2V$, (d) $3/2V$.

4. When a source moves with a speed greater than the phase velocity of wave in the medium, the wave-front of the wave is (a) spherical, (b) cylindrical, (c) conical, (d) plane.

5. If V is the supersonic speed of a jet plane and V_s is the velocity of sound wave, the Mach Number of the plane is (a) $\frac{V_s}{V}$, (b) $\frac{V}{V_s}$, (c $V-V_s$, (d) V_s+V.

6. Suppose you play a mischief and tie a can to the tail of a dog and make the dog run fast. Will the dog hear a note of the (a) same, (b) double, (c) half, (d) zero pitch as that of the can ?

(**Ans. 1.** a, **2.** a, **3.** a, **4.** c, **5.** b, **6.** a)

(B+C)

1. a) Explain the Doppler's principle.

(b) Derive an expression for the change in apparent frequency of a note due to the relative motions of the source, the observer, and the medium.

2. Mention some of the applications of Doppler's effect.

(**Hint** : to determine the speed of aeroplanes, to determine the speed of stars and planets along the line of sight.)

(D)

1. A train is whistling while approaching a tunnel at a speed of 36 kph. The driver hears the echo of the whistle reflected from the tunnel and estimates it to be 850 Hz. Find the actual frequency of the whistle. The speed of sound in air is 330 ms^{-1}.

(Ans. 800 Hz)

2. A spectrum line of wavelength 4×10^{-7}m in the spectrum of light from a star is found to be displaced from its normal position towards the red end of the spectrum by an amount 10^{-10} m. What velocity of the star in the line of sight would account for this ?

(Ans $7{\cdot}5\times10^{4}ms^{-1}$.; receding)

3. Two whistles are sounding with frequencies of 548 and 552 Hz. A man in the direct line between them walks towards the lower pitched whistle at 1·5 $m_s{-}i$

If the velocity of sound is 330 ms^{-1}, what is the beat frequency that he hears ?

(Ans. 1 Hz)

4. A train approaches a stationary observer at a speed of 75 kph sounding a whistle of 1000 Hz. What will be the apparent frequency of the whistle to the observer ? Velocity of sound = 332 ms^{-1}. (Ans. 1067 Hz)

(E)

1. An astronaut sends radio signals to the ground control from a space ship moving away from the earth. Will the ground control receive signals of............. (shorter, longer) wavelength ?

2. The speed of a supersonic jet plane is 2 Mach. This means that the speed of the plane is...... (double, half) of the speed of sound.

3. The Doppler's effect is greater when the source approaches the observer than when the observer approaches the source with the same speed. True or false ?

4. If in the example 1, the ground control sends a message at 11 m band to the astronaut, will his receiver be tuned at ....(at the same, less or greater) meter band ?

(Ans. 1. longer, 2. double, 3. true, 4. greater.)

CHAPTER 8

ULTRASONICS : SONAR

8.1. Ultrasonics

Ultrasonics are longitudinal waves beyond the upper limit of human audibility. The approximate upper limit of frequencies to which the human ear can respond is about 20,000 Hz. Hence we may say that ultrasonics are longitudinal waves of frequencies greater than 20,000 Hz. The corresponding wavelength at standard temperature and pressure is 1·66 cm. Man cannot hear ultrasonics, but many inferior animals can hear them. Birds, bats, (a mammal) can not only hear ultrasonics, but can also produce them. In the dark, they can fly freely without dashing against any obstacle because during flight they constantly send forward ultrasonic signal and if any obstacle is there, they hear the echoes of the ultrasonic waves and at once they can change their course of flight.

8.2. Production of Ultrasonics

Ultrasonics are produced by two methods—Magnetostriction method and Piezoelectric method.

(*a*) *Magnetostriction generator*. The principle of this method is that when a rod of ferromagnetic material, specially nickel, is placed parallel to a magnetic field, it suffers an increase in its length depending on the strength of the field. So if an alternating magnetic field is applied to a nickel rod, it will vibrate with double the frequency of the oscillating field. If, however, the rod is magnetised first by a steady current and then an alternating current is superimposed on it, the rod will oscillate with the frequency of the alternating current. But this will not generate ultrasonics, because its vibrations, being forced, amplitudes are very small. By changing the frequency of the alternating current when it is matched with the natural frequency of the rod, the latter will vibrate with the maximum amplitude giving rise to a very strong beam of ultrasonics. The natural frequency of a rod of length l and clamped at the centre of the rod is given by,

$n=\frac{1}{2l}\sqrt{\frac{E}{\rho}}$; where l is the length of the

rod, E = axial Young's modulus of the material of the rod and ρ is the density of the material of the rod. The high frequency alternating current needed to generate ultrasonics is obtained from an oscillator. Just as a simple pendulum is a mechanical oscillator which would oscillate, once set to oscillations, for an indefinite period if there is no air resistance, in electricity a circuit consisting of a capacitor and an inductive coil (*L-C* circuit) is an electrical oscillator. The discharge of a capacitor through an inductor coil sets up an alternating current of undying amplitude in the circuit, if there is no resistor. But due to the resistance of the coil, the oscillations gradually decrease. A triode valve, by virtue of its amplifying action, provides energy to make up the loss of electrical energy in the form of heat energy in the resistance of the coil. When this is done at the opportune moment, oscillations in the circuit are maintained. The frequency of the alternating current is given by

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$$

where L is the coefficient of self-induction of the coil in henry and C is the capacitance of the capacitor in farad. In practice L and C are very small and so the frequency of the alternating current is very high. By using a gang capacitor the frequency of the alternating current can be changed and set to any desired value. Fig. 8.1 shows the complete outfit for generating ultrasonics by this method. A nickel rod of suitable length is clamped rigidly at the centre. The oscillating circuit consisting of L and C is connected in between the plate and the cathode of the triode through a *H. T.* battery. A feed-back coil is placed in between the grid and the cathode of the triode. A steady direct current called the polarising current is passed through a separate coil, wound over the entire rod. By an adjustment of the capacitance of the capacitor C the frequency of oscillations of the inductive-capacitance circuit can be made exactly

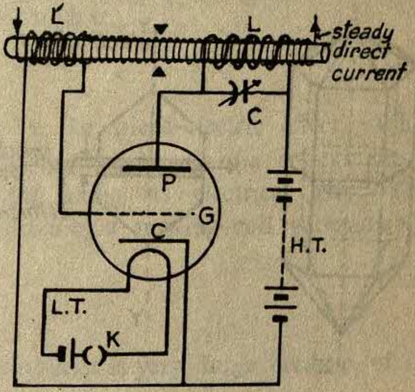


Fig. 8.1

the same as that of the rod. The rod oscillating in resonance produces a very strong beam of ultrasonic waves.

(*b*) *Piezo-electric generator.* This method is based on the Piezo-electric effect. In 1880 P. Curie discovered that if on some crystals such as quartz, tourmaline, Rochelle salt etc. a mechanical pressure or tension is exerted in a specific direction a maximum p.d. is developed in a direction perpendicular to it. This is known as the Piezo-electric effect. The two axes are respectively called the *mechanical axis* and the *electric axis*.

The effect is reversible. If an electric field is applied along the electric axis, there will be contraction or extension along the mechanical axis. If the applied electric field be alternating at very high frequency, the crystal will be set to vibrations with the frequency of the applied field. Further, if the frequency of the applied electric field is matched with the natural frequency of the crystal, it will vibrate with large amplitude.

The effect is most prominent in quartz. Quartz crystalises as hexagonal prisms with a pyramid at each end. The vertical axis ZZ' (Fig. 8.2 a) is called the optic axis of the crystal. The section of the crystal perpendicular to the z-axis is a hexagon. The line perpendicular to two opposite sides of the hexagon is the mechanical axis of the crystal and the line perpendicular to it (and obviously passing through opposite corners) is its electrical axis. A rectangular cut of the crystal is taken in such a way that its thickness 't' is along the electrical axis, the length 'l' is parallel to the mechanical axis and the breadth (b) is parallel to the z-axis. The natural frequency of the crystal cut along its thickness is

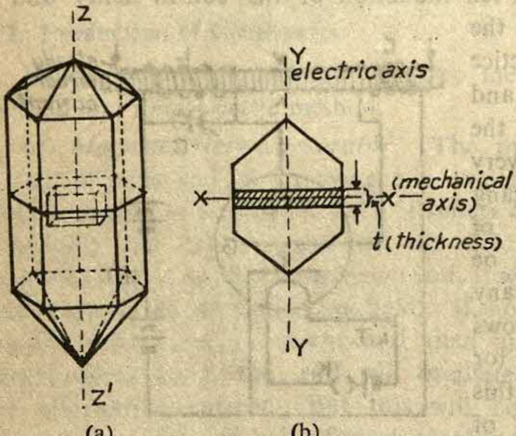


Fig. 8.2

$$n_t = \frac{1}{2t}\sqrt{\frac{E}{\rho}}$$ where E is the Young's modulus

of the crystal and ρ is its density. The frequency of the lengthwise vibration is

$$n_l = \frac{1}{2l}\sqrt{\frac{E}{\rho}}.$$

The complete outfit for generating ultrasonics by this effect is shown in Fig. 8.3. The alternating emf from an oscillator is applied between two parallel plates when an alternating electric field is produced between the plates. The crystal cut Q is subjected to this alternating electric field by placing it between the plates. The resonant circuit LC is tuned to the natural frequency of the crystal. R_g is the gridleak resistance. C_1 is a capacitor (by-pass) which bypasses the high frequency alternating current through it.

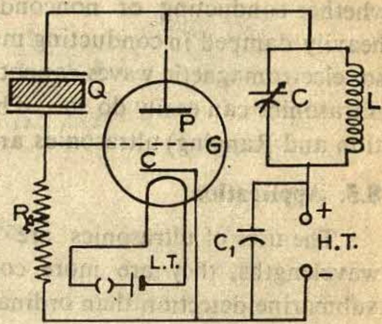


Fig. 8.3

8.3. Detection of Ultrasonics

Ultrasonics are detected by the piezo-electric effect. When ultrasonics are incident along the mechanical axis of a crystal cut, it develops a feeble alternating emf along its electrical axis. This feeble emf is amplified and detected by a moving coil galvanometer and a diode.

8.4. Properties of Ultrasonics

(i) The energy current in ultrasonics is very large because of its high frequency.

(ii) Because of its short wavelength the diffraction of an ultrasonic wave is very small. It can give a well collimated beam without spreading out of its rays even after a very large distance.

(iii) They have got very strong disruptive effects called the cavitation effect. Let us explain this property by simple figures. If an ultrasonic wave has a frequency 3×10^8 Hz and an energy current $10^3\,\mathrm{Wm^{-2}}$, calculations based on S.H.M. will show that in such a wave particles of the medium will have a maximum acceleration of

10^6 times the acceleration due to gravity and a pressure variation of ± 5 atmospheres. Such large stresses in an ultrasonic wave propagation through a liquid cause a tearing apart of the liquid with production of hollow spaces filled with vapour. These hollow spaces collapse with great violence. This is what is called the cavitation effect.

(iv) They can propagate in all three media—liquid, gas and solid—whether conducting or nonconducting. Electromagnetic waves are heavily damped in conducting media. Sea water is conducting and so electromagnetic waves cannot propagate through sea water, but ultrasonics can easily do so. This is why in sonar (Sound Navigation and Ranging) ultrasonics are used with greater advantage.

8.5. Application

The uses of ultrasonics are numerous. Because of their short wavelengths, they are more convenient in echo sounding or anti-submarine detection than ordinary sound waves.

In recent years, ultrasonics have become a new tool in industry, in medicine, in agriculture etc.

(i) *In industry ultrasonics are used* more conveniently than X-rays in flaw detection in metal structure.

(ii) *Its cavitation effect. provides the following uses*—(a) The production of finely dispersed emulsions, (b) two liquids which are ordinarily immissible can be made to form a uniform mixture by ultrasonic agitation, (c) they can produce 'coagulation', that is, small particles in suspension in a liquid can be made to form big particles and settle at the bottom by supersonic agitation, (d) they accelerate the speed of chemical reactions, (e) cleaning of clothes, surgical instruments etc.

(iii) *Biological and medical uses.* Ultrasonics can destroy bacteria and corpuscles of the blood. Neuralgia, rheumatism etc. can be cured by exposure to ultrasonics.

(iv) *Agriculture use.* When small plants are exposed to ultrasonics, they grow rapidly.

Examples :

1. *Calculate the length of a nickel rod required to generate ultrasonics of frequencies 40,000 Hz. Young's modulus* $=2{\cdot}07\times10^{11}$ Nm^{-2} *and its density* $=8900$ kgm^{-3}.

Sol. We have $n=\frac{1}{2l}\sqrt{\frac{E}{D}}$

$$\therefore \quad l=\frac{1}{2n}\sqrt{\frac{E}{D}}=\frac{1}{2\times 40000}\sqrt{\frac{2\cdot07\times10^{11}}{8900}}=\cdot060 \text{ m. Ans.}$$

2. *The thickness of an X-cut crystal is ·20 mm. Calculate the frequency of the thickness vibrations of the crystal. Density of quartz = 2660 kgm⁻³ and Young's modulus = 7·3 × 10¹⁰ Nm⁻².*

Sol. We have $n=\frac{1}{2l}\sqrt{\frac{E}{D}}=\frac{1}{2\times\cdot2\times10^{-3}}\sqrt{\frac{7\cdot3\times10^{10}}{2660}}$

or $n=1\cdot3\times10^{7}$ Hz. **Ans.**

QUESTIONS

(A)

1. Ultrasonics are (a) longitudinal waves of frequencies less than 20,000 Hz, (b) transverse waves of frequencies less than 20,000 Hz, (c) longitudinal waves of frequencies greater than 20,000 Hz, (d) transverse waves of frequencies greater than 20,000 Hz.

2. If f is the upper limit of human audition, then ultrasonics are (a) longitudinal waves of frequencies less than f, (b) transverse waves of frequencies less than f, (c) longitudinal waves of frequencies greater than f, (d) transverse waves of frequencies greater than f.

3. Piezo-electricity is the electric charge produced (a) by pressure, (b) by heat, (c) by friction, (d) by transport of atoms.

4. If 20,000 Hz is the upper limit of the human auditory system then ultrasonics are (a) longitudinal waves of wavelength greater than 1·5 cm, (b) longitudinal waves of wavelength less than 1·5 cm, (c) electromagnetic waves of wavelength greater than $1\cdot5\times10^{4}$ m, (d) electromagnetic waves of wavelength less than $1\cdot5\times10^{4}$ m.

5. Ultrasonics are waves (a) travelling with the speed of sound wave, (b) travelling with a speed greater than that of sound, (c) travelling with the speed of light.

6. Ultrasonics are used in sonar with greater advantage, because (a) ultrasonic waves are of very high frequency, (b) ultrasonics are not absorbed by sea water, (c) ultrasonics can be easily be produced, (d) ultrasonics are electromagnetic in nature.

(**Ans. 1.** c, **2.** c, **3.** a, **4.** b, **5.** a, **6.** b.)

(B+C)

1. What are ultrasonic waves ? How are they produced ? Mention some important uses of ultrasonics.

2. What is the magnetostriction effect ? Explain how it is utilised in the production of ultrasonics.

3. What is the piezo-electric effect ? Explain how it is used to produce and detect ultrasonics.

(E)

1. Sonar stands for.................. .

2. Piezo-electricity is the phenomenon of generation of electricity by............................(pressure, heat).

3. Magnetostriction is the phenomenon of increase in length due to the action of..................(a) magnetic field, an electric field).

(Ans. 1. Sound Navigation And Ranging. 2. pressure. 3. a magnetic field.)

CHAPTER 9

RECORDING AND REPRODUCTION OF SOUND

9.1. Introduction

Recent developments in recording and reproduction of sound have contributed to a great extent to the entertainment of the most people. Developments have proceeded along three lines :

(i) Disc Recording and Reproduction,

(ii) Tape Recording and Reproduction, and

(iii) Film Recording and Reproduction.

The basic instruments for recording is 'microphone' and that for reproduction is a loud speaker or simply 'speaker'.

9.2. Microphones

(*i*) *Carbon microphone*. It works on the theory that the resistance of loosely packed carbon granules depends on the pressure on the package. When they are loosely packed, the resistance is large, but when it is compressed, the resistance decreases by an amount depending on the degree of compression. It consists of a carbon diaphragm and a block of carbon having carbon granules in between. The rest of the metal case, in which the block and the diaphragm are enclosed, is filled with flannel packing to prevent sharp resonant vibrations. Two terminals having electrical connections with the block and the diaphragm are provided at the back of the metal case. A battery is connected between the terminals and the primary of a transformer. When a sound wave is incident on the diaphragm, the vibration causes a change in resistance of the layer of carbon granules. This, in its turn, causes a change in the current in the circuit. A transformer is used to induce a current in the secondary in exact tally with the variation in pressure of the incident sound wave.

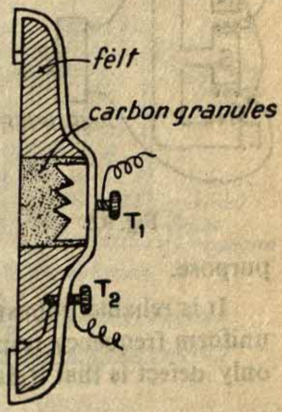


Fig. 9.1

It is robust, reliable and cheap. It gives a large out-put without amplification.

Its frequency response is very poor. There is a constant background hiss.

(*ii*) *Capacitor microphone.* The principle of this microphone is that the capacitance of a parallel plate capacitor depends on the distance between the plates. So if one of the plates of a parallel plate capacitor is made of a metal diaphragm and the other is a fixed plate, then on speaking in front of the diaphragm, the capacitance of the capacitor will change and this in turn will cause a change in p.d. across the high resistance put in series with the capacitor.

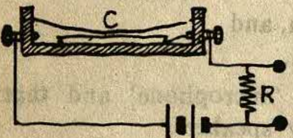


Fig. 9.2

It works over a wide range of frequencies without distortion. There is no background hiss. This is very small in size.

(*iii*) *Moving coil microphone.* It works on the principle that when a coil moves in a magnectic field, an induced emf is produced in the coil, whose magnitude and direction respectively depend on the velocity and direction of motion of the coil. It consists of a pot magnet that is, a magnet having a central south pole surrounded by a north pole from all sides. A cylindrical paper frame carrying a coil of thin insulated copper wire is floated from a diaphragm in the air gap between the poles. When something is spoken in front of the diaphragm its vibrations cause a motion of the coil in the magnetic field and consequently an alternating emf is produced in the coil. This is amplified and used for subsequent purpose.

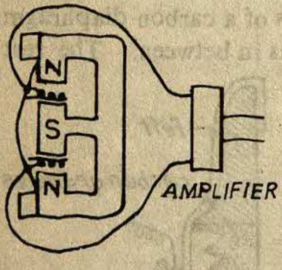


Fig. 9.3

It is reliable, robust and free from background noise. It has a uniform frequency response and it needs no polarising current. The only defect is that it flutters in the wind.

9 3. Loudspeakers

The function of a loudspeaker is to convert the out-put from a microphone into a loud sound before a large gathering. The principle of working of a loudspeaker is that a current coil experiences a

mechanical force when placed in a magnetic field. It consists of a pot-magnet, that is, a magnet having a central south pole surrounded by a north pole from all sides. A cylindrical paper frame carrying a coil of insulated copper wire called the speech coil is floated in the air gap between the poles from the apex of a conical diaphragm made of stiff paper. The diaphragm is attached at its periphery to non-magnetic metallic frame. The varying speech currents from microphone circuit are fed in the speech coil. The speech coil experiences a varying force in the direction perpendicular to the magnetic field. The whole diaphragm is thus caused to vibrate in the corresponding manner and thus it reproduces the original sound.

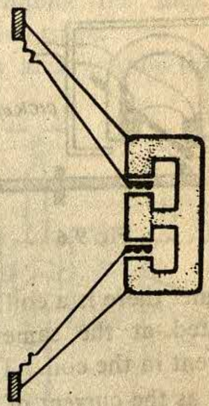


Fig. 9.4

9.4. Disc Recording and Reproduction

Sound is recorded on a disc by cutting spiralling grooves by a sapphire cutter. The cutter is attached to the armature of a coil '*C*' of insulated copper wire. This armature is capable of turning to and fro about fulcrum. The upper end of the armature lies in between the poles of an electro-magnet. The whole arrangement is enclosed in a box called the recording head. To record any sound, the out-put from the microphone circuit is passed through the coil *C*, when the style is vibrated in exact correspondence with the fluctuations of the current from the microphone. The recording head is so arranged that the cutter penetrates a little into the rotating paraffin disc *D*. For long playing records (LP), the disc is rotated at $33\frac{1}{3}$ r.p.m. and for extended play (EP) records recording is done at 45 r.p.m. As the cutter cuts the grooves, the recording head moves slowly radially towards the centre of the

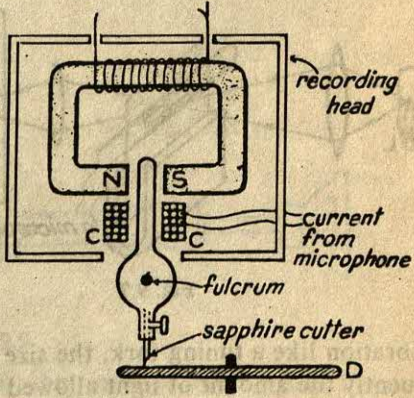


Fig. 9.5

disc. After recording is done in a paraffin disc, records are produced by stamping the 'mother disc' prepared from the paraffin disc.

To reproduce the sound from the grooves of the records a 'pick up' having almost a similar arrangement as the recording head is set on the grooves. The 'pick up' consists of a small soft iron rod arranged between the poles of a permanent magnet. The rod is provided with a sharp sapphire or diamond point which is set in the groove. Round the poles of the magnet there is a coil of insulated copper wire. When the record is rotated at the same speed, the rod vibrates producing an induced current in the coil. The current in the coil varies exactly in the same way as the current of the recording microphone. This current, when amplified and fed in a loudspeaker, reproduces the original sound.

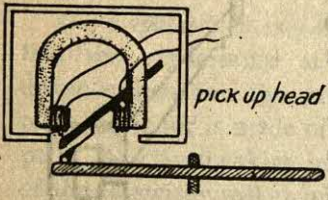


Fig. 9.6

9.5. Film Recording and Reproduction

There are two methods of recording on film—(i) Variable Density Method and (ii) Variable Area Method.

(i) Variable Density Method. Light from a recording lamp L is passed through a narrow slit in an opaque screen. In front of the slit, there is a loop of aluminium ribbon. The edges of the ribbon define the upper and lower edges of the slit, so that when the loop is in vibration like a tuning fork, the size of the slit changes and consequently the amount of light allowed to pass. The light on the other side of the screen is collected by a lens and concentrated on the sound track of the film. The ribbon is placed in a magnetic field between the poles of a permanent horse-shoe magnet. In the process of recording, the current from the microphone circuit is passed through the loop and the film is moved at a constant speed. Due to

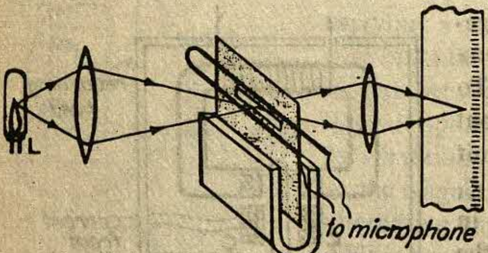


Fig. 9.7

the variable current through the loop, it experiences a fluctuating force and consequently it is set in vibration causing a change in area and therefore in the intensity of light in exact correspondence with the fluctuating microphone current. On developing the film a permanent record of sound is obtained in the sound track.

(*ii*) *Variable Area Method.* In the variable area method light from a recording lamp is reflected from the mirror of a Duddel oscillograph. The beam of light then passes through a slit of fixed area and focussed on the side track of the film. The fluctuating current from the microphone circuit is passed through the phosphor bronze loop of the oscillograph and the film is moved at a constant speed behind the slit. The fluctuating currents cause the mirror to vibrate according to the variation in the strengths of the currents. The resulting sound track has uniform density, but with serrated edges.

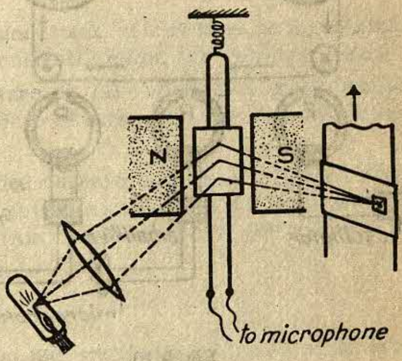


Fig. 9.8

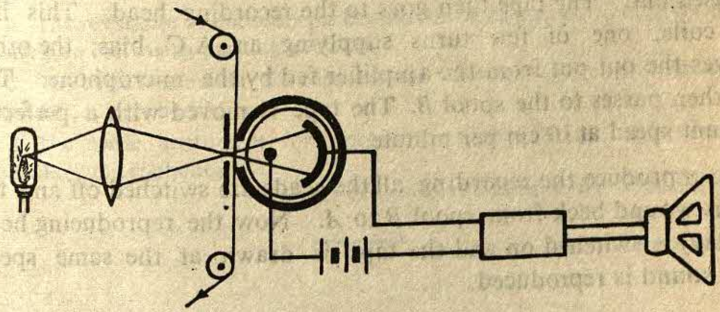


Fig. 9.9

To reproduce the sound from the recorded track, light from a bright source is focussed on the sound track. On the other side of the film a photo cell is placed. The photoelectric current depends on the amount of light falling on the cathode of the cell. The varying current thus produced is amplified and fed in a loudspeaker, where the original sound is reproduced.

9.6. Tape Recording and Reproduction

The principle of tape recording is that the magnetisation of a magnetic substance (here a plastic tape having a layer of cellulose acetate impregnated with ferric oxide) is proportional to the flux density of the magnetic field and the principle of reproduction is that the induced emf in a coil is proportional to the rate of change of magnetic flux. In a tape-recorder, there are three magnets with three coils wound over them called 'heads'. The three heads are—(i) the erasing head (E), (ii) the recording head (R), (iii) the reproducing head (P). The tape comes off the spool A and passes past the erasing head, where anything already recorded is erased out. The tape then goes to the recording head. This has two coils, one of few turns supplying an A.C. bias, the other receives the out-put from the amplifier fed by the microphone. The tape then passes to the spool B. The tape is moved with a perfectly constant speed at 10 cm per minute.

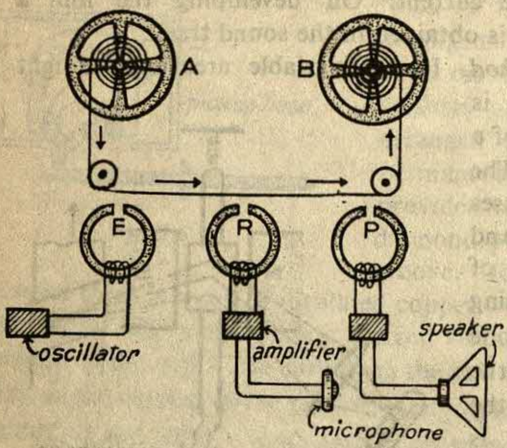


Fig. 9.10

To reproduce the recording, all the heads are switched off and the tape is wound back from spool B to A. Now the reproducing head (or play) is switched on and the tape is drawn at the same speed when sound is reproduced.

QUESTIONS

(A)

1. A carbon microphone works on (a) the principle of electromagnetic induction, (b) magnetic effect of current, (c) variation of resistance of a pack of carbon granules with pressure, (d) magnetic effect of current and electromagnetic induction.

2. A condenser microphone works on (a) variation of capacitance with distance between plates, (b) magnetic effect of current, (c) electromagnetic induction, (d) none of these.

3. A 'speaker' works on (a) electromagnetic induction, (b) magnetic effect of current, (c) variation of resistance with temperature, (d) photoelectric effect.

4. Sound is recorded on cine films on the principle (a) magnetic effect of current, (b) photoelectric effect, (c) electromagnetic effect, (d) thermionic effect.

5. Sound is reproduced from the sound track of cine films on (a) magnetic effect, (b) photoelectric effect, (c) electromagnetic effect, (d) thermionic effect.

6. Sound is recorded on magnetic tape on (a) variation of magnetisation with magnetic induction, (b) magnetic effect of current, (c) photoelectric effect, (d) none of these.

7. Sound is reproduced from recorded magnetic tape on (a) magnetic effect of current, (b) electromagnetic induction, (c) thermionic effect, (d) Seebeck effect.

(Ans. 1. c, **2.** a, **3.** b, **4.** a, **5.** b, **6** a, **7.** b)

(B+C)

1. Give an account of the production and reproduction of sound in motion pictures.

2. Explain recording and reproduction on discs.

3. Explain working of (a) carbon microphone, (b) a loudspeaker, (c) a condenser microphone and mention their relative merits and demerits.

(E)

1. The resistance of a loosely packed carbon granules ....(increases or decreases) on increasing packing pressure.

2. The basic instrument for reproduction of sound is........(galvanometer, loudspeaker, microphone).

3. The basic instrument for recording sound is...... (galvanometer, loudspeaker, microphone).

(Ans. 1. decreases, **2.** loudspeaker, **3.** microphone)

CHAPTER 10

ACOUSTICS OF BUILDINGS**

10.1. Elements of Acoustics of Buildings

The fitness of lecture halls, cinema houses, auditoriums etc. for music, speech, concerts etc. depends largely on their design. If the design be such that there is uniformity in rendering of speeches, music, concerts etc., they are said to be acoustically good; if not, they are acoustically bad. There are two main causes of bad acoustics of a building—echoes and reverberation.

Echo is the repetition of original sound after it is dead. This is formed due to reflection of the original sound wave. A man cannot distinguish between two sounds reaching his ear within 1/10 of a second, that is, impression of a sound wave remains in normal human ear for 1/10 of a second and he cannot hear any of them if they arrive within 1/10 of a second. So in large auditoriums it is just likely that the echo of a syllable will be mixed up with the next syllable directly reaching the listener and thus be a cause of bad acoustics of the auditorium.

Reverberation is the continuation of the original sound intensity level above the threshold of audibility after the source is cut off. Suppose that a source of sound of constant output, such as a loud speaker excited by an oscillator is started in an auditorium. The source supplies energy at a constant rate to the auditorium, part of which is absorbed by absorbing materials in the auditorium (including doors, windows and ventilators which are perfect absorber of sound) and the rest is reflected back to the auditorium. So the energy density in the auditorium increases slowly and soon the energy density attains a steady value because an equilibrium is established between the rate of supply of energy and the rate of loss of energy by absorption. This is shown in the Fig. 10.11. Though we have considered here a continuous source of constant output, when we utter syllables the energy density corresponding to each syllable increases in the same way. Now suppose that the source is cut off and just imagine for the time being that there is no absorbing material in the auditorium. What happens to the energy density ?

Since there is no supply of energy and also no absorption of energy, the total amount of sound energy in the auditorium will remain constant for all the time. But in fact some energy is absorbed by the walls, doors, windows, ventilators etc. So the energy density level will remain only for some time above the threshold level. This is called reverberation of the auditorium. The time taken by the energy density to fall by a factor 10^{-6} or say, the time taken by intensity level to fall through 6 bels or 60 decibels is called time of reverberation of the building. Now let us see how this becomes the cause of bad acoustics of an auditorium. Suppose the time of reverberation of an auditorium is 3 seconds and a speaker is speaking at the rate of 3 syllables per second. So the sound waves corresponding to 9 syllables will exist in the auditorium above the threshold level. Under such conditions the listener will be utterly confused.

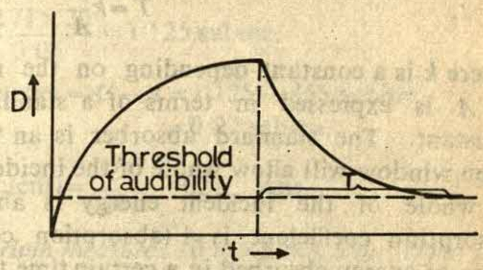


Fig. 10.11

10.2. Sabine's Law

W. C. Sabine was a professor of Physics in Harvard University. The auditorium of that university was acoustically so bad that a speaker could hardly make himself intelligible to the audience on the first row. He took up the problem and conducted experiments to find the factors on which the time of reverberation depends, because he could guess in the very beginning that the problem was due to excessive reverberation. Introducing cushion he found that the time of reverberation (T) is *inversely proportional* to the length of the cushion. Since absortion caused by a cushion is proportional to its length, his result may also be stated as 'the time of reverberation is inversely proportional to the absorption caused by the materials of the auditorium'. By conducting experiments in different halls having approximately the same absorbing material, he found that the time of reverberation is proportional to the volume (V) of the auditorium. Thus Sabine's results state,

$T \propto 1/A$ when V is constant

$\propto V$ when A (absorption) is constant.

$$\therefore \qquad T = k\frac{V}{A} \qquad \cdot\cdot \quad (10.1)$$

where k is a constant depending on the material of the absorber. If A is expressed in terms of a standard absorber, k will be a constant. The standard absorber is an 'open window' because an open window will allow whole of the incident energy to pass out, as if whole of the incident energy is absorbed by it. That is, its absorption coefficient is 1 (absorption coefficient is defined as the ratio of energy absorbed in a certain time to the energy incident on it in the same time). If S area of an open window causes the same absorption as the given material, then its absorption $(A) = S$ open window unit (O. W. U.) or sabine. Also if S is the area of the absorbing material and a is its absorption coefficient, then $A = aS$ O. W. U. or sabine. The above relation (Eq.10.1) is called Sabine's Law.

The value of the constant k is about ·171 in SI when A is expressed in O.W.U. or sabine.

10.3. Optimum reverberation

From the above discussion it appears that if there is no reverberation, the auditorium will be an ideal one. This is true no doubt. But such an auditorium will appear acoustically 'dead' to the speaker as he will have no response from it. It is to be noted that a speaker, singer, or an instrumentalist feels a stimulating effect when there is a certain amount of reflection, because a little effort on his part produces a powerful sound. Thus we see that for clarity's sake there should be no reverberation but for speaker or singer there must be some reverberation. Obviously the two are contradictory to each other and so there must be a compromise between the two. The reverberation arising out of compromise between the two opposing factors is called 'optimum reverberation'.

Examples

1. *The time of reverberation of an auditorium of volume* 20 m^3 *is* 4 *s. When a carpet of area* 10 m^2 *is introduced, the time of reverberation is reduced to* 3·04 *s. Calculate the absorption coefficient of the carpet.*

Sol. We have $\quad T = \dfrac{\cdot 171 V}{A}$ by Sabine's law

or $$A=\frac{\cdot171V}{T}=\frac{\cdot171\times20}{4}=\cdot855 \text{ sabine.}$$

When the carpet is introduced, let A' be the absorption.

$$\therefore \quad A'=\frac{\cdot171\times20}{3\cdot04}=1\cdot125 \text{ sabine.}$$

$\therefore$ Absorption of the carpet $=A'-A=1\cdot125-\cdot855$ sabine $=0\cdot27$ sabine.

$$\therefore \quad a \text{ (absorption coefficient)}=\frac{\cdot27}{10}=\cdot027. \quad \textbf{Ans.}$$

2. *The floor of an auditorium measures* 10 *m* $\times$ 5 *m* $\times$ 3 *m. What is the reverberation time if the average absorption coefficient of all surfaces is* ·05 *?*

Sol. We have, $T=\dfrac{\cdot171V}{A}$ by Sabine's Law.

Here A (absorption) $=\cdot05(5\times10+10\times3+3\times5)\times2=9\cdot5$ sabine.
V (volume) $=10\times5\times3=150$ m³.

$$\therefore \quad T=\frac{\cdot171\times150}{9\cdot5}=2\cdot7 \text{ s.} \quad \textbf{Ans.}$$

QUESTIONS

(B+C)

1. Discuss the defects in a hall from the acoustic point of view.
2. Explain how reverberation causes bad acoustics of a building.
3. State Sabine's Law. What is meant by 'open window unit' ?
4. Explain reverberation and optimum reverberation.

(D)

1. The time of reverberation of an auditorium of volume 150 m³ is 2 s. When 500 audiences are seated in auditorium, the time of reverberation is reduced to ·2 s. What is absorpiion caused by each audience ? (**Ans.** ·23 sabine)

2. A cinema house has dimensions 20 m $\times$ 12 m $\times$ 3m. What is the time of reverberation of the house if the average coefficient of absorption is ·03 ? If 1000 audiences enter into the house, what is the time of reverberation if absorption of one audience is ·5 sabine. (**Ans.** 6·1 s; ·24 s

3. The time of reverberation of an auditorium is 3 s. It is to be corrected for optimum reverberation time ·2 s. Calculate the area of cushion required to be introduced if its absorption coefficient is ·3 and the volume of the auditorium is 720 m³. (**Ans.** 1915·2 m²)

MISCELLANEOUS

1. 'Parsec' is the unit of (a) time, (b) distance, (c) mass, (d) charge.

(Hint. A *Parsec* is the distance at which one astronomical unit would subtend an angle of 1 sec of arc. 1 A. U. = average distance of the earth from the sun $=1{\cdot}496\times10^{11}$ m.) (Ans. b)

2. 'Lightyear' is the unit of (a) time, (b) distance, (c) mass, (d) luminous intensity. (Ans. b)

3. There is a limit beyond which further polishing increases rather decreases resistance. Explain why ?

(Hint. Due to molecular attraction.)

4. Why glass rod becomes smooth on heating ?

(Hint. Due to surface tension.)

5. Calculate the difference in the apparent weights of a body at the highest and lowest point moving in a circle of radius 5 m in 10 s in terms of the weight W of the body. (Ans. $\cdot 4W$)

6. Calculate the height of the 'apple' India's first stationary satellite above the surface of the earth. Radius of the earth = 6400 km and $g=9{\cdot}8$ ms^{-2}.
(Ans. 36×10^3 km)

7. State the reasons why the value of the acceleration due to gravity is different at various places.

(Hint. Variation of the radius of the earth and rotation of the earth.)

8. Why one should take short steps rather than long ones when walking on ice.

(Ans. To develop greater normal reaction so that enough friction is developed necessary for walking.)

9. State whether whole of the work can be converted into heat.
(Ans. Yes)

10. Name the motion in which acceleration is perpendicular to the velocity.
(Ans. circular)

11. A uniform rope of length l and mass m lies on a smooth horizontal table with its length perpendicular to the edge of the table and a small part of the rope hanging over the edge. Find the velocity of the rope when the length of the hanging part is x, and the acceleration.

$$\left(\text{Ans. } v=\sqrt{\frac{gx^2}{l}};\ a=\frac{gx}{l}\right)$$

12. State whether whole of the heat can be converted in to work.
(Ans. No.)

13. What is more elastic— rubber or steel, air or water ?
(Ans. Steel and water)

14. Can a body have energy without having momentum ? Can a body have momentum without having energy ? **(Ans. Yes; No.)**

15. Where is 'Weightlessness' ? At the centre of the earth ? In a spaceship ? On the surface of the earth ? **(Ans. Yes; Yes; No)**

16. Infra-red radiations have (a) maximum heating effect, (b) maximum illuminance, (c) maximum chemical effect, (d) maximum penetrating power. **(Ans. a)**

17. Two pulleys of masses 12 kg and 8 kg are connected by a fine string passing over a smooth fixed pulley. Over the 8 kg pulley a fine string is passed and two masses m kg and 12 kg are attached to the ends of it. Over the 12 kg pulley another fine string is passed and two masses 3 kg and 6 kg are attached to its ends. Determine 'm' so that the string over the fixed pulley remains stationary. **(Ans. 12 kg)**

18. Two equal weights P and Q connected by a string passing over a smooth pulley are moving with a common velocity, P descending and Q ascending. If P be suddenly stopped and immediately let go, find the time that elapses before the string is again taut.

(Ans. $t=u/g$, where u is the common speed)

(Hint. The string will be taut again when upward distance of Q is equal to the downward distance of P.)

19. Quality of a musical note depends on (a) the wave form, (b) frequency, (c) amplitude, (d) relative motion between the source and the listener. **(Ans. a)**

20. Mountain roads rarely go straight up the slope but wind up gradually. Explain why ?

(Ans. otherwise one will get tired because of greater rate of doing work. Dynamically we can say that the inclined road gives mechanical advantage to the man or vehicle moving on it.)

21. A gas expands by 100 times. Does it approach ideal state or diverge from it ? **(Ans. Becomes more ideal.)**

22. In a game of tug of war two opposing teams are pulling the rope with equal but opposite forces of 500 kg at each end of the rope. What is the tension of the rope ? (a) 1000 kg, (b) zero, (c) 500 kg, (d) 250 kg. **(Ans. c)**

23. A gas bubble, from an explosion under water, oscillates with a period T proportional to $P^a\ d^b\ E^c$, where P is the static pressure, d is the density of water and E is the energy of explosion. Find the values of a, b and c.

(I. I. T. 1981)

(Ans. $a=-\frac{5}{6}, b=\frac{1}{2}, c=\frac{1}{3}$)

24. A body of mass 1 kg, initially at rest, explodes and breaks into three fragments in the ratio 1 : 1 : 3. The two pieces of equal masses fly off perpendicular to each other with a speed of 30 ms^{-1} each. What is the velocity of the heavier fragment ? **(I. I. T. 1981)**

(Ans. $10\sqrt{2}\ ms^{-1}$)

25. When a person walks on a rough surface, the frictional force exerted by the surface on the person is opposite to the direction of his motion. True or False ? (Ans. False)

26. The root mean square speeds of the molecules of different ideal gases, maintained at the same temperature, are the same. True or false ? (I. I. T. 1981) (Ans. False, because it depends on the molecular mass)

27. A cylindrical tube, open at both ends, has a fundamental frequency 'f' in air. The tube is dipped vertically in water so that half of it is in water. The fundamental frequency of the air-column is (a) $f/2$, (b) $3f/4$, (c) f, (d) $2f$. (I. I. T. 1981) (Ans. a)

28. A vessel containing water is given a constant acceleration towards the right along a straight horizontal path. Will the surface (a) remain horizontal, (b) turn clockwise, (c) turn anti-clockwise, (d) oscillate ? (I. I. T. 1981) (Ans. b)

29. The temperature of a solid depends upon (a) its colour, (b) its volume, (c) its mass, (d) its specific heat capacity, (e) the amplitude of oscillations of its atoms. (Ans. e)

30. If a machine is lubricated (a) the mechanical advantage of the machine increases, (b) the mechanical efficiency of the machine increases, (c) both mechanical advantage and efficiency increase, (d) its efficiency increases, but its mechanical advantage decreases. (Ans. c)

31. A constant volume gas thermometer works on (a) the principle of Archimedes, (b) Boyle's law, (c) Pascal's law, (d) Charles' law. (Ans. d)

32. A man stands on the ground at a fixed distance from a siren which emits sound of a fixed amplitude. The man hears the sound to be louder on a clear night than on a clear day. True or false ? (Ans. True)

33. A spring of force constant k is cut into three pieces (equal). What is the force constant of each part ? (Ans. $3k$)

34. A car rounds a curve at 30 kph. If it rounds the curve at 60 kph, its tendency to overturn is (a) halved, (b) doubled, (c) tripled, (d) quadrupled. (I. I. T. 1978) (Ans. d)

35. The state of equilibrium of a man standing on one leg is .... (I. I. T. 1978) (Ans. unstable)

36. As a boat moves from fresh water into salt water, its water mark will be......... (I.I.T. 1978) (Ans. raised)

37. When a ball is thrown up, the magnitude of its momentum decreases and then increases. Does this violate the conservation principle of momentum ? (I. I. T. 1979) (Ans. No, the total momentum of the earth and the ball is conserved. The conservation principle is not applicable to the ball alone.)

38. A right circular cylinder of radius r and mass m is suspended by a cord that is wound around its surface. If it is allowed to fall, prove that its centre of gravity will follow a vertical rectilinear path and find the acceleration along this path. Determine also the tensile force in the cord. (Ans. 2/3 g; 1/3 mg)

(**Hint :** Consider linear and angular motions of the cylinder separately.)

39. In a slippery curved road a car has greater tendency to skid or to overturn ? (Ans. to skid)

40. In a curved rough road a car has greater tendency to skid or to overturn ? (Ans. to overturn)

41. At the moment of vertical take off the total mass of a rocket is 3×10^6 kg. The exhaust gases come out at the rate of 29400 kgs^{-1} with a velocity 1500 ms^{-1} relative to the rocket. Calculate the initial acceleration of the rocket. (Ans. 4·9 ms^{-2})

(Hint. $Mdv/dt=F_{ext}+v_{rel}\,dM/dt$; $F_{ext}=-Mg$)

42. A weight suspended from a spring of 10 cm length produces an extension of 3 cm. Now another spring identical with the first but of length 5 cm is also used to support the same weight along with the first. The point of suspension of the second spring is 6 cm below the point of suspension of the first spring. What is the extension produced in the first spring ? (Ans. 1·7 cm)

43. A shaft of radius r rotates with constant angular speed in bearings for which the coefficient of friction is μ. Through what angle θ will it rotate after the driving torque is removed ? (Ans. $\theta=r\omega^2/4\mu g$)

44. A circular roller o f radius r is contacted at the top and bottom points of its circumference by two conveyor belts. If the belts move with velocity v_1 and v_2 in the same direction, find the linear speed v_c of the roller and also its angular velocity ω. (Ans. $v_c=(v_1+v_2)/2$; $\omega=(v_1-v_2)/2r$)

(Hint. Velocity v_1 at the top is $v_1=v_c+\omega r$ and v_2 at the bottom is $v_2=v_c-\omega r$)

45. A right circular disc of radius r resting on a smooth table spins with angular velocity ω about its vertical axis. What new angular velocity ω' will the disc have if a point on its circumference is suddenly pinned to the table ? (Ans. $\omega'=\omega/3$)

(**Hint.** Apply the principle of conservation of angular momentum.)

46. Two teams having a tug of war must always pull equally hard on one another. The team that pushes harder on the ground wins. Why ?

(Ans. The component of the reaction of the ground will help pulling the tug to the side.)

47. Why two blankets are perferable for putting on in winter night instead of one having the same thickness as that of that two ?

(Ans. To prevent loss of heat by conduction by entrapped air in between the blankets.)

48. Explain why the voices of bathers whose heads are naturally a little above the water surface, can be plainly heard on the shore.

(Ans. Sound is reflected and refracted exactly in the same way as light. But for sound air is acoustically denser than water because sound travels faster in water than in air. It is just the opposite for light. Light travels faster in air than in water. If C is the critical angle for air-water interface then. 1440 $\sin C=340 \sin \pi/2$ or $C=13°$.

Thus, sound rays propagating in air, when incident at an angle greater than 13° at the air-water interface, the rays are totally reflected. This explains the above fact.)

49. Voices are clearly heard from a distance at night than during the daytime. Explain.

(**Ans.** In the daytime air is hottest near the ground level and it becomes progressively cold upwards. Therefore the velocity of sound is the greatest near the ground ($\because C \propto \sqrt{T}$) and it decreases upwards. The plane wave-fronts are thus bent upwards. At night the situation is just the reverse and so the plane wave-fronts are bent downwards.

50. Two S. H. M's one along the x-axis and another along the y-axis combine to form the figure 8. What is the ratio of the frequencies of the two S. H. M's ? (**Ans.** $f_x : f_y = 2 : 1$)

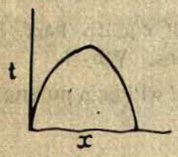
51. At mid-day the sun's attraction on a body on the surface is opposite to that of the earth and at mid-night it is just the opposite. So a body should weigh more at mid-night than at mid-day. Is it true ? If not, explain the fact.

(**Ans.** It is not true. The attraction of the sun on a body on the surface of the earth provides the centripetal force to maintain the motion of the body in the orbit of the earth round the sun.)

52. If there were no annual motion of the earth round the sun, will the above statement be correct ?

(**Ans.** Yes. No force would then be needed for circular motion round the sun. So a body would weigh more at mid-night than at mid-day.)

53. Is the time variation of position shown in the figure observed in nature ? (I. I. T. 1979)



(**Ans.** No. Because at a given time a body cannot have two displacements. The given figure shows that the body can have two diplacements simultaneously. This is physically impossible.)

54. If you want to stop a car in the shortest distance on a slippery road, should you (a) push hard on the brakes to lock the wheels, (b) push just hard enough to prevent slipping, or (c) pump the brakes ?

(**Ans.** b, because if the brakes are applied to lock the wheels, the car will slide over the road. Hence sliding friction will arise. If, however, the brakes are pushed hard enough to prevent slipping, the velocity of the points of contact of the wheels with the ground is zero. Hence now a static friction will arise between the wheels and the ground. D, stopping distance$=v^2/2\,\mu g$. We know $\mu_k < \mu_s$. $\therefore\ D_k > D_s$)

55. Your car skids on a slippery curved highway. Should you turn the front wheels in the direction of skid or in the opposite direction (a) when you want to avoid collision with an oncoming car, (b) when no car is near but you want to

regain control of the steering ?

(Ans. (a) in the opposite direction, (b) is the direction of skid.)

56. Suppose you are standing on a vast sheet of ice which is supposed to be perfectly smooth. How can you move forward ? How can you move backward ?

(Ans. By kicking feet. By kicking feet in the backward direction or by spitting in the forward direction.)

57. Dimensionally S_t (distance described in t-th second)$=u+\frac{1}{2}a(2t-1)$ seems to be incorrect. In fact, this is a correct relation. Can you explain where the flaw lies ?

(Ans. $S_t=(ut+\frac{1}{2}at^2)-(u\,\overline{t-1}+\frac{1}{2}a\,\overline{t-1}^2)=u(t-\overline{t-1})-\frac{1}{2}(t^2-\overline{t-1}^2)a$, Thus u is multiplied by 1 second and not by simply 1. So also a is multiplied by the difference of two squared terms in time, that is, by $(2t-1)$ second^2 and not by simply $(2t-1)$.

58. A balloon filled with hydrogon is at a height h above the ground. Its potential energy is (a) greater than the one on the ground, (b) less than the one on the ground, (c) is equal to that on the ground, (d) zero.

(Ans. b, because potential energy is the work done by an external agent)

59. Action and reaction of two parts of a system such as brake and wheel of a car cancel out for the system. How then car is stopped on applying brakes ?

(Ans. The car is stopped due to the action of an external body, namely, the earth. On applying brakes the rotational motion of the wheels is stopped and the car slides on the ground. This gives rise to enormous friction which stops the car within the shortest distance.)

60. Does the centre of mass of a solid body necessarily lie within the body ? If not, give an example. (Ans. No. Example, a ring)

61. Which is more likely to break, hammock (a hanging bed in ships) stretched tightly between two pillars or one that sags quite a bit ?

(Ans. When stretched tightly. $T=\dfrac{W}{2\cos\theta}$. Justify the answer from this formula.)

62. A ladder is at rest with its upper end against a wall and the lower end on the ground, Is it more likely to slip when a man stands on it at the bottom or at the top ?

(Ans. When the man stands at the top because his weight will have greater moment about the lower point.)

63. A block A of mass m rests on the top rough surface of a fixed block C. It is connected by a light string to a spring of force constant $k=1960$ N/m after passing it over a frictionless pulley. A third block of mass 2 kg is suspended from the spring and the whole system is allowed to move. It is found that the whole system moves with a constant speed. Calculate the mass m, assuming that the coefficient between surfaces is ·2. (I. I. T. 1982) (Ans. 10 kg)

64. Inside a spaceship what difficulties would you encounter ? In walking ? In jumping ? In drinking ? (Ans. In drinking)

65. The gravitational force exerted by the sun on the moon is much greater than the gravitational force exerted by the earth on the moon. Why then doesn't the moon escape from the earth during a solar eclipse ?

(**Ans.** Because both the forces provide necessary centripetal forces. The sun's attraction provides the centripetal force for the circular motion of the moon round the sun and the earth's attraction provides the necessary centripetal force for circular motion round the earth.)

66. The gravitational attraction of the sun and the moon on the earth. produce tides. The sun's tidal effect is about half as great as that of the moon's The direct pull of the sun on the earth, however, is about 175 times that of the moon. Why is it then that the moon causes the larger tides ?

(**Hint**. The tidal effect is proportional to difference between gravitational attraction on the top and the bottom of water ($\triangle F$) and $\triangle F \propto \frac{1}{r^3}$ where r is the distance of the sun.

67. It is found that liquid will flow faster and more smoothly from a sealed can when two holes are made in the can than when one hole is made. Why ?

(**Ans.** Because with one hole the flow is turbulent on account of high speed of flow. When two holes are made, the speed of flow comes down and the flow becomes streamline.)

68. If A is the area of exhaust pipe of a rocket, p is the pressure inside the exhaust chamber and p_0 is the pressure outside the pipe, then the thrust on the rocket is (a) $A\ (p-p_0)$, (b) $1/2A\ (p-p_0)$, (c) $2A\ (p-p_0)$, (d) $A(p_0-p)$.

(**Ans.** c. Apply Bernoulli's theorem to the flow of gas through the pipe and make permissible approximations. The thrust is $v_{rel} \frac{dM}{dt}$.

It is not $A(p-p_0)$ because this is the thrust due to fluid at rest. Here the thrust is not due to fluid at rest.)

69. When the stopper is removed from a filled basin, the water runs out while circulating like a small whirlpool. The angular velocity of a fluid element about a vertical axis through the orifice is greatest at the orifice. Explain.

Ans. Consider an element of fluid which is moving with speed v along a curved streamline. Here the pressure must vary from streamline to streamline to provide centripetal force necessary for curved motion. Now

centripetal force $= Adp$ where $A =$ area of the element $= (A\rho\, dr) \times \frac{v^2}{r}$

or $\frac{dp}{dr} = \frac{\rho v^2}{r}$. By Bernoulli's theorem, $\frac{1}{2} v^2 + \frac{p}{\rho} =$ a constant.

Differentiating w. r. t. r, $v\frac{dv}{dr} + \frac{1}{\rho}\frac{dp}{dr} = 0$

or $\frac{dv}{v} = -\frac{dr}{r}$, or $vr =$ a constant, $\therefore\ v \propto \frac{1}{r}$.

70. If a planet of given density were made large, its force of attraction for an object on its surface would increase because of the planet's greater mass but would decrease because of the greater distance from the centre of the planet. Which effect predominates ?

(**Ans.** Mass effect would predominate.

$$F=\frac{G4\pi/3R^3\rho m}{R^2}; \qquad \therefore\ F\propto R)$$

71. A meteorite burns in the atmosphere before it reaches the earth's surface. What happens to its momentum ?

(**Ans.** The momentum is transferred to the earth through the atmospheric particles.)

72. Will the velocity of a rocket increase if the outflow velocity of the gases with respect to the rocket is smaller than the velocity of the rocket itself, so that the gases ejected from the nozzle follow the rocket ?

(**Ans.** Yes, because of the pressure of the ejected gases.)

73. Does a homogeneous disc revolving about its axis have any momentum ? The axis of the disc is stationary.

(**Ans.** No. One half will have as much positive momentum as the other half will have negative momentum. Hence on the whole the disc will have no momentum.)

74. A homogeneous brick lies on an inclined plane. What half of the brick (upper or lower) exerts a greater pressure on it ?

(**Ans.** The lower half.)

75. At what coefficient of friction will a man not slip when he runs along a straight rough path ? The maximum angle between the vertical line and the line connecting the man's centre of gravity with the point of support is α.

(**Ans.** At $\mu \geqslant \tan\alpha$. If F is the reactional force along the toe-to-c.g. line, then for equilibrium of the man, $F\sin\alpha = f_{\text{friction}}$ and $F\cos\alpha = mg$.

The man will not slip so long frictional force does not reach its limiting value, that is,

$f_{\text{friction}} \leqslant f_{\text{lim.fric}}$ or $F\sin\alpha \leqslant \mu F\cos\alpha$).

76. In what conditions will bodies inside a spaceship be weightless, that is, cease to exert any pressure on the walls of the cabin ?

(**Ans.** When the engine is shut off and there is no resistive force on the motion of the satellite Then there will be no extra force other than the gravitational pull which provides the necessary centripetal force and so the bodies inside the cabin will be weightless.)

77. Two air bubbles in water are at a certain distance apart. Will they attract or repel each other ?

(**Ans.** They attract each other.)

78. If an air bubble and an iron ball of the same radius are situated at a certain distance apart in water, will they attract or repel each other ?

(**Ans.** They repel each other.)

79. Two sattelites move along a circular orbit in the same direction at a small distance from each other. A container has to be thrown from the first satellite onto the second one. When will the container reach the second satellite faster : if is thrown in the direction of motion of the first satellite or in the opposite direction ? The velocity with which the container is thrown is small in comparison to that of the satellites.

(**Ans.** The container should be thrown in the direction of motion of the first satellite because in this case the orbit of the container will be an ellipse bigger than the circular orbit of the satellites touching at the point where the container is thrown. If the velocity of the container be such that during one revolution of the container, satellite makes one revolution and describes the additional distance between the satellites, then the container will reach the second satellite in the shortest time. On the other hand if the container is thrown in the opposite direction, it will orbit in a smaller ellipse inside the circular orbit of the satellites. Therefore, they can meet at the point of contact between the orbits only after a great number of revolutions.)

80. A block of wood floats vertically with its longer edge vertical. How will the level of the water in the glass change if the same edge becomes horizontal ?

(**Ans.** No change in the level because quantity of water displaced will remain the same.)

81. A vessel filled with water falls with an acceleration $a<g$.. How does the pressure p in the vessel change with depth ?

(**Ans.** $p=\rho h(g-a)$. This can be seen by considering downward motion of a column of liquid.)

82. A vessel with a body floating in it falls with an acceleration $a<g$. Will the body rise to the surface ?

(**Ans**. No. The body does not rise to the surface.)

83. A trough filled with water is supported on a horizontal knife edge. Will the equilibrium be disturbed if a small wooden block is gently placed on the surface of water ?

(**Ans.** Equilibrium will not be disturbed, since according to the Pascal's law the pressure on the bottom of the vessel will be the same everywhere.)

84. What would happen if the forces of interaction between the molecules of water disappear suddenly ?

(**Ans.** The water would be converted into an ideal gas.)

85. Will the energy of the air in a room increase if a stove is heated in it ?

(**Ans.** No. The energy per unit mass of air $=cT$ where c is a constant dan T is the absolute temperature of the room By the gas equation $\frac{p}{\rho T}=$ a constant, or $\rho T \propto p$. Thus the energy of all the air in the room is determined by the pressure only. The pressure in the room is equal to the atmospheric pressure and hence the energy of the room does not change.)

86. Is it more difficult to compress a litre of air to three atmospheres, or a litre of water ?

(**Ans.** It is more difficult to compress a litre of air, because more work has to be done in this case.)

87. Why moisture is retained longer in soil if it is harrowed ?

(**Ans.** In the compact mass of soil, that is, when it is not harrowed there are capillaries in the soil. Water rises in them to the surface from where it (water) is intensively evaporated. Harrowing destroys all the capillaries and so the moisture is retained longer.)

88. To remove grease spot from a cotton shirt it is better to apply petrol to the edges of the spot, while the spot itself should never be wetted with petrol ? Why not ?

(**Ans.** The surface tension of pure petrol is less than that of petrol in which grease is dissolved. When the petrol is applied to the edges, due to difference in surface tension of pure petrol and impure petrol the spot contracts to the centre. If the spot is wetted, it will spread over the fabric.)

89. Why do drops of water appear at the end of a burning piece of fire-wood ?

(**Ans.** The capillary forces move the water from the hotter region (where surface tension is greater) to the colder region (where surface tension is less.)

90. Why does a stream of steam issuing from a boiling kettle become visible when the burner is switched off, while no steam was visible before ?

(**Ans.** Steam becomes visible only when a small cloud of minute drops is formed after condensation. So long burner is on, the kettle is enveloped by hot air and, therefore, steam cannot condense and be visible. When the burner is switched off, the hot envelope of air dissapear. This favours condensation of steam and so it becomes visible.)

91. Why is iron or steel used as reinforcement in structures, while other materials such as duralumin are hardly used ?

(**Ans.** Because the coefficient of expansion of concrete is very close to that of iron or steel.)

92. Equal quantities of salt are dissolved in two identical vessels filled with water. In one case the salt is one large crystal and in the other powder. In which case will the temperature of the solution be higher after the salt is completely dissolved, if in both the cases the salt and the water originally had the same temperature ?

(**Ans.** The water in the second case will be at higher temperature. When a crystal is dissolved, extra energy is required to destroy its crystal lattice. The energy needed is derived from the water and hence there is greater fall in temperature. The effect, however, will be extremely small.)

93. Why boilers are generally spherical in shape ?

(Ans. The pressure of steam in a boiler tends to blow it up. This is prevented by elastic force. Let us consider a boiler of any arbitrary shape. Divide the boiler into two halves by a plane and consider the stability of one half (like that of equilibrium one half of a soap bubble in surface tension) of the boiler. Let f be the elastic force per unit length of the perimeter of the section and A be the area of the section. The resultant inward elastic force$=f.l$ where l is the length of the perimeter and the resultant outward force$=A.p$ where p is the pressure of the steam. For stability of the boiler we have $A.p=f.l$ or $f=\frac{A.p}{l}$.

Obviously for a given value of the pressure of the steam that shape will be most advantageous for which f is a minimum. As is seen the force f will be minimum with the smallest ratio between the cross-sectional area and the perimeter. As is known, this ratio is a minimum for a circle. It is also known that a circular section can be obtained cutting a sphere with any plane. Therefore, a sphere is the most advantageous shape for the boiler.)

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